Note: there is a newer edition of the AMath 250 course notes. For edition 2.1, effective Fall 2025, see

https://www.math.uwaterloo.ca/~jjwest/course_notes/AMath250_Course_Notes_2.1.pdf

Introduction to Differential Equations

Course Notes for AMath 250

Edition 2.0

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Acknowledgements

Original Version: I would like to thank Professor Bev Marshman for a critical reading of the manuscript and Professors Frank Goodman and Pino Tenti for helpful discussions concerning the physical interpretation of differential equations, and for allowing me to use their notes concerning dimensional analysis. Thanks are also due to Ann Puncher for her excellent work in typesetting the notes.

J. Wainwright

Fall 2014 edition: I would like to thank David Harmsworth for some of the new problems including "the wine cask problem" and for his suggestions, especially those relating to dimensional analysis. Thanks also to Professor Brian Ingalls and again to Professor Bev Marshman for more helpful suggestions.

J. West

Edition 2.0, summer 2023: Many thanks to Thomas Karabela and Faisal Al Faisal for the LaTeX assistance moving to the new format, to various students for catching minor errors in the previous edition, and to David Harmsworth for his thoughtful suggestions and notes on the method of variation of parameters. Finally, a big thank you to my department chair Siv Sivaloganathan, who helped arrange my teaching assignments so that I had sufficient time to make several other improvements to the course notes.

J. West

Preface

Goals of the course

The course AMath 250 is intended to serve three purposes:

i) to provide an accessible introduction to the world of differential equations for students who do not intend to specialize in this area,

ii) to provide a prerequisite course (for those unable to take AMath 251) for the more specialized third and fourth year courses in ordinary differential equations, partial differential equations, and dynamical systems (AMath 351, AMath 353 and AMath 451),

iii) to provide an introduction to the discipline of Applied Mathematics, namely, the formulation and analysis of mathematical models of real-world phenomena. Since many models are based on differential equations, an introductory course in DEs provides a natural vehicle for this purpose.

What you need to know

Success in the course depends on having a good knowledge of *single variable calculus* – derivatives, antiderivatives, qualitative curve sketching and improper integrals, in particular. In the final chapter, some knowledge of *linear algebra* is also required – matrices, eigenvalues and eigenvectors – but only for the two-dimensional case. You'll find that the exponential function plays a major role in the subject of differential equations, and so it is important that you have a good grasp of *exponentials* and *logarithms*. Somewhat surprisingly, *complex numbers* are used in the course, even though all the unknown functions are real-valued. The reason for the appearance of complex numbers is that the roots of real polynomials are in general complex. So in the course you'll find yourself using the famous *Euler formula*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Some concepts from physics arise in the applications, the most important being *Newton's Second Law of Motion*. However, in order to keep the course accessible, the background needed for the applications, many of which arise in everyday life, will be given in the course.

Learning the course material

The lecture notes give a discussion of the theoretical matters, and contain a selection of worked examples. Some of the solutions to the examples are given in full detail, but when reading many parts of the notes you will need a pencil and paper in order to fill in some of the missing steps. Each section contains exercises with answers to help you get started in learning the course. *We recommend that you do all of these exercises.* The notes also contain a Review Problem Set, in case your knowledge of Calculus needs refreshing. There are five problem sets, one for each chapter in the notes. The answers to the odd-numbered problems are given.

Chapter 1

First Order Differential Equations

1.1 DEs and Mechanics

1.1.1 Newton's Second Law of Motion

Newton's Second Law of Motion leads to differential equations when applied to various problems in mechanics. We consider the simplest case of motion of a particle in a straight line. The relevant physical quantities are

$$\begin{aligned} m, &\quad \text{the mass of the particle (which may vary with time),} \\ v(t), &\quad \text{the velocity of the particle at time } t, \\ F, &\quad \text{the total force acting (time-dependent, in general).} \end{aligned}$$

Newton's Second Law states that the rate of change of momentum $mv(t)$ equals the total force, i.e.

$$\frac{d}{dt}[mv(t)] = F. \tag{1.1}$$

If the mass is constant this equation can be written

$$m\frac{dv}{dt} = F, \tag{1.2}$$

i.e. *mass times acceleration equals force acting*. Before (1.1) or (1.2) can be used to describe the motion of a particle, the force F has to be specified. In the example to follow, F depends on t through the velocity v, i.e. $F = F(v)$, in which case (1.2) assumes the form

$$m\frac{dv}{dt} = F(v),$$

which is a first-order differential equation (DE) for the unknown function $v(t)$.

