Lecture 1a

Vectors in $\mathbb{R}^2$

(pages 1-4)

By this point in your life you have learned to do a great many things with numbers. In this course, we are going to look at doing things with groups of numbers. To start things slowly, we begin by looking at pairs of numbers. And we begin our study of pairs with a definition from the text:

Definition: $\mathbb{R}^2$ is the set of all vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where x_1 and x_2 are real numbers. Equivalently, we write

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} | x_1, x_2 \in \mathbb{R} \right\}$$

EXAMPLE: Some examples of vectors in $\mathbb{R}^2$ are:

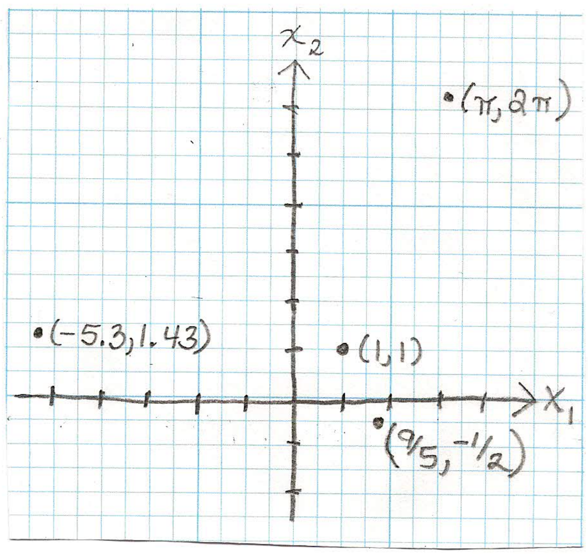
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -5.3 \\ 1.41 \end{bmatrix}, \quad \begin{bmatrix} \pi \\ 2\pi \end{bmatrix}, \quad \begin{bmatrix} 9/5 \\ -1/2 \end{bmatrix}$$

Notation: We use $\vec{x}$ to denote vectors in $\mathbb{R}^2$, and will write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Moreover, the numbers x_1 and x_2 are called the **components** of the vector $\vec{x}$. Individually, we say that x_1 is the first component of $\vec{x}$ and x_2 is the second component of $\vec{x}$.

EXAMPLE: The first component of the vector $\begin{bmatrix} -5.3 \\ 1.41 \end{bmatrix}$ is -5.3, and the second component of the vector $\begin{bmatrix} 9/5 \\ -1/2 \end{bmatrix}$ is $-1/2$.

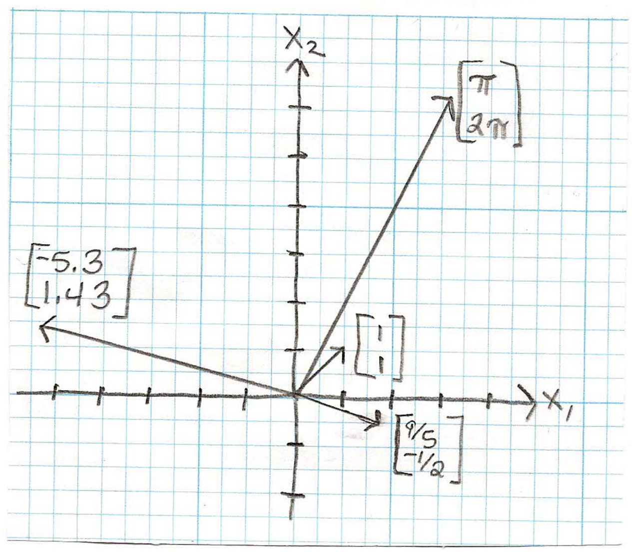
You may be used to using the notation $\mathbb{R}^2$ to denote the set of Cartesian coordinates, that is, the set of points in the plane. Instead of overriding that definition, we will in fact embrace it, as we will identify the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with the corresponding point (x_1, x_2) in the plane.

EXAMPLE: We can graph the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as the point $(1, 1)$, the vector $\begin{bmatrix} -5.3 \\ 1.41 \end{bmatrix}$ as the point $(-5.3, 1.41)$, the vector $\begin{bmatrix} \pi \\ 2\pi \end{bmatrix}$ as the point $(\pi, 2\pi)$, and the vector $\begin{bmatrix} 9/5 \\ -1/2 \end{bmatrix}$ as the point $(9/5, -1/2)$.



But vectors will be more than just points. Eventually we will come to think of vectors as having a length and direction, and this notion of a vector is usually visualized as a directed line segment from the origin to the point (x_1, x_2).

