Lecture 4g

The Determinant and Invertibility

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We've done a lot with calculating the determinant, but when the determinant was first introduced it was as a value that would determine whether or not a system of linear equations was consistent with a unique solution solution. Well, we've already seen that a system is consistent with a unique solution if and only if the coefficient matrix is invertible. And so, we can connect the determinant of A with the invertibility of A as follows:

Theorem 5.2.5: If A is an $n \times n$ matrix, then the following are equivalent:
(1) $\det A \neq 0$
(2) $\text{rank}(A) = n$
(3) A is invertible
Notice that this is an extension of the Invertible Matrix Theorem (Theorems 3.5.4 and 3.5.6).

Proof of Theorem 5.2.5: We've already shown that the statement $\text{rank}(A) = n$ is equivalent to the statement A is invertible (Invertible Matrix Theorem), so all we need to show now is that $\det A \neq 0$ if and only if $\text{rank}(A) = n$. The first thing to notice is something I mentioned at the end of the last lecture: if B is row equivalent to A, then $\det A$ is a non-zero scalar multiple of $\det B$, and thus $\det B = 0$ if and only if $\det A = 0$. (This is because whatever row operations are used to get B from A can only result in multiplying the determinant by a non-zero scalar.) As such, if B is the *reduced* row echelon form of A, then $\det B \neq 0$ if and only if $\det A \neq 0$. But the reduced row echelon form of an $n \times n$ matrix is an upper-triangular matrix that has either a 1 or 0 on the main diagonal. Thus, the only way for such a matrix to have a non-zero determinant is if it has only 1s on the main diagonal–which means that it is the identity matrix. And so we have shown that $\det A \neq 0$ if and only if its reduced row echelon form is the identity matrix. And we already know that an $n \times n$ matrix has reduced row echelon form I if and only if its rank is n. (Intriguingly, this is never actually stated as a theorem in the textbook, but I hope that it is clear.) So we know that $\det A \neq 0$ if and only if $\text{rank}(A) = n$.

Example: $\begin{vmatrix} 2 & 4 \\ -4 & -8 \end{vmatrix} = (2)(-8) - (-4)(4) = 0$, so $\begin{bmatrix} 2 & 4 \\ -4 & -8 \end{bmatrix}$ is not invertible.

$\begin{vmatrix} 2 & 4 \\ -4 & -7 \end{vmatrix} = (2)(-7) - (-4)(4) = 2$, so $\begin{bmatrix} 2 & 4 \\ -4 & -7 \end{bmatrix}$ is invertible.

We will develop more relationships between the inverse of a matrix and the determinant in Section 5.3.