Remark

Newton's second law has been tested in countless experiments, and accurately predicts the motion of particles, subject to one limitation, namely that the velocity of the particle is small compared to the velocity of light i.e.

$$\frac{v}{c} \ll 1.$$

The velocity of light is $c \approx 3 \times 10^8 m/s \approx 10^9$ km/hr. For sufficiently high velocities, as occur for example in high energy particle accelerators, Newton's second law has to be replaced by its relativistic counterpart, which is part of Einstein's theory of relativity.

Example: Terminal velocity of a skydiver

Gravity exerts a downward force on the skydiver, and air resistance exerts an upward force, which increases as the velocity increases. Eventually we expect that air resistance will balance the force due to gravity, so that the total force acting is zero. Then by (1.2) the skydiver will fall with constant velocity, called the *terminal velocity*. In order to describe the motion in detail we need to specify the force due to air resistance. This is a complicated matter; we make the simple assumption that this force is proportional to the velocity and acts so as to oppose the motion. Taking the downward direction as the positive direction, the total force acting on the skydiver will be

$$F = mg - \alpha v, \tag{1.3}$$

where g is the constant acceleration due to gravity (at the earth's surface). The constant α, which depends on the physical characteristics of the skydiver or falling object, is called the *drag coefficient*. □

Remark

Strictly speaking, the acceleration due to gravity is not constant, and depends on the distance from the centre of the earth. Since the height h above the earth's surface is small compared to the earth's radius R, i.e.

$$\frac{h}{R} \ll 1,$$

it is a reasonable approximation to treat g as a constant. □

Example continued

With (1.3), Newton's law (1.2) assumes the form

$$m\frac{dv}{dt} = mg - \alpha v, \tag{1.4}$$

a first order DE for $v(t)$. This DE can be solved by separation of variables (see Section 1.2.2), but for now we draw a conclusion directly from the DE. As stated earlier we expect the skydiver to eventually reach a constant terminal velocity. Since a constant velocity gives $\frac{dv}{dt} = 0$, equation (1.4) implies that

$$v_{\text{terminal}} = \frac{mg}{\alpha}. \tag{1.5}$$

Note that the terminal velocity depends on the three physical parameters m, g and α, being inversely proportional to the drag coefficient α (opening a parachute will *increase* α, thereby *reducing* v_{terminal}). There is one other parameter associated with the physical system, namely the skydiver's initial velocity $v(0)$, i.e. the vertical velocity when leaving the plane at time $t = 0$. It is of interest that v_{terminal} does not depend on $v(0)$, although we expect that $v(t)$, the velocity at time t, will depend on $v(0)$. The relation between $v(t)$ and $v(0)$ will become clear when you solve the DE (1.4) [see Problem Set 1].

Remark

A more realistic model for a skydiver is $m\frac{dv}{dt} = mg - \beta v^2$, where the drag force is assumed to be proportional to the *square* of the speed. See Problem Set 2 #8 for more detail.

1.1.2 Dimensions of physical quantities

In analyzing any physical system it is essential to keep track of the physical dimensions of the various variables and parameters. In mechanics, the primary physical variables are taken to be

$$\text{mass} \quad M, \qquad \text{length} \quad L, \qquad \text{time} \quad T$$

and physical quantities have dimensions of the form

$$M^m L^\ell T^t,$$

where the exponents m, ℓ and t are integers. For example, velocity v has dimensions LT^{-1} (i.e. $m\,s^{-1}$, $km\,hr^{-1}$, depending on the units). We indicate the dimensions by writing

$$[\text{velocity}] = LT^{-1}.$$

Since acceleration is the rate of change of velocity with respect to time, we have

$$[\text{acceleration}] = LT^{-2}.$$

By Newton's Second Law,

$$\text{force} = (\text{mass})(\text{acceleration}),$$

and so

$$[\text{force}] = [\text{mass}][\text{acceleration}] = MLT^{-2}. \tag{1.6}$$

Remark

There are other primary physical quantities as well, including temperature θ, electrical charge Q, and luminous intensity I. Most problems in this course will involve M, L, T, and sometimes θ or Q.

