Lecture 1e

Vector Equation of a Line in $\mathbb{R}^3$

(page 11)

In $\mathbb{R}^2$, we needed a point and a direction vector to create a line, and the same is true in $\mathbb{R}^3$. However, in $\mathbb{R}^2$, the concept of direction could also be expressed as a slope, whereas in $\mathbb{R}^3$ there is not a single number that could correspond to the notion of "slope". As such, there is no single equation which can describe a line in $\mathbb{R}^3$. (Equations of the form $ax + by + cz = d$ turn out to describe a plane, NOT a line.) Instead, we were always forced to describe a line in $\mathbb{R}^3$ with parametric equations of the form

$$x_1 = a_1 + tb_1,\ x_2 = a_2 + tb_2,\ x_3 = a_3 + tb_3,\ t \in \mathbb{R}$$

The values a_1, a_2, and a_3 correspond to a point on the line (which we can see by plugging in $t = 0$ to the equations), while the values b_1, b_2, and b_3 correspond to the direction the line travels in the x_1, x_2, and x_3 directions (respectively). This means that the b values combine to form our direction vector! So, instead of using three parametric equations, we combine them to form the single vector equation of a line in $\mathbb{R}^3$:

$$\vec{x} = \vec{a} + t\vec{b},\ t \in \mathbb{R}$$

where $\vec{a}$ is a point on the line, and $\vec{b}$ is a direction vector for the line.

Example: Find a vector equation of the line that passes through the point $P(-3, -5, 1)$ with direction vector $\vec{d} = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$.

With no work at all, we get that the equation is $\vec{x} = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} + t \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$, $t \in \mathbb{R}$.

Now, seldom are we handed a direction vector, but we are often trying to find the equation of a line through two points, say $P(p_1, p_2, p_3)$ and $Q(q_1, q_2, q_3)$. When creating the parametric equations of the line through these points, we would have computed that $b_1 = q_1 - p_1$, $b_2 = q_2 - p_2$, and $b_3 = q_3 - p_3$. This means that our direction vector is $\vec{b} = \begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{bmatrix}$. Notice however that we have $\vec{b} = \vec{PQ}$. As such, a vector equation of the line through P and Q is

$$\vec{x} = \vec{p} + t\vec{PQ},\ t \in \mathbb{R}$$

Example Find a vector equation for the line that passes through $P(-5, 2, 10)$ and $Q(3, -4, -4)$.

First we calculate $\vec{PQ} = \begin{bmatrix} 3 \\ -4 \\ -4 \end{bmatrix} - \begin{bmatrix} -5 \\ 2 \\ 10 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ -14 \end{bmatrix}$. Then we get that an equation for our line is $\vec{x} = \begin{bmatrix} -5 \\ 2 \\ 10 \end{bmatrix} + t \begin{bmatrix} 8 \\ -6 \\ -14 \end{bmatrix}$.

Now, we always say "a" vector equation, instead of "the" vector equation, as there are many (in fact, infinitely many) equations that describe the same line. For starters, we can replace the "point" in our equation with any point on the line. Then, we can also replace our direction vector with any non-zero scalar multiple of that direction vector. For example, the direction vector given in the example above could easily have a factor of 2 pulled out, giving us another vector equation for the line:

$$\vec{x} = \begin{bmatrix} -5 \\ 2 \\ 10 \end{bmatrix} + t \begin{bmatrix} 4 \\ -3 \\ -7 \end{bmatrix},\ t \in \mathbb{R}$$

And of course we could have used the point Q instead, to get

$$\vec{x} = \begin{bmatrix} 3 \\ -4 \\ -4 \end{bmatrix} + t \begin{bmatrix} 4 \\ -3 \\ -7 \end{bmatrix},\ t \in \mathbb{R}$$

We could even have used the vector $\vec{QP}$ instead of $\vec{PQ}$ as our direction vector, which simply results in multiplying the direction vector by (-1). To say the least, this makes it difficult to mark and provide answers to assignment questions about vector equations of lines, as there are many correct variations. So if your answer doesn't match up with the answer I give, that doesn't necessarily mean that you've given an incorrect answer. Want to know the best way to see if your answer is truly the same as mine? Pick two points on my line, and make sure that they are on your line. After all, two points is all it takes to define the line! (If the question was to find an equation for a line through two points, all you need to do is verify that your equation produces those two points.)