Lecture 4c

The Determinant of an $n \times n$ Matrix

(pages 258-9)

Just as we calculated the determinant of a 3×3 matrix by looking at smaller 2×2 matrices within it, we calculate the determinant of a 4×4 matrix by looking at 3×3 matrices within it. And in general, the determinant of an $n \times n$ matrix depends on the determinant of $(n-1) \times (n-1)$ matrices within it, which in turn depends on the determinants of $(n-2) \times (n-2)$ matrices within them, and so on. And so, to define the determinant in the general $n \times n$ case, we will generalize our definitions from the 3×3 case.

Definition: Let A be an $n \times n$ matrix. Let $A(i, j)$ denote the $(n-1) \times (n-1)$ submatrix obtained from A by deleting the i-th row and the j-th column.

Example: Let $A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 3 & -3 & 4 & -4 \\ -5 & -5 & 6 & 6 \\ -7 & 8 & 7 & -8 \end{bmatrix}$. Then

$$A(1,1) = \begin{bmatrix} - & - & - & - \\ - & -3 & 4 & -4 \\ - & -5 & 6 & 6 \\ - & 8 & 7 & -8 \end{bmatrix} = \begin{bmatrix} -3 & 4 & -4 \\ -5 & 6 & 6 \\ 8 & 7 & -8 \end{bmatrix}$$

$$A(1,2) = \begin{bmatrix} - & - & - & - \\ 3 & - & 4 & -4 \\ -5 & - & 6 & 6 \\ -7 & - & 7 & -8 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -4 \\ -5 & 6 & 6 \\ -7 & 7 & -8 \end{bmatrix}$$

$$A(1,3) = \begin{bmatrix} - & - & - & - \\ 3 & -3 & - & -4 \\ -5 & -5 & - & 6 \\ -7 & 8 & - & -8 \end{bmatrix} = \begin{bmatrix} 3 & -3 & -4 \\ -5 & -5 & 6 \\ -7 & 8 & -8 \end{bmatrix}$$

$$A(1,4) = \begin{bmatrix} - & - & - & - \\ 3 & -3 & 4 & - \\ -5 & -5 & 6 & - \\ -7 & 8 & 7 & - \end{bmatrix} = \begin{bmatrix} 3 & -3 & 4 \\ -5 & -5 & 6 \\ -7 & 8 & 7 \end{bmatrix}$$

$$A(2,3) = \begin{bmatrix} 1 & 2 & - & -2 \\ - & - & - & - \\ -5 & -5 & - & 6 \\ -7 & 8 & - & -8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ -5 & -5 & 6 \\ -7 & 8 & -8 \end{bmatrix}$$

$$A(4,3) = \begin{bmatrix} 1 & 2 & - & -2 \\ 3 & -3 & - & -4 \\ -5 & -5 & - & 6 \\ - & - & - & - \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 3 & -3 & -4 \\ -5 & -5 & 6 \end{bmatrix}$$

Definition: Let A be an $n \times n$ matrix, and let a_{ij} be an entry of A. Then the **cofactor** of a_{ij} is defined to be

$$C_{ij} = (-1)^{i+j}\det A(i,j)$$

Example: Let A be as in the previous example. Using the results of this example, we see that:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -3 & 4 & -4 \\ -5 & 6 & 6 \\ 8 & 7 & -8 \end{vmatrix}$$

Of course, to calculate $\begin{vmatrix} -3 & 4 & -4 \\ -5 & 6 & 6 \\ 8 & 7 & -8 \end{vmatrix}$, we need to expand it by cofactors as well:

$$\begin{vmatrix} -3 & 4 & -4 \\ -5 & 6 & 6 \\ 8 & 7 & -8 \end{vmatrix} =$$

$$-3(-1)^{1+1} \begin{vmatrix} 6 & 6 \\ 7 & -8 \end{vmatrix} + (4)(-1)^{1+2} \begin{vmatrix} -5 & 6 \\ 8 & -8 \end{vmatrix} + (-4)(-1)^{1+3} \begin{vmatrix} -5 & 6 \\ 8 & 7 \end{vmatrix} =$$

$$-3((6)(-8) - (7)(6)) - 4((-5)(-8) - (8)(6)) - 4((-5)(7) - (8)(6)) =$$

$$-3(-90) - 4(-83) - 4(-83) = 634$$

And so we finally have that $C_{11} = (1)^{1+1}(634) = 634$.

