Lecture 4e

Elementary Row Operations and the Determinant

(pages 264-8)

We saw in the previous lecture that it is much easier to calculate the determinant of a triangular matrix, or better yet, a matrix with a row or column of all zeros. And we've already seen how to use elementary row operations to find a row equivalent matrix that either has a row of all zeros, or at least is upper triangular. The purpose of this lecture is to see how elementary row operations affect the determinant, so that we only need to calculate these easier determinants. In particular, this lecture is going to look more at the theory of the process (mostly stating theorems and giving proofs, with a couple of simple examples). But as I found that getting through all this material can be mentally taxing, I have separated out the more practical "How do we do this" examples into the next lecture. As such, there is not an assignment at the end of this lecture.

If you recall, there are three types of elementary row operations: multiply a row by a non-zero scalar, interchange two rows, and replace a row with the sum of it and a scalar multiple of another row. We will look at the effect that each of these operations has on the determinant.

Theorem 5.2.1: Let A be an $n \times n$ matrix and let B be the matrix obtained from A by multiplying the i-th row of A by r. Then $\det B = r\det A$.

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$, so that B is obtained from A by multiplying the first row of A by 2. Notice that

$$\begin{aligned} \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} &= ((2)(5) - (3)(4)) \\ &= (2(1)(5) - 2(3)(2)) = 2((1)(5) - (3)(2)) \\ &= 2\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} \\ &= 2(-1) = -2 \end{aligned}$$

Proof of Theorem 5.2.1: If we expand along the i-th row of B to calculate its determinant, we get

$$\det B = b_{i1}C_{i1} + \cdots + b_{in}C_{in}$$

But the reason we have chosen the i-th row of B is that we know that $b_{ij} = ra_{ij}$ for $j = 1, \ldots n$. Moreover, since the submatrices $B(i, j)$ will all have row i

removed, and since this is the only place where B differs from A, we see that $A(i,j) = B(i,j)$. Thus, the cofactor C_{ij} for b_{ij} is the same as the cofactor C^*_{ij} for a_{ij}. So we have that

$$\det B = ra_{i1}C_{i1} + \cdots + ra_{in}C_{in} = r(a_{i1}C^*_{i1} + \cdots + a_{in}C^*_{in}) = r\det A$$

Theorem 5.2.2: Suppose that A is an $n \times n$ matrix and that B is the matrix obtained from A by swapping two rows. Then $\det B = -\det A$.

Example: Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & 3 \\ 0 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}$, so that B is obtained from A by swapping the first and third rows. We see that

$$\begin{aligned}
&\det B \quad \text{(expanding along the second row)} \\
&= 0 + 2(-1)^{2+2}\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix} + (-2)(-1)^{2+3}\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \\
&= 0 + 2(-1)^{2+2}((3)(1) - (1)(3)) + (-2)(-1)^{2+3}((3)(0) - (1)(1)) \\
&= 0 + 2(-1)^{2+2}(-1)((1)(3) - (3)(1)) + (-2)(-1)^{2+3}(-1)((1)(1) - (3)(0)) \\
&= -1\left(0 + 2(-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} + (-2)(-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}\right) \\
&= -\det A \text{ (expanded along the second row)}
\end{aligned}$$

Performing the calculations, we get that $\det A = 2$ and $\det B = -2$.

Proof of Theorem 5.2.2: In our example, we expanded along row 2 because this was a row that had NOT been switched. If we look further at the example, we see that the submatrices of B used to calculate the determinant could be obtained from the corresponding submatrices of A by swapping the rows. Generalizing to any $n \times n$ matrix B, we will find that we are looking at submatrices that are a row swap from the corresponding submatrices of A. As such, we will want to prove this theorem by induction.

Base case: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and let $B = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$. Then B is the *only* matrix that can be obtained from A by swapping rows. And we see that $\det B = a_{21}a_{12} - a_{11}a_{22} = -(a_{11}a_{22} - a_{21}a_{12}) = -\det A$.

Induction hypothesis: For all $k \times k$ matrices A, if B is obtained from A by swapping two rows, then $\det B = -\det A$.

Induction step: Let A be a $(k+1) \times (k+1)$ matrix, and let B be a matrix obtained from A by swapping two rows. Let i be such that the i-th row of A was NOT swapped when making B. (So, row i of A and row i of B are the same.) If we expand along row i, we get

$$\det B = b_{i1}C_{i1} + \cdots + b_{i(k+1)}C_{i(k+1)}$$

To compute $\det B$, we will need to look at the submatrices $B(i,j)$. Our choice of i means that $B(i,j)$ can be obtained from $A(i,j)$ by swapping the same rows as we swapped to get B from A. This means that $B(i,j)$ is a $k \times k$ matrix that is obtained from $A(i,j)$ by swapping two rows, and thus, by our inductive hypothesis, $\det B(i,j) = -\det A(i,j)$. So we now have that

$$\begin{aligned}
\det B &= b_{i1}(-1)^{i+1}\det B(i,1) + \cdots + b_{i(k+1)}(-1)^{i+k+1}\det B(i,k+1) \\
&= a_{i1}(-1)^{i+1}\det B(i,1) + \cdots + a_{i(k+1)}(-1)^{i+k+1}\det B(i,k+1) \\
&= a_{i1}(-1)^{i+1}(-1)\det A(i,1) + \cdots + a_{i(k+1)}(-1)^{i+k+1}(-1)\det A(i,k+1) \\
&= -(a_{i1}(-1)^{i+1}\det A(i,1) + \cdots + a_{i(k+1)}(-1)^{i+k+1}\det A(i,k+1)) \\
&= -\det A
\end{aligned}$$

There is an interesting consequence of Theorem 5.2.2:

Corollary 5.2.3: If two rows of A are equal, then $\det A = 0$.

