Lecture 1b

Vector Equation of a Line in $\mathbb{R}^2$

(pages 5-6)

The equation $x_2 = (0.5)x_1 + 1$ defines the x_2 component of a point on the line in terms the x_1 component. But when we graph the line, we think less of this equation, and more of the following two facts: the x_2-intercept is 1 and the slope is 0.5. And so, when graphing, we may start at the point $(0, 1)$, and then move "up one, right two" to find a second point on the line (the point $(2, 2)$), and then connecting those points gives us our line. In fact, though, we could have used any multiple of "up one, right two" to find a second point on the line. So, we can think of the line $x_2 = (0.5)x_1 + 1$ as starting at the point $(0, 1)$, and adding all scalar multiples of the directions "up one, right two". Well, if we decide to think of vectors as directions instead of points, then the direction "up one, right two" corresponds to the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Then we can turn the equation $x_2 = (0.5)x_1 + 1$ into the equation $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for all $t \in \mathbb{R}$. Notice that picking $t = 0$ gives us our starting point of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

<u>Definition</u>: A line through $\vec{p}$ with direction vector $\vec{d}$ is the set

$$\{\vec{p} + t\vec{d} \mid t \in \mathbb{R}\}$$

which has the vector equation

$$\vec{x} = \vec{p} + t\vec{d}\ ,\ t \in \mathbb{R}$$

Note that in the case when $\vec{p} = \vec{0}$, we end up with the equation $\vec{x} = t\vec{d}$, which is simply all scalar multiples of $\vec{d}$. This ends up being the equation of the line through the origin that goes through $\vec{d}$. Another thing to note is that since the direction vector $\vec{d}$ corresponds to the slope of a line, we see that two lines are parallel if and only if their direction vectors are NON-ZERO MULTIPLES of each other.

Sometimes it is useful to write the components of a vector equation separately. For example, the equation $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the same as saying $x_1 = 2t$ and $x_2 = 1 + t$ (for the same t). In this case, t is called a "parameter", and the equations are known as parametric equations.

Definition The **parametric equation** of the line $\vec{x} = \vec{p} + t\vec{d}$ is the collection of equations

$$\begin{matrix} x_1 = p_1 + td_1 \\ x_2 = p_2 + td_2 \end{matrix} \quad t \in \mathbb{R}$$

Now, if we solve both of these equations for t, we get

$$t = \frac{x_1 - p_1}{d},\ t = \frac{x_2 - p_2}{d_2}$$

But since $t = t$, we get

$$\frac{x_1 - p_1}{d} = \frac{x_2 - p_2}{d_2}$$

and then solving for x_2 in terms of x_1, we get equation $x_2 = p_2 + \frac{d_2}{d_1}(x_1 - p_1)$. This is the type of equation we started with!

Definition: The **scalar form** of the equation of a line is $x_2 = p_2 + \frac{d_2}{d_1}(x_1 - p_1)$, where $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ is a point on the line, and $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ is a direction vector for the line.

EXAMPLE The vector equation of a line passing through the point $P(-2, 2)$ with direction vector $\vec{d} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ has vector equation

$$\vec{x} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad t \in \mathbb{R}$$

It has parametric equations

$$\begin{matrix} x_1 & = & -2 + 2t \\ x_2 & = & 2 + 3t \end{matrix} \quad t \in \mathbb{R}$$

Solving these parametric equations for t gives us

$$t = \frac{x_1 + 2}{2} = \frac{x_2 - 2}{3}$$

Solving for x_2 gives us the scalar form of the line: $x_2 = \frac{3}{2}(x_1 + 2) + 2$, or $x_2 = \frac{3}{2}x_1 + 5$.