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 4 & -4 \\ -5 & 6 & 6 \\ -7 & 7 & -8 \end{vmatrix}, \text{ where}$$

$$\begin{vmatrix} 3 & 4 & -4 \\ -5 & 6 & 6 \\ -7 & 7 & -8 \end{vmatrix} =$$

$$3(-1)^{1+1} \begin{vmatrix} 6 & 6 \\ 7 & -8 \end{vmatrix} + (4)(-1)^{1+2} \begin{vmatrix} -5 & 6 \\ -7 & -8 \end{vmatrix} + (-4)(-1)^{1+3} \begin{vmatrix} -5 & 6 \\ -7 & 7 \end{vmatrix} =$$

$$3((6)(-8) - (7)(6)) - 4((-5)(-8) - (-7)(6)) - 4((-5)(7) - (-7)(6)) =$$

$$-270 - 323 - 28 = -626$$

And so we see that $C_{12} = (-1)^{1+2}(-626) = 626$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & -3 & -4 \\ -5 & -5 & 6 \\ -7 & 8 & -8 \end{vmatrix}$, where

$\begin{vmatrix} 3 & -3 & -4 \\ -5 & -5 & 6 \\ -7 & 8 & -8 \end{vmatrix} =$

$3(-1)^{1+1}\begin{vmatrix} -5 & 6 \\ 8 & -8 \end{vmatrix} + (-3)(-1)^{1+2}\begin{vmatrix} -5 & 6 \\ -7 & -8 \end{vmatrix} + (-4)(-1)^{1+3}\begin{vmatrix} -5 & -5 \\ -7 & 8 \end{vmatrix} =$

$3((-5)(-8) - (8)(6)) + 3((-5)(-8) - (-7)(6)) - 4((-5)(8) - (-7)(-5)) =$

$-24 + 246 + 300 = 522$

And so we see that $C_{13} = (1)^{1+3}(522) = 522$

$C_{14} = (-1)^{1+4}\begin{vmatrix} 3 & -3 & 4 \\ -5 & -5 & 6 \\ -7 & 8 & 7 \end{vmatrix}$, where

$\begin{vmatrix} 3 & -3 & 4 \\ -5 & -5 & 6 \\ -7 & 8 & 7 \end{vmatrix} =$

$3(-1)^{1+1}\begin{vmatrix} -5 & 6 \\ 8 & 7 \end{vmatrix} + (-3)(-1)^{1+2}\begin{vmatrix} -5 & 6 \\ -7 & 7 \end{vmatrix} + (4)(-1)^{1+3}\begin{vmatrix} -5 & -5 \\ -7 & 8 \end{vmatrix} =$

$3((-5)(7) - (8)(6)) + 3((-5)(7) - (-7)(6)) + 4((-5)(8) - (-7)(-5)) =$

$-249 + 21 - 300 = -528$

And so we see that $C_{14} = (-1)^{1+4}(-528) = 528$

I will not show the calculations for C_{23} and C_{43}, as the above calculations are all we will need to find the determinant of A. To that end, we have the following definition:

Definition: The **determinant** of an $n \times n$ matrix A is defined by

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Example: Continuing from our previous examples, we now see that $\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14} = (1)(634) + (2)(626) + (-1)(522) + (-2)(528) = 308$.

So, calculating the determinant is a very long process. Much longer than just row reducing the matrix to find the solution in the first place! But over the next few lectures we'll explore some ways that can make calculating a determinant easier.