Proof of Corollary 5.2.3: Suppose that two rows of A are equal, and let B be the matrix obtained from A by swapping these identical rows. Then $B = A$, so $\det B = \det A$. But, by Theorem 2, $\det B = -\det A$. So we have that $\det A = -\det A$. The only number that is equal to its negative is zero, so we have shown that $\det A = 0$.

And now, we turn our attentions to the final row operation:

Theorem 5.2.4: Suppose that A is an $n \times n$ matrix and that B is obtained from A by adding r times the i-th row of A to the k-th row. Then $\det B = \det A$.

Example: Let $A = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 4 & -2 \\ 3 & 1 & 4 \end{bmatrix}$, and let $B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ be the matrix obtained from A by adding 2 times row 1 to row 2. Then we see that

$$\begin{aligned}
\det B \quad & (\text{ expanding along the third row}) \\
&= 3(-1)^{3+1}\begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} + (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} - 4(-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}
\end{aligned}$$

while

$$\begin{aligned}
\det A \quad & (\text{ expanding along the third row}) \\
&= 3(-1)^{3+1}\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} + (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix} - 4(-1)^{3+3}\begin{vmatrix} 1 & -1 \\ -2 & 4 \end{vmatrix}
\end{aligned}$$

In comparing the necessary calculations for $\det A$ and $\det B$, we see that the submatrices involved in the calculation of det$B$ can be obtained from the corresponding submatrices of det$A$ by adding 2 times row 1 to row 2–the same row operation we used to get B from A. Moreover, we note that

$$\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = (-1)(-2) - (4)(1) = -2 \quad \text{and} \quad \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} = (-1)(0) - (2)(1) = -2$$

$$\begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix} = (1)(-2) - (-2)(1) = 0 \quad \text{and} \quad \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = (1)(0) - (0)(1) = 0$$

$$\begin{vmatrix} 1 & -1 \\ -2 & 4 \end{vmatrix} = (1)(4) - (-2)(-1) = 2 \quad \text{and} \quad \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = (1)(2) - (0)(-1) = 2$$

And so we see that $\det A = \det B = 3(-1)^{3+1}(-2) + (-1)^{3+2}(0) - 4(-1)^{3+3}(2) = -14$.

Note: I expanded along the third row to parallel the technique used to prove Theorem 5.2.4. It would have been much quicker to calculate the determinant of B (and thus, of A) to expand along the second row. In fact, the entire reason we are looking at row operations is so that we can make such simpler calculations!

<u>Proof of Theorem 5.2.4</u>: This theorem also needs to be proved by induction.

Base case: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and let $B = \begin{bmatrix} a_{11} & a_{12} \\ ra_{11} + a_{21} & ra_{12} + a_{22} \end{bmatrix}$. Then

$$\begin{aligned} \det B &= a_{11}(ra_{12} + a_{22}) - a_{12}(ra_{11} + a_{21}) \\ &= ra_{11}a_{12} + a_{11}a_{22} - ra_{11}a_{12} - a_{21}a_{12} \\ &= a_{11}a_{22} - a_{21}a_{12} \\ &= \det A \end{aligned}$$

Similarly, if $B = \begin{bmatrix} ra_{21} + a_{11} & ra_{22} + a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$\begin{aligned} \det B &= (ra_{21} + a_{11})a_{22} - a_{21}(ra_{22} + a_{12}) \\ &= ra_{21}a_{22} + a_{11}a_{22} - ra_{21}a_{22} - a_{21}a_{12} \\ &= a_{11}a_{22} - a_{21}a_{12} \\ &= \det A \end{aligned}$$

And so we see that if A is a 2×2 matrix, and B is obtained from A by adding a multiple of one row to another, then $\det B = \det A$.

Induction hypothesis: For all $k \times k$ matrices A, if B is obtained from A by adding a multiple of one row to another, then $\det B = \det A$.

Induction step: Let A be a $(k+1) \times (k+1)$ matrix, and suppose that B is the matrix obtained from A by adding r times row i to row j, and let $l \neq i, j$. Then we can compute the determinant of B by expanding along row l, getting

$$\det B = b_{l1} C_{l1} + \cdots + b_{l(k+1)} C_{l(k+1)}$$

Our choice of l gets us that $b_{lm} = a_{lm}$ for all $m = 1, \ldots, k+1$. So, now let's consider the submatrices $B(l, m)$ and $A(l, m)$. Because $l \neq i, j$, rows i and j both appear in $B(l, m)$ and $A(l, m)$ (with their entries from column m omitted). This means that $B(l, m)$ can be obtained from $A(l, m)$ by adding r times row i to row j. And since $A(l, m)$ and $B(l, m)$ are $k \times k$ matrices, we use our induction hypothesis and get that $\det A(l, m) = \det B(l, m)$. And so we see that

$$\begin{aligned} \det B &= b_{l1}(-1)^{l+1} \det B(l, 1) + \cdots + b_{l(k+1)}(-1)^{l+k+1} \det B(l, (k+1)) \\ &= a_{l1}(-1)^{l+1} \det A(l, 1) + \cdots + a_{l(k+1)}(-1)^{l+k+1} \det A(l, (k+1)) \\ &= \det A \end{aligned}$$