Lecture 4a

The Determinant of a 2×2 Matrix

(pages 255-6)

Suppose we were trying to solve a system of two linear equations in two unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

We would row reduce the augmented matrix

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

Because we can't guarantee that any of the a_{ij} are not zero, instead of the standard way of row reducing, aimed at getting us to reduced row echelon form, consider the following:

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \begin{array}{c} a_{21}R_1 \\ a_{11}R_2 \end{array} \sim \left[\begin{array}{cc|c} a_{11}a_{21} & a_{12}a_{21} & b_1a_{21} \\ a_{11}a_{21} & a_{11}a_{22} & b_2a_{11} \end{array}\right] \; R_2 - R_1$$
$$\sim \left[\begin{array}{cc|c} a_{11}a_{21} & a_{12}a_{21} & b_1a_{21} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & b_2a_{11} - b_1a_{21} \end{array}\right]$$

If the number $a_{11}a_{22} - a_{12}a_{21}$ is not zero, then our augmented matrix can not have a bad row, and therefore we know that our system is consistent. (The textbook claims that this is an "if and only if" result, but unfortunately it is wrong. Consider, for example, the system $x_1 + x_2 = 2$, $2x_1 + 2x_2 = 4$, which has general solution $\vec{x} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, even though (1)(2)-(2)(1)=0.) And so, the value $a_{11}a_{22} - a_{12}a_{21}$ can be used to "determine" whether or not a system has a solution. (We will eventually develop the result that the system has a unique solution if and only if $a_{11}a_{22} - a_{12}a_{21}$, so that's what it is really designed to determine.) And so, it became known as the determinant...

<u>Definition</u>: The **determinant** of a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is defined by

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Example: The determinant of $\begin{bmatrix} 1 & 5 \\ 2 & -3 \end{bmatrix}$ is $(1)(-3) - (2)(5) = -3 - 10 = -13$

The determinant of $\begin{bmatrix} 2 & 10 \\ 1 & 5 \end{bmatrix}$ is $(2)(5) - (1)(10) = 10 - 10 = 0$.

The determinant of $\begin{bmatrix} 1 & 0 \\ -2 & 7 \end{bmatrix}$ is $(1)(7) - (-2)(0) = 7 + 0 = 7$

Notation: We write $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ to mean $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Note that although this is "absolute value" type notation, a determinant CAN BE NEGATIVE. One must also be careful not to confuse this determinant notation with regular matrix notation. (When typed, it is usually easy to see the difference, but be careful when handwriting.)

Example: $\begin{vmatrix} -2 & 3 \\ -1 & 6 \end{vmatrix} = (-2)(6) - (-1)(3) = -12 + 3 = -9$