Lecture 1c

Directed Line Segments

(pages 7-8)

So far, we have always had our vectors "start" at the origin, and "end" at the point corresponding to our vector. But if we are thinking of vectors as a direction instead of as a point, then it shouldn't really matter where we start or end. And so, we will expand our study of vectors to now consider the following:

<u>Definition</u> The **directed line segment** from a point P in $\mathbb{R}^2$ to a point Q in $\mathbb{R}^2$ is drawn as an arrow with starting point P and tip Q. It is denoted by $\vec{PQ}$.

Example: Let $P = (4, 4)$, $Q = (-3, 6)$, $R = (-6, -2)$, and $O = (0, 0)$. Then the following figure illustrates the directed line segments $\vec{OP}$, $\vec{PQ}$, and $\vec{QR}$.

Now, the directed line segment $\vec{OP}$ is the same as our visualization of the vector $\vec{p} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. (Sidenote: For some reason, "points" get capital letters, while "vectors" get lowercase letters. It will happen frequently that the point "P" will suddenly become the vector "$\vec{p}$". We will often switch back and forth between considering an element of $\mathbb{R}^2$ to be a point or a vector, and perhaps this change in notational convention helps keep straight which particular aspects we are trying to emphasize or make use of. Nevertheless, the point P, vector $\vec{p}$, and directed line segment $\vec{OP}$ are all the same element of $\mathbb{R}^2$.)

Definition A directed line segment that starts at the origin and ends at a point P is called the **position vector** for P.

And so, you may be asking, what do directed line segments have to do with our study of vectors? Sure, those position vectors gave us a nice parallelogram rule for addition, and putting vectors end-to-end when adding was fun too. But if vectors are the same as points in $\mathbb{R}^2$, then what is $\vec{PQ}$?

Well, part of the idea behind viewing points as vectors is to realize that to describe a point in $\mathbb{R}^2$, you need two pieces of information: its distance from the origin, and which direction to travel that distance. So the key bits of information about a vector are its length and direction. But directed line segments do not increase our collection of lenghts/directions–they simply move vectors around so that they start at a point other than the origin. And so, we shall consider two directed line segments to be (more or less) the same if they have the same length and direction. That is...

Definition We define two directed line segments $\vec{PQ}$ and $\vec{RS}$ to be **equivalent** if $\vec{q} - \vec{p} = \vec{s} - \vec{r}$, in which case we shall write $\vec{PQ} = \vec{RS}$. In the case where $R = O$, we get that $\vec{PQ}$ is equivalent to $\vec{OS}$ if $\vec{q} - \vec{p} = \vec{s}$.

Okay, but I still haven't explained why we care about directed line segments, much less if they are equivalent. Well, it comes down to finding the equation of a line. Because far too often in life we find ourselves looking for the equation of a line that doesn't have the decency to go through the origin. But now, we can take any two points on the line, and find the position vector equivalent to their directed line segment. And point plus position vector gives us the equation of a line!

Example: Find a vector equation of the line through $P = (4, 4)$ and $Q = (-3, 6)$.

Well, $\vec{PQ}$ is a direction vector for the line, and $\vec{PQ} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$, and the line goes through $(4, 4)$, so a vector equation of the line is $\vec{x} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} + t \begin{bmatrix} -7 \\ 2 \end{bmatrix}$, $t \in \mathbb{R}$. Of course, the line also goes through $(-3, 6)$, so $\vec{x} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} + t \begin{bmatrix} -7 \\ 2 \end{bmatrix}$, $t \in \mathbb{R}$, is also a vector equation for our line. And, come to think of it, $\vec{QP}$ is also a direction vector for our line, and since $\vec{QP} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} - \begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, we could also use either $\vec{x} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} + t \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, $t \in \mathbb{R}$, or $\vec{x} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} + t \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, $t \in \mathbb{R}$ as our vector equation of the line.

Course Author's note: I've included these four different equations simply as an example of different ways that the problem can be solved. When doing assignments, quizzes, or exams, you only need to provide one valid equation.

Example: Let L_1 be the line through $O = (0, 0)$ and $P = (4, 4)$, and let L_2 be the line through $Q = (-3, 6)$ and $R = (-6, -2)$. Are L_1 and L_2 parallel?

Instead of finding the equations of these lines, all we really need to know is whether or not $\vec{OP}$ is a scalar multiple of $\vec{QR}$. Since $\vec{OP} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, and $\vec{QR} = \begin{bmatrix} -6 \\ -2 \end{bmatrix} - \begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$, we see that the lines are not parallel.