Lecture 4b

The Determinant of a 3×3 Matrix

(pages 256-8)

As I mentioned in the previous lecture, the determinant of a 2×2 matrix is a value that determines whether or not the related system of equations has a unique solution. It turns out that the notion of such a "determinant" value is not exclusive to systems of two equations. It is rather tedious to show the origin of these values for larger systems, so instead we'll simply teach you how to calculate the determinant.

And our first step in that process will be to look at the determinant of a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. We build this determinant from the determinant of 2×2 matrices found within A:

Definition: Let A be a 3×3 matrix. Let $A(i, j)$ denote the 2×2 submatrix obtained from A by deleting the i-th row and the j-th column.

Example: Let $A = \begin{bmatrix} 2 & -4 & 1 \\ 8 & 0 & -3 \\ 7 & 4 & -5 \end{bmatrix}$. Then

$$A(1,1) = \begin{bmatrix} - & - & - \\ - & 0 & -3 \\ - & 4 & -5 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 4 & -5 \end{bmatrix}$$

$$A(1,2) = \begin{bmatrix} - & - & - \\ 8 & - & -3 \\ 7 & - & -5 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ 7 & -5 \end{bmatrix}$$

$$A(1,3) = \begin{bmatrix} - & - & - \\ 8 & 0 & - \\ 7 & 4 & - \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 7 & 4 \end{bmatrix}$$

$$A(2,1) = \begin{bmatrix} - & -4 & 1 \\ - & - & - \\ - & 4 & -5 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 4 & -5 \end{bmatrix}$$

$$A(3,2) = \begin{bmatrix} 2 & - & 1 \\ 8 & - & -3 \\ - & - & - \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 8 & -3 \end{bmatrix}$$

Definition: Let a_{ij} be an entry in a 3×3 matrix A. Then the **cofactor** of a_{ij} is

$$C_{ij} = (-1)^{(i+j)} \det A(i, j)$$

Example: Let A be as in our previous example. Using the results of our previous example, we have:

$$C_{11} = (-1)^{1+1}\det A(1,1) = (1)\begin{vmatrix} 0 & -3 \\ 4 & -5 \end{vmatrix} = (0)(-5) - (-3)(4) = \ 12$$

$$C_{12} = (-1)^{1+2}\det A(1,2) = (-1)\begin{vmatrix} 8 & -3 \\ 7 & -5 \end{vmatrix} = (-1)((8)(-5) - (7)(-3)) = \ 19$$

$$C_{13} = (-1)^{1+3}\det A(1,3) = (1)\begin{vmatrix} 8 & 0 \\ 7 & 4 \end{vmatrix} = (8)(4) - (7)(0) = \ 32$$

$$C_{21} = (-1)^{2+1}\det A(2,1) = (-1)\begin{vmatrix} -4 & 1 \\ 4 & -5 \end{vmatrix} = (-1)((-4)(-5)+(1)(4)) = \ -24$$

$$C_{32} = (-1)^{3+2}\det A(3,2) = (-1)\begin{vmatrix} 2 & 1 \\ 8 & -3 \end{vmatrix} = (-1)((2)(-3) - (1)(8)) = \ 14$$

<u>Definition</u>: The determinant of a 3×3 matrix A is defined to be

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Example: Again, let A be as in the previous examples. Then we can use our previous results to calculate

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 2(12) - 4(19) + 1(32) = -20$$

Instead of computing the cofactors first, and then the determinant, we usually think of the cofactors as part of the determinant calculation. As such, a determinant calculation would look like this.

Example:

$$\begin{vmatrix} 2 & 3 & -2 \\ -1 & 4 & 3 \\ 5 & -3 & -8 \end{vmatrix} =$$

$$2(-1)^{1+1}\begin{vmatrix} 4 & 3 \\ -3 & -8 \end{vmatrix} + 3(-1)^{1+2}\begin{vmatrix} -1 & 3 \\ 5 & -8 \end{vmatrix} + (-2)(-1)^{1+3}\begin{vmatrix} -1 & 4 \\ 5 & -3 \end{vmatrix} =$$

$(2)(1)((4)(-8) - (-3)(3)) + (3)(-1)((-1)(-8) - (5)(3)) + (-2)(1)((-1)(-3) - (4)(5)) =$

$2(-23) - 3(-7) - 2(-17) =$

9

Because of the multiplicative nature of determinant calculations, this process becomes much simpler when the entries of the matrix are zero.

Example:

$$\begin{vmatrix} 0 & 2 & 3 \\ 1 & -1 & 0 \\ 4 & 0 & 7 \end{vmatrix} =$$

$$(0)(-1)^{1+1}\begin{vmatrix} -1 & 0 \\ 0 & 7 \end{vmatrix} + (2)(-1)^{1+2}\begin{vmatrix} 1 & 0 \\ 4 & 7 \end{vmatrix} + (3)(-1)^{1+3}\begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} =$$

$$0 - 2((1)(7) - (4)(0)) + 3((1)(0) - (4)(-1)) =$$

$$-2(7) + 3(4) =$$

$$-2$$