Lecture 1d

Vectors in $\mathbb{R}^3$

(pages 9-11)

Well, $\mathbb{R}^2$ is nice, but there's more to explore in this world. Like a third dimension, for example! And all of the work that we've done so far easily extends to $\mathbb{R}^3$. (And to general $\mathbb{R}^n$, but we'll be there soon enough...)

Definition $\mathbb{R}^3$ is the set of all vectors of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, where x_1, x_2, and x_3 are real numbers. Alternatively,

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} | x_1, x_2, x_3 \in \mathbb{R} \right\}$$

As in $\mathbb{R}^2$, we equate the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ with the point (x_1, x_2, x_3), as well as with a directed line segment from the origin to the point (x_1, x_2, x_3). However, plotting points in three dimensions on two dimensional paper poses challenges, so we won't be doing much graphing. But having the visual in your head can often be helpful. To that end, the book notes that we are adopting the convention that our coordinate axes form a "right hand system". Another way to visualize a right hand system is to think that you have a sheet of paper on a table in front of you, with the x_1 axis pointing right and the x_2 axis pointing up as normal, and then the x_3 axis would be sticking up from the table. When sketching points in $\mathbb{R}^3$, we rotate our right hand system so that the x_1 axis points down and to the left and the x_2 axis points down and to the right. The standard axis set-up is illustrated in Figure 1.1.7 on page 9 of the text.

But, getting back to the notion of a vector, instead of just a point, we define addition and scalar multiplication of vectors componentwise as we did in $\mathbb{R}^2$.

Definition If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$, $t \in \mathbb{R}$, then we define addition of vectors by

$$\vec{x}+\vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}$$

and we define scalar multiplication of the vector $\vec{x}$ by the scalar t by

$$t\vec{x} = t\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} tx_1 \\ tx_2 \\ tx_3 \end{bmatrix}$$

Examples:

$$\begin{bmatrix} 2 \\ 7 \\ -5 \end{bmatrix} + \begin{bmatrix} 1 \\ -12 \\ 8 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 7-12 \\ -5+8 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 3 \end{bmatrix}$$

$$-5\begin{bmatrix} 4 \\ -10 \\ -3 \end{bmatrix} = \begin{bmatrix} (-5)(4) \\ (-5)(-10) \\ (-5)(-3) \end{bmatrix} = \begin{bmatrix} -20 \\ 50 \\ 15 \end{bmatrix}$$

As in $\mathbb{R}^2$, we still have a parallelogram (or end-to-end) rule for addition. (This comes from the fact that any two vector will lie in a plane.) We will continue our notational conventions, so that $(-1)\vec{x} = -\vec{x}$ and $\vec{x}-\vec{y} = \vec{x}+(-1)\vec{y}$. We also consider more general directed line segments, and we still get that the directed line segment from P to Q will be notated $\vec{PQ}$, and $\vec{PQ} = \vec{q}-\vec{p}$.

Example: Consider the points $P(1,5,7)$, $Q(2,4,3)$ and $R(3,3,-1)$ in $\mathbb{R}^3$. They correspond to the vectors $\vec{p} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$, $\vec{q} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ and $\vec{r} = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$. We have $\vec{PQ} = \vec{q}-\vec{p} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$ and $\vec{RQ} = \vec{q}-\vec{r} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$. And so we see that $\vec{PQ} = -\vec{RQ}$.

Also, we extend our notation $\vec{0}$ to now equal $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. As we would never try to add a vector from $\mathbb{R}^2$ to a vector from $\mathbb{R}^3$, we do not need to change our notation when we change dimension. Instead, context will dictate which version of these vectors we mean.

Example: Let $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$. Then $2\vec{x} + 5e_1 = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$, while $2\vec{y} + 5e_1 = 2\begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ -4 \\ 10 \end{bmatrix}$.