Lecture 4h

The Determinant of a Product

(pages 270-1)

One of the unexpected consequences of the ease with which one can compute a determinant using elementary row operations is the fact that the determinant preserves multiplication. That is $\det(AB) = (\det A)(\det B)$. It is not at all obvious how these two facts are related, but the following Lemma gives you a glimpse at the reason why.

Lemma 5.2.6: If E is an $n \times n$ elementary matrix and C is any $n \times n$ matrix, then

$$\det(EC) = (\det E)(\det C)$$

Proof of Lemma 5.2.6: Consider the three different types of elementary row operation. If E corresponds to a row swap, then we can row reduce E to I using the same row swap, and thus $\det E = -\det I = -1$. And since EC is a matrix obtained from C by a row swap, we know $\det(EC) = (-1)(\det C)$. Thus, $\det(EC) = (-1)(\det C) = (\det E)(\det C)$.

Similarly, if E corresponds to multiplying a row by r, then we can row reduce E to I by multiplying the same row by $(1/r)$. Thus $\det I = (1/r)\det E$, or $\det E = r\det I = r$. And since EC is a matrix obtained from C by multiplying a row by r, we know that $\det(EC) = r(\det C)$. Thus, $\det(EC) = (r)(\det C) = (\det E)(\det C)$.

Finally, if E corresponds to adding a r times row i to row j, then we can row reduce E to I by adding $-r$ times row i to row j. Since row operations of this type do not change the determinant, we have $\det E = \det I = 1$. And since EC is a matrix obtained from C by adding r times row i to row j, we have that $\det C = \det(EC)$. Thus, $\det(EC) = (1)(\det C) = (\det E)(\det C)$.

So, no matter what type of elementary matrix E is, we have shown that $\det(EC) = (\det E)(\det C)$.

Theorem 5.2.7: If A and B are $n \times n$ matrices, then $\det(AB) = (\det A)(\det B)$

Proof of Theorem 5.2.7: We break this proof into two cases: either A can be written as a product of elementary matrices, or it can't.

Case 1–A can be written as a product of elementary matrices: Let $A = E_1 \cdots E_k$, where $E_1, \ldots, E_k$ are elementary matrices. Then

$$\begin{aligned}
\det(AB) &= \det(E_1 \cdots E_k B) \\
&= (\det E_1)(\det(E_2 \cdots E_k B)) \\
&= \cdots \\
&= (\det E_1)(\det E_2) \cdots (\det E_k)(\det B) \\
&= (\det(E_1 E_2))(\det E_3) \cdots (\det E_k)(\det B) \\
&= \cdots \\
&= (\det(E_1 \cdots E_k))(\det B) \\
&= (\det A)(\det B)
\end{aligned}$$

Case 2–A can not be written as a product of elementary matrices: Then the rank of A is not n. Using the invertible matrix theorem, we get two facts. The first is that A is not invertible, which means $\det A = 0$, and thus that $(\det A)(\det B) = 0$. The second is that the range of A (thinking of it as a linear mapping) is not $\mathbb{R}^n$. This means that there is some $\vec{y} \in R^n$ such that the equation $A\vec{z} = \vec{y}$ has no solution. But this means that the equation $A(B\vec{x}) = \vec{y}$ cannot have a solution (since if it did, $B\vec{x}$ would be a $\vec{z}$ such that $A\vec{z} = \vec{y}$). And thus, $(AB)\vec{x} = \vec{y}$ does not have a solution, so the range of AB is not $\mathbb{R}^n$. And so, AB is not invertible, and thus $\det(AB) = 0$. But this means that $\det(AB) = 0 = (\det A)(\det B)$.

Note: The statement $\det(A+B) = \det(A) + \det(B)$ IS NOT TRUE.

Example: Let $A = \begin{bmatrix} 2 & 4 & -2 \\ 0 & -5 & 2 \\ 0 & 8 & -4 \end{bmatrix}$, and $B = \begin{bmatrix} 5 & 0 & 1 \\ 2 & 1 & 4 \\ 6 & 3 & 6 \end{bmatrix}$. Then:

And $AB = \begin{bmatrix} 0 & -5 & 0 \\ 2 & 1 & -8 \\ -8 & -4 & 8 \end{bmatrix}$, and $A+B = \begin{bmatrix} 7 & 4 & -2 \\ 2 & -4 & 6 \\ 6 & 11 & 2 \end{bmatrix}$, and

$\det A =$ (expanding along the first column) $2(-1)^{1+1}\begin{vmatrix} -5 & 2 \\ 8 & -4 \end{vmatrix} + 0 + 0 = 2(20-16) = 8$

$\det B =$ (expanding along the first row) $5(-1)^{1+1}\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix} + 0 + (1)(-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = 5(6-12) + (6-6) = -30$

$\det(AB) =$ (expanding along the first row) $0 + (-5)(-1)^{1+2}\begin{vmatrix} 2 & -8 \\ -8 & 8 \end{vmatrix} + 0 = 5(16-64) = -240 = 8(-30) = (\det A)(\det B)$.

$\det(A+B) =$ (expanding along the first row) $7(-1)^{1+1}\begin{vmatrix} -4 & 6 \\ 11 & 2 \end{vmatrix} + 4(-1)^{1+2}\begin{vmatrix} 2 & 6 \\ 6 & 2 \end{vmatrix} - 2(-1)^{1+3}\begin{vmatrix} 2 & -4 \\ 6 & 11 \end{vmatrix} = 7(-8-66) - 4(4-36) - 2(22+24) = -518 + 128 - 92 = -482 \neq 8 - 30$.