In calculating the dimensions of force, we have implicitly used certain consistency requirements that have to be satisfied by equations having physical content. These are as follows.

D1: One can only add, subtract or equate physical quantities that have the same dimension.

This statement is referred to as the *Principle of Dimensional Homogeneity.*

D2: Quantities having different dimensions can only be combined by multiplication and division, and the dimensions of a product or quotient are given by

$$[AB] = [A][B], \qquad \left[\frac{A}{B}\right] = \frac{[A]}{[B]},$$

where the right sides are simplified using the usual laws of exponents.

These consistency requirements are of importance for the following reasons.

1) D1 and D2 act as a check on the working in a physical problem. For example, if we have obtained the formula

$$A = B + CD, \tag{1.7}$$

then if we are correct the dimensions of A, B, C and D must satisfy

$$[A] = [B] = [C][D]. \tag{1.8}$$

Of course if (1.8) holds this does not imply that (1.7) is correct: the point is that if (1.8) does not hold then (1.7) is incorrect. □

2) D1 and D2 enable us to calculate the dimension of a physical quantity.

As an illustration, we calculate the dimensions of the drag coefficient α in (1.3). The drag force f_d is given by $f_d = -\alpha v$. Hence by D1,

$$[\alpha v] = [\text{force}],$$

and by D2,

$$[\alpha][v] = [\text{force}].$$

Using (1.6) and the laws of exponents,

$$[\alpha] = \frac{[\text{force}]}{[v]} = \frac{MLT^{-2}}{LT^{-1}} = MT^{-1}. \qquad \square$$

3) D2 enables us to form *dimensionless quantities*, which are essential for writing physical relations in their simplest forms.

   To indicate that a quantity A is dimensionless, we write

$$[A] = 1.$$

   For example, any number is dimensionless, e.g. $[\pi] = 1$. In discussing the validity of Newton's second law we imposed the restriction $\frac{v}{c} \ll 1$, which involves the dimensionless quantity $\frac{v}{c}$:

$$\left[\frac{v}{c}\right] = \frac{[v]}{[c]} = \frac{LT^{-1}}{LT^{-1}} = 1. \qquad \square$$

Remark

If t is time, i.e. $[t] = T$, then the expression $\sin(\pi t)$ does not make sense, since the argument of $\sin(\quad)$ must be dimensionless. However, if ν is a frequency i.e. $[\nu] = T^{-1}$, then $2\pi\nu t$ is dimensionless, and it makes sense to write $\sin(2\pi\nu t)$. Likewise, the arguments of all other elementary functions must be dimensionless, apart from the power function $f(\quad) = (\quad)^n$, n an integer (see D2). $\square$

Exercise 1:

If a particle moves in a straight line with constant acceleration g, starting with velocity v_0 and position s_0 at time $t = 0$, the distance travelled at time t is

$$s = v_0 t + \tfrac{1}{2} g t^2 + s_0.$$

Verify that this formula is dimensionally consistent.

Exercise 2:

The kinetic energy of a particle of mass m moving with constant velocity v in a straight line is given by

$$E = \tfrac{1}{2} m v^2.$$

Calculate the dimensions of energy.