EXAMPLE: Now we look at our vectors as directed line segments...



Please keep in mind that all three ways of viewing a vector: as a pair of numbers, as a point, and as a directed line segment; are valid and useful. And all three will be used in this course!

But the main way that vectors are different from points in the plane is that we define operations on them.

Definition: If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$, then we define addition of vectors by

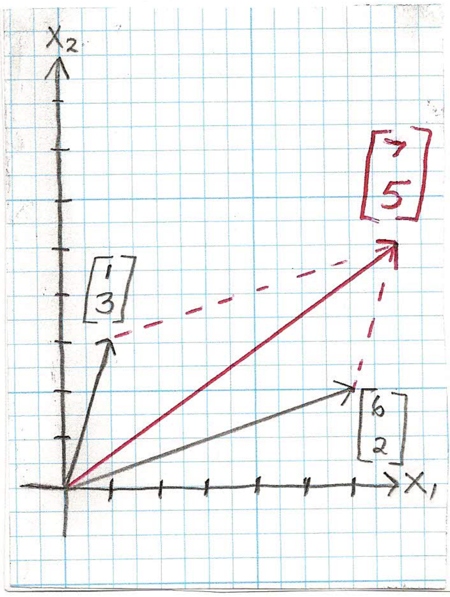
$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

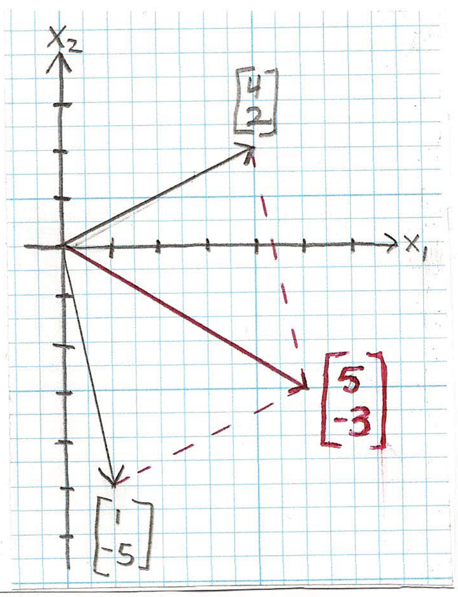
We say that addition is defined "componentwise", as we add matching components. Most importantly, we notice the value of the second components does not effect the calculation of the first components, and vice versa.

EXAMPLE: $\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+6 \\ 3+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$

EXAMPLE: $\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 4+1 \\ 2-5 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

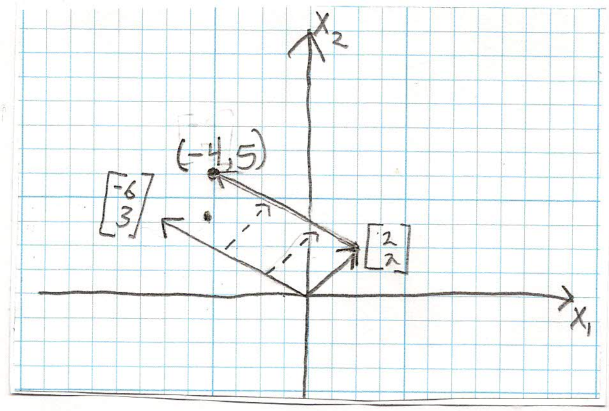
If we view the vectors in these examples as directed line segments, we notice that addition also has a graphical interpretation.



