Lecture 4d

Calculating a Determinant

(pages 259-61)

The definition we have used for a determinant is known as expanding by cofactors. Even more specifically, we have expanded by cofactors along the first row. But the first step in (potentially) making the calculation of a determinant easier is to notice that we can in fact expand by cofactors along any row, or any column.

Theorem 5.1.1: Suppose that A is an $n \times n$ matrix. Then the determinant of A may be obtained by a **cofactor expansion** along any row or any column. In particular, the expansion of the determinant along the i-th row of A is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The expansion of the determinant along the j-th column of A is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdot + a_{nj}C_{nj}$$

Proof of Theorem 5.1.1: The proof of theorem 1 lies outside the scope of this course.

Example: To compute the determinant of $\begin{bmatrix} -1 & 2 & 1 \\ 0 & 5 & 0 \\ 0 & 3 & -4 \end{bmatrix}$, I could expand along the first column and get:

$$\begin{vmatrix} -1 & 2 & 1 \\ 0 & 5 & 0 \\ 0 & 3 & -4 \end{vmatrix} =$$

$$(-1)(-1)^{1+1}\begin{vmatrix} 5 & 0 \\ 3 & -4 \end{vmatrix} + 0 + 0 =$$

$$-1((5)(-4) - (3)(0)) = 20$$

Or, I could expand along the second row, and get:

$$\begin{vmatrix} -1 & 2 & 1 \\ 0 & 5 & 0 \\ 0 & 3 & -4 \end{vmatrix} =$$

$$0 + (5)(-1)^{2+2}\begin{vmatrix} -1 & 1 \\ 0 & -4 \end{vmatrix} + 0 =$$

$$5((-1)(-4) - (0)(1)) = 20$$

These would be the best options, since the column and row in question each contains two zero entries, thus eliminating our need to calculating the corresponding cofactor. To that end, we notice the following:

Theorem 5.1.2: If one row (or column) of an $n \times n$ matrix A contains only zeros, then $\det A = 0$.

Proof of Theorem 5.1.2: Simply expand along the column or row of all zeros. Then you will have either

$$\det A = 0C_{i1} + 0C_{i2} + \cdots + 0C_{in} = 0$$

if A has a row of all zeros, or if we expand along a column of all zeros we get

$$\det A = 0C_{1j} + 0C_{2j} + \cdots + 0C_{nj} = 0$$

Example: $\begin{vmatrix} 1 & 0 & 2 & 0 & 1 \\ -2 & 0 & 2 & -2 & 2 \\ 1 & 0 & 3 & 0 & 1 \\ 3 & 0 & -3 & 3 & 3 \\ 4 & 0 & 1 & 0 & -4 \end{vmatrix} = 0$, because the second column of the matrix is all zeros.

You should note that Theorem 5.1.2 is not an "if and only if" statement. That is, it is possible–in fact, quite common–for a matrix to have a determinant of zero even if it does not have a row or column of all zeros. We have already some examples of these in the textbook and the lectures.

So, while having a row or column of all zeros makes the determinant calculation really easy, simply having a lot of zeros will definitely make things easier. In the specific case where a matrix is either upper-triangular (all the entries below the main diagonal are zeros) or lower-triangular (all the entries above the main diagonal are zero), there is an easy way to calculate the determinant.

Theorem 5.1.3: If A is an upper- or lower-triangular matrix, then the determinant of A is the product of the diagonal entries of A. That is

$$\det A = a_{11}a_{22}\cdots a_{nn}$$

Proof of Theorem 5.1.3: The proof of Theorem 3 can be done by induction on the size of the matrix. (Which makes sense, since the determinant is defined inductively.) So, as our base case, let's consider the determinant of a 2×2 upper-triangular matrix. Such a matrix would look like $\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$. The determinant of this matrix is $((a_{11})(a_{22}) - (0)(a_{12})) = a_{11}a_{22}$, as desired. The determinant of a 2×2 lower-triangular matrix $\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}$ is $((a_{11})(a_{22}) - (a_{21})(0)) = a_{11}a_{22}$, as desired.

And so, we have seen that the determinant of either an upper- or lower-triangular 2×2 matrix is the product of its diagonal entries. Now, by way of induction, let us assume that the determinant of either an upper- or lower-triangular $k \times k$ matrix is the product of its diagonal entries, and let's look at a $(k+1) \times (k+1)$ upper-triangular matrix A. We can expand A along its first column, and get

$$\det A = a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{(k+1)1}C_{(k+1)1}$$

Because of the structure of an upper-triangular matrix, we know that $a_{j1} = 0$ for $j \neq 1$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

And thus our calculation of the determinant is

$$\det A = a_{11}C_{11} + 0C_{21} + \cdot + 0C_{(k+1)1} = a_{11}C_{11}$$

But what is C_{11}? Thanks to the structure of an upper-triangular matrix, we get that the submatrix $A(1,1)$ is a $k \times k$ upper-triangular matrix.

$$\begin{bmatrix} - & - & \cdots & \cdots & - \\ - & a_{22} & a_{23} & \cdots & a_{2n} \\ - & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ - & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

Thus, by our inductive hypothesis, we know that the determinant of $A(1,1)$ is the product of its diagonal entries. Because $A(1,1)$ was obtained from A by removing the first row and the first column, the diagonal entries of $A(1,1)$ are $a_{22}, a_{33}, \ldots, a_{(k+1)(k+1)}$, and thus $\det A(1,1) = a_{22}a_{33}\cdots a_{(k+1)(k+1)}$. And thus, $C(1,1) = (-1)^{1+1}a_{22}a_{33}\cdots a_{(k+1)(k+1)} = a_{22}a_{33}\cdots a_{(k+1)(k+1)}$. And so, at last we see that $\det A = a_{11}C_{11} = a_{11}a_{22}a_{33}\cdots a_{(k+1)(k+1)}$, as desired.

The proof in the case that A is a lower-triangular matrix is almost identical.

Consider the following concrete example:

Example: $\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & -7 \end{vmatrix} =$ (expanding along the first column)

$(1)(-1)^{1+1}\begin{vmatrix} 2 & 3 & -1 \\ 0 & 5 & -2 \\ 0 & 0 & -7 \end{vmatrix} + 0 + 0 + 0 =$ (expanding along the first column)

$1\left(2(-1)^{1+1}\begin{vmatrix} 5 & -2 \\ 0 & -7 \end{vmatrix} + 0 + 0\right) =$

$1(2((5)(-7) - (-2)(0)) = (1)(2)(5)(-7) = -70$

There is one last theorem in this section, that has nothing to do with zero entries. But it turns out that the inductive approach used to prove Theorem 5.1.3 can also be used to show the following:

Theorem 5.1.4: If A is an $n \times n$ matrix, then $\det A = \det A^T$.

Proof of Theorem 5.1.4: The base case of a 2×2 matrix is easy to see, since $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$ and $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc = ad - cb$. The inductive steps are rather tedious, though, so we will not cover the rest of the proof.