Answer: $[\text{Energy}] = ML^2T^{-2}$. $\square$

1.1.3 Newton's Law of Gravitation

Will a projectile fired vertically upwards on the earth's surface (or on the surface of the moon) eventually fall back to earth, or will it continue travelling away indefinitely? The answer is that it depends on the velocity with which the projectile is fired. If this initial velocity exceeds a certain threshold called the *escape velocity*, then the projectile will travel away indefinitely and "escape" from the earth's gravitational field. As with the skydiver, this problem is governed by Newton's Second Law, and also involves gravity. The key difference is the distance scale. In the present case the distance from the earth's surface will not be small compared to the earth's radius, and so it is unreasonable to treat the acceleration due to gravity as a constant. We thus need to use *Newton's Law of Gravitation*. On the other hand, in giving a simple analysis, it is reasonable to neglect air resistance, since the thickness of the earth's atmosphere is small compared to the earth's radius (of course air resistance is totally absent on the moon).

Newton's Law of Gravity states that the force of attraction between two point particles of mass m_1 and m_2 is proportional to the masses and inversely proportional to the square of the distance r between them. For motion in one dimension the force is given by

$$F = \frac{Gm_1 m_2}{r^2}, \tag{1.9}$$

where G is a constant of proportionality called the *gravitational constant*.

Exercise:

Calculate the dimensions of G.

Answer: $[G] = M^{-1}L^3T^{-2}$.

Remark

We can idealize the projectile as a point particle, but not the earth. However, it can be shown that the gravitational force exerted by a finite homogeneous sphere on a particle is the same as if all the mass of the sphere was concentrated at the centre of the sphere. Thus, as regards Newton's Law of gravitation, the earth can be idealized as a point particle. □

For a vertically moving projectile the total force acting is

$$F = -\frac{Gm\mathcal{M}}{r^2}, \tag{1.10}$$

where m is the mass of the projectile and $\mathcal{M}$ is the earth's mass, and r is the distance of the projectile from the centre of the earth. We have chosen the positive direction to be vertically up.

Remark

The acceleration g due to gravity near the earth's surface can be related to the gravitational constant G. Set $r = R$, the earth's radius, in (1.10) to obtain

$$F = -\frac{Gm\mathcal{M}}{R^2} = -mg,$$

giving

$$g = \frac{G\mathcal{M}}{R^2}. \quad \square \tag{1.11}$$

The equation of motion for the projectile is now obtained by substituting (1.10) in Newton's Second Law (1.2), giving

$$m\frac{dv}{dt} = -\frac{Gm\mathcal{M}}{r^2},$$

where v is the velocity of the projectile. We simplify this equation by cancelling m, and using (1.11) to express G in terms of g, giving

$$\frac{dv}{dt} = -\frac{gR^2}{r^2}. \tag{1.12}$$

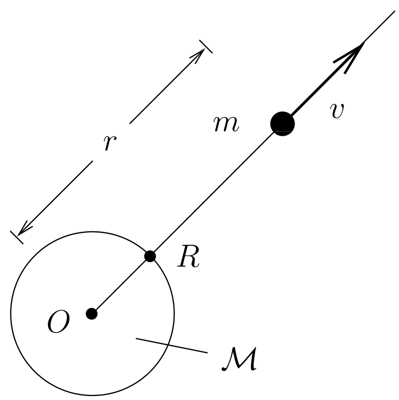


Figure 1.1: A planet and projectile.

Remark

In cancelling m, we are tacitly using a fundamental experimental fact about gravity. The mass m that appears in Newton's Second Law (1.2) is the *inertial mass* m_I of the particle, while the mass m that appears in Newton's Law of Gravity is the *gravitational mass* m_G. It has been determined experimentally to a high degree of accuracy that

$$m_I = m_G,$$

although no explanation is known. □

Equation (1.12) contains two unknown functions $v(t)$ and $r(t)$. We can write it as a DE in one unknown in two ways.

First, since $v = \frac{dr}{dt}$, we can write

$$\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2},$$

which is a *second order* DE for $r(t)$. The initial conditions at the launch at time $t = 0$ are

$$r(0) = R, \qquad \frac{dr}{dt}(0) = v_{\text{init}},$$

where v_{init} is the initial velocity.