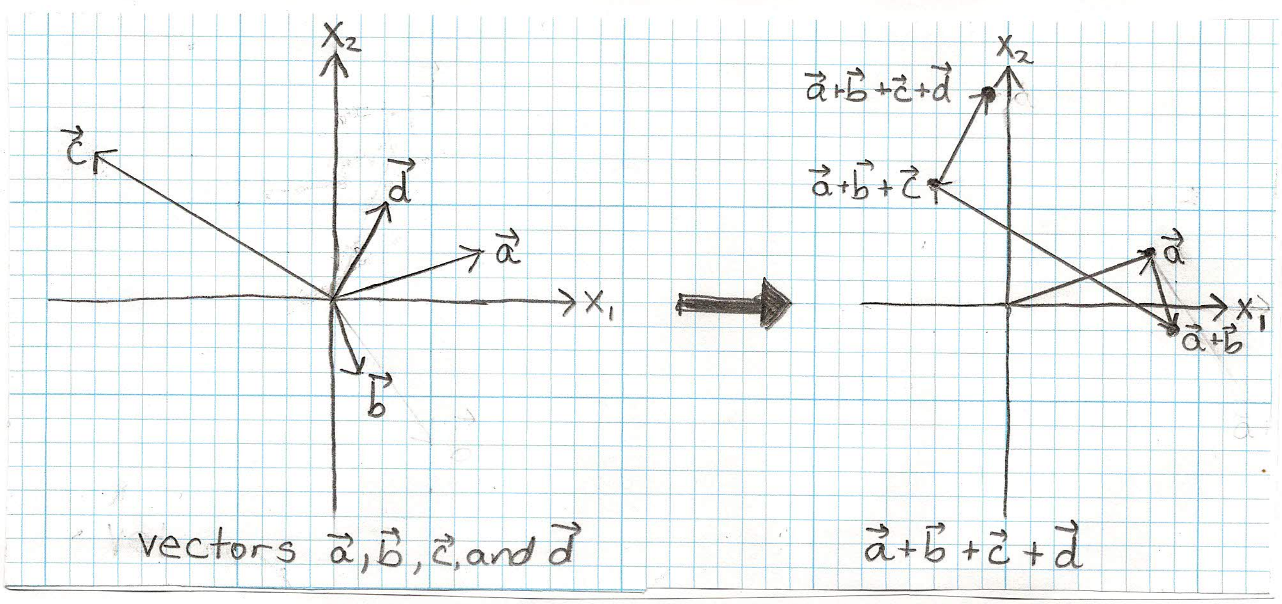


The graphical interpretation of vector addition is known as the parallelogram rule for addition, as it is always the case that the point $(x_1 + y_1, x_2 + y_2)$ is the point opposite from the origin on the parallelogram defined by $\vec{x}$ and $\vec{y}$.

Another way to visualize vector addition is to think of actually sliding $\vec{y}$ until it is at the end of $\vec{x}$, so that we are adding $\vec{y}$ to the end of $\vec{x}$.



I call this the end-to-end method of adding vectors, and it can be particularly useful if you need to add several vectors to each other.



We also define an operations known as scalar multiplication:

Definition: If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ and $t \in \mathbb{R}$, then we define scalar multiplication of a vector by the scalar t by

$$t\vec{x} = t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix}$$

In this case, we take a real number (known as a scalar), and multiply it by a vector. Again we see that scalar multiplication is defined componentwise. That is, the scalar is multiplied by each component, and the value of the second component does not effect the calculation of the first component, and vice versa. Note also that our definition of scalar multiplication is consistent with the view that $2\vec{x}$ should equal $\vec{x} + \vec{x}$ (and $3\vec{x} = \vec{x} + \vec{x} + \vec{x}$, etc.), since

$$2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} x_1 + x_1 \\ x_2 + x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

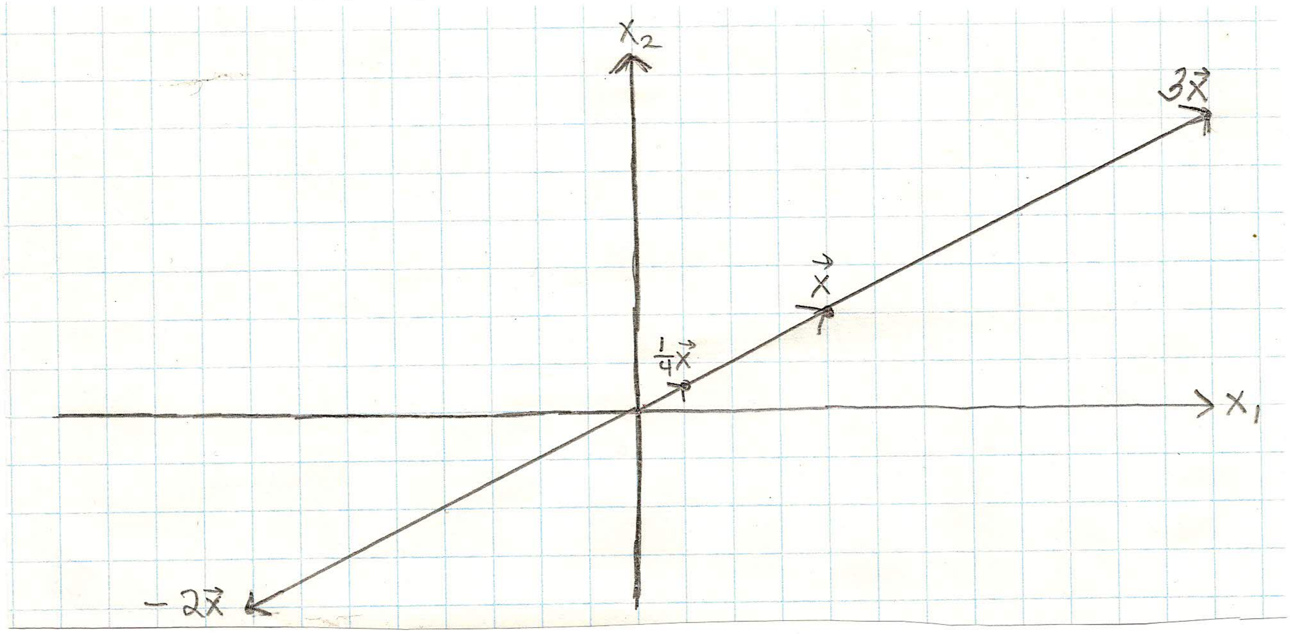
EXAMPLE If $\vec{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, then