Second, if we consider velocity v as a function of distance r, we can write (1.12) as a first order DE, which is more convenient for determining the escape velocity. By the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \frac{dv}{dr}v,$$

so that (1.12) becomes

$$v\frac{dv}{dr} = -\frac{gR^2}{r^2}, \tag{1.13}$$

a first order DE for $v = v(r)$, with initial condition

$$v(R) = v_{\text{init}}. \tag{1.14}$$

The problem now is: for which values of v_{init} will the velocity $v(r)$ satisfy $v(r) > 0$ for all $r \geq R$? The escape velocity is the smallest value of v_{init} with this property. In order to solve this problem we need to solve the DE (1.13) [see Problem Set 1].

1.2 Mathematical aspects of first order DEs

1.2.1 Types of first order DEs

The general form of a first order DE is

$$\frac{dy}{dx} = f(x, y), \tag{1.15}$$

where f is a function of two variables.[1] The variable y represents the *unknown function*, $y = y(x)$, and x is the *independent variable*. A *solution of the DE* (1.15) is a differentiable function ϕ such that $y = \phi(x)$ satisfies (1.15) for all x in some interval.

Exercise

Verify that

$$y = Ce^{-x^2} + x^2 - 1, \quad x \in \mathbb{R},$$

where C is a constant, is a solution of the DE

$$\frac{dy}{dx} = -2xy + 2x^3. \qquad \square$$

In general, it is not possible to actually find solutions of the DE (1.15), even though we know they exist. Fortunately, the first order DEs that arise in many applications are of two special types that can be solved, namely *separable and linear*.

Separable first order DEs

The general form of a first order *separable DE* is

$$\frac{dy}{dx} = A(x)B(y), \tag{1.16}$$

where $A(x)$ and $B(y)$ are arbitrary functions.

[1] It is usually assumed that f has continuous partial derivatives (i.e. is of class C^1); technical details such as these are not important in this course.

Example 1:

The skydiver DE in Section 1.1.1,

$$m\frac{dv}{dt} = mg - \alpha v, \tag{1.17}$$

is of the form

$$\frac{dv}{dt} = A(t)B(v),$$

with

$$A(t) = 1, \quad B(v) = g - \frac{\alpha}{m}v,$$

and hence is *separable*. $\square$

Remark

In example 1, notice that one of the functions is actually a constant. In general, looking at (1.16), we mean that $A(x)$ is a function of *at most* x and $B(y)$ is a function of *at most* y, i.e. one or both functions might be constant.

Example 2:

The escape velocity DE in Section 1.1.3,

$$v\frac{dv}{dr} = -\frac{gR^2}{r^2}, \tag{1.18}$$

is of the form

$$\frac{dv}{dr} = A(r)B(v),$$

with

$$A(r) = \frac{R^2}{r^2}, \quad B(v) = -\frac{g}{v},$$

and hence is *separable*. $\square$

Linear first order DEs

The general form of a first order *linear* DE is

$$\frac{dy}{dx} + k(x)y = f(x), \tag{1.19}$$

where $k(x)$ and $f(x)$ are arbitrary functions.

Example

The DE

$$\frac{dy}{dx} + xy = e^{-x}$$

is *linear*, but the DE

$$\frac{dy}{dx} = xy^2$$

is *non-linear*. Note that the skydiver DE (1.17) is linear, but the escape velocity DE (1.18) is non-linear. □

There are two special types of linear DEs:

- *homogeneous*,

$$\frac{dy}{dx} + k(x)y = 0 \tag{1.20}$$

  This DE is also separable.

- *constant coefficient*,

$$\frac{dy}{dx} + k_0 y = f(x), \qquad k_0 = \text{constant}. \tag{1.21}$$

A linear DE that is homogeneous AND has a constant coefficient can be written in the form

$$\frac{dy}{dx} = ky, \qquad (\text{where } k = -k_0.) \tag{1.22}$$

This is *the world's simplest and most important first order DE*. It is so simple that it can be *solved by inspection*: the derivative of the unknown function y is k times y, and the only functions with this property are

$$y = Ce^{kx}, \qquad C = \text{constant}. \tag{1.23}$$

This is important enough to write out formally.

Proposition

Any solution of the DE

$$\frac{dy}{dx} = ky, \qquad k = \text{constant}$$

is given by

$$y = Ce^{kx},$$

where C is a constant.

Aside: If $k > 0$, this DE describes *exponential growth* and if $k < 0$, it describes *exponential decay*.

Proof: Multiply (1.22) by e^{-kx} and rewrite as

$$\frac{d}{dx}\left(e^{-kx}y\right) = 0.$$

Hence,

$$e^{-kx}y = C,$$

giving (1.23). □

Exercise

Write the general form of a linear DE that is also separable.

Answer: $\dfrac{dy}{dx} = k(x)(ay+b)$, where a and b are constants.

1.2.2 Solving separable DEs

A separable DE

$$\frac{dy}{dx} = A(x)B(y) \tag{1.24}$$

can be solved by *separation of variables*, as follows. Divide (1.24) by $B(y)$ and take the antiderivative with respect to x:

$$\int \frac{1}{B(y)}\frac{dy}{dx}dx = \int A(x)dx.$$

By the change of variable algorithm, this can be written

$$\int \frac{1}{B(y)}dy = \int A(x)dx.$$

Provided that both antiderivatives can be evaluated in terms of elementary functions, one obtains a *one-parameter family of solutions* i.e. depending on one constant of integration.
□

Equilibrium solutions:

A separable DE may have certain exceptional solutions that can be found by inspection. If the function $B(y)$ in (1.24) is zero at $y = b$, i.e.

$$B(b) = 0,$$

then *the constant function* $y = b$ *is a solution of the DE* (1.24). Because the unknown function is a constant function, this solution is called an *equilibrium solution* – one thinks of the physical system as being in a state of equilibrium. For example the separable DE

$$\frac{dy}{dx} = y(1-y)$$

has equilibrium solutions $y = 0$ and $y = 1$. When solving a separable DE, *always begin by finding the equilibrium solutions* (if any), because they are excluded by the general procedure since one divides by $B(y)$.

Example

The separable DE

$$\frac{dy}{dx} = \frac{xy}{1+y^2}$$

has an equilibrium solution $y = 0$. To find the general solution, rewrite the DE as

$$\frac{1+y^2}{y}\frac{dy}{dx} = x.$$

This leads to

$$\int \left(\frac{1}{y} + y\right) dy = \int x dx,$$

giving

$$\ln \mid y \mid + \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C. \qquad \square$$

Remark

When solving a separable DE it may not be possible to isolate y in the general solution, as happens in the above example.

Exercise

Solve the DE

$$\frac{dy}{dx} = 2xe^{-y}.$$

Answer: $y = \ln(x^2 + C)$, where C is a constant.

1.2.3 Solving linear DEs

A linear DE

$$\frac{dy}{dx} + k(x)y = f(x)$$

can be solved by finding an auxiliary function called an *integrating factor*. The method is illustrated in the following example.

Example

Solve the DE

$$\frac{dy}{dx} - y = -e^{-x}. \tag{1.25}$$

Solution: Multiply throughout by a function $I(x)$,

$$I\frac{dy}{dx} - Iy = -Ie^{-x}. \tag{1.26}$$

Choose I to satisfy

$$\frac{dI}{dx} = -I. \tag{1.27}$$

The reason for doing this is that (1.26) becomes

$$I\frac{dy}{dx} + y\frac{dI}{dx} = -Ie^{-x},$$

which, using the Product Rule for derivatives, can be written

$$\frac{d}{dx}(Iy) = -Ie^{-x}. \tag{1.28}$$

The DE (1.27) for I (the "world's simplest") can be solved by inspection:

$$I = Ce^{-x}.$$

Since we only want a particular solution, we choose $I = e^{-x}$ (i.e. $C = 1$) for simplicity. The DE (1.28) assumes the form

$$\frac{d}{dx}\left(e^{-x}y\right) = -e^{-2x}. \tag{1.29}$$

Take the antiderivative of both sides with respect to x:

$$\int \frac{d}{dx}\left(e^{-x}y\right) dx = \int -e^{-2x} dx,$$

giving

$$e^{-x}y = \tfrac{1}{2}e^{-2x} + C.$$

Solving for y:

$$y = \tfrac{1}{2}e^{-x} + Ce^{x}, \tag{1.30}$$

where C is an arbitrary constant. *Equation* (1.30) *gives the family of all solutions of the DE* (1.25). □