$$3\vec{x} = 3 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \cdot 4 \\ 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

$$\frac{1}{4}\vec{x} = \frac{1}{4} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} (1/4)(4) \\ (1/4)(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

$$-2\vec{x} = -2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \cdot 4 \\ -2 \cdot 2 \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \end{bmatrix}$$

While vector addition represents putting vectors end-to-end, scalar multiplication represents shrinking or stretching a vector, or reversing the direction when we multiply by a negative.



We end this lecture with a bit more notation.

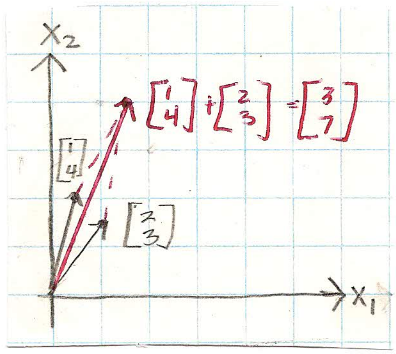
Notation: We will write $-\vec{x}$ to mean the scalar product $(-1)\vec{x}$, and $\vec{x} - \vec{y}$ to mean the sum $\vec{x} + (-1)\vec{y}$.

Notation: The vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ occurs so frequently that we give it its own name: $\vec{0}$. There are two other vectors that will also appear frequently in our studies, so they are given special names too. They are $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The special feature of these vectors is that any vector $\vec{x}$ satisfies $\vec{x} = x_1\vec{e_1} + x_2\vec{e_2}$.

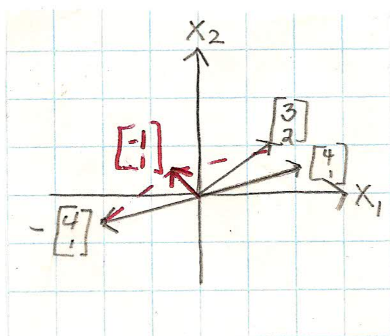
ASSIGNMENT 1a p.12 A1, A2, D1

Solution to Assignment 1a

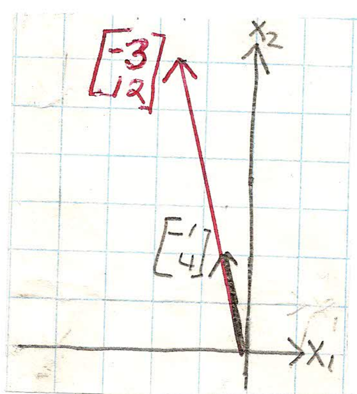
A1(a) $\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 4+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$



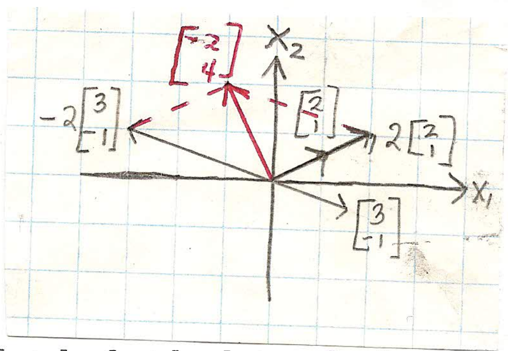
A1(b) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-4 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



A1(c) $3\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} (3)(-1) \\ (3)(4) \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \end{bmatrix}$



A1(d) $2\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} (2)(2)-(2)(3) \\ (2)(1)-(2)(-1) \end{bmatrix} = \begin{bmatrix} 4-6 \\ 2-(-2) \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$



A2(a) $\begin{bmatrix} 4 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4-1 \\ -2+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

A2(b) $\begin{bmatrix} -3 \\ -4 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -3-(-2) \\ -4-5 \end{bmatrix} = \begin{bmatrix} -1 \\ -9 \end{bmatrix}$

A2(c) $-2\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} (-2)(3) \\ (-2)(-2) \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$

A2(d) $\frac{1}{2}\begin{bmatrix} 2 \\ 6 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/2+4/3 \\ 6/2+3/3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 4 \end{bmatrix}$

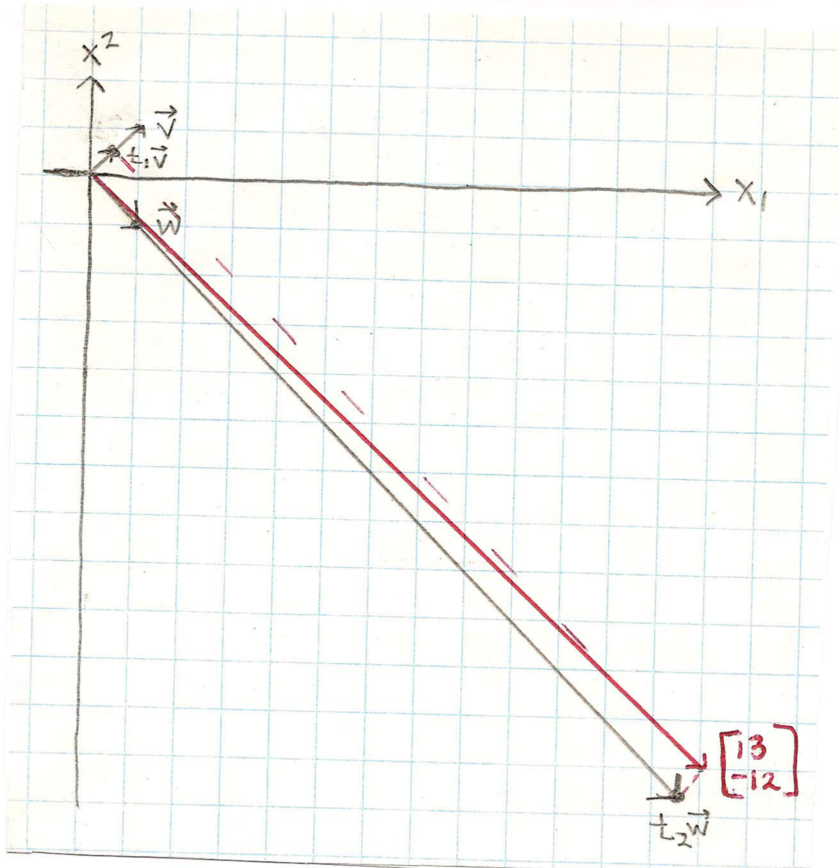
A2(e) $\frac{2}{3}\begin{bmatrix} 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1/4 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2/3 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$

A2(f) $\sqrt{2}\begin{bmatrix} \sqrt{2} \\ \sqrt{3} \end{bmatrix} + 3\begin{bmatrix} 1 \\ \sqrt{6} \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{6} \end{bmatrix} + \begin{bmatrix} 3 \\ 3\sqrt{6} \end{bmatrix} = \begin{bmatrix} 5 \\ 4\sqrt{6} \end{bmatrix}$

D1(a) We need to find real numbers t_1 and t_2 such that $t_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 13 \\ -12 \end{bmatrix}$. To do this, I will break the vector equation into its two components, getting the following two equations

$$t_1 + t_2 = 13 \qquad\qquad t_1 - t_2 = -12$$

Adding the two equations together, we get $2t_1 = 1$, so $t_1 = 1/2$. This means that $1/2 + t_2 = 13$, so $t_2 = 25/2$.



D1(b) We need to find real numbers t_1 and t_2 such that $t_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. To do this, I will break the vector equation into its two components, getting the following two equations

$$t_1 + t_2 = x_1 \qquad\qquad t_1 - t_2 = x_2$$

Adding the two equations together, we get $2t_1 = x_1 + x_2$, so $t_1 = (x_1 + x_2)/2$. This means that $(x_1 + x_2)/2 + t_2 = x_1$, so $x_1 + x_2 + 2t_2 = 2x_1$, and thus $t_2 = (x_1 - x_2)/2$. (Note that this result agrees with our result for part (a).)

D1(c) We have $t_1 = (x_1 + x_2)/2 = (\sqrt{2} + \pi)/2$ and $t_2 = (\sqrt{2} - \pi)/2$.