Remark

The function $I(x)$ in the previous solution is called an *integrating factor* for the DE. It is always determined by solving a separable DE, which in the above example could be solved by inspection (see equation (1.27)). The purpose of finding the integrating factor is to write the given DE in the form (1.28), since in this form it can be solved directly by taking the antiderivative of both sides. Note that any DE of the form

$$\frac{d(I(x)y)}{dx} = g(x)$$

can be solved for y directly, by taking the antiderivative of both sides with respect to x. □

Exercise 1:

Solve the linear DE

$$x\frac{dy}{dx} - y = x^3, \quad x > 0.$$

Comment: It is essential to divide through by x so as to put the DE into standard form $\frac{dy}{dx} + k(x)y = f(x)$, before attempting to find the integrating factor.

Answer: The integrating factor is $I(x) = \frac{1}{x}$, and the general solution is $y = \frac{1}{2}x^3 + Cx$. □

Exercise 2:

Show that, in general, the integrating factor for the DE $\frac{dy}{dx} + k(x)y = f(x)$ is $I(x) = e^{\int k(x)\ dx}$.

1.2.4 Qualitative sketches of families of solutions

Two basic problems in the theory of DEs are

A: Solve the DE, i.e. find all solutions of the DE (only possible for certain classes of DE).

B: Describe the behaviour of typical solutions of a DE, e.g. how does the solution $y(x)$ behave as $x \to +\infty$?

In 1.2.2 and 1.2.3 we discussed the two most important special classes of first order DEs that can be solved. In order to understand the behaviour of solutions, however, it is necessary to give *a qualitative sketch of the family of solutions*, which will depend on one parameter (the constant of integration). In this section, we discuss how to draw such a sketch.

When sketching the solutions one can obtain useful information directly from the DE, without solving it, as follows. We think of the DE

$$\frac{dy}{dx} = f(x, y)$$

as specifying a *slope* at each point of the xy-plane, namely the slope of the tangent line to the solution $y = y(x)$ through that point. In other words, if $y = y(x)$ is the solution curve through the point (x_0, y_0), then the slope of the curve at (x_0, y_0) is

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f(x_0, y_0).$$

Example 1:

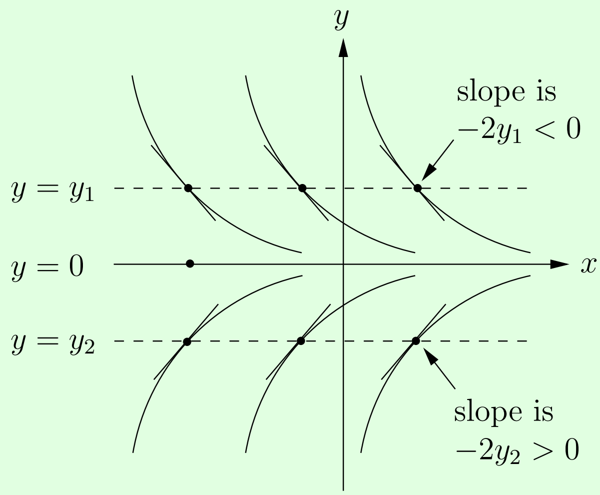
For the DE

$$\frac{dy}{dx} = -2y \tag{1.31}$$

the family of solutions is

$$y = Ce^{-2x}$$

(by inspection). Note that $y = 0$ is an equilibrium solution, with slope 0 at each point. The special property of this DE is that the slope at (x, y) depends on y but not on x. □



Here is an example where the pattern of the solution curves is more complicated, and where we make use of the DE itself.

Example 2:

Give a qualitative sketch of the solution curves of the DE

$$\frac{dy}{dx} - y = -e^{-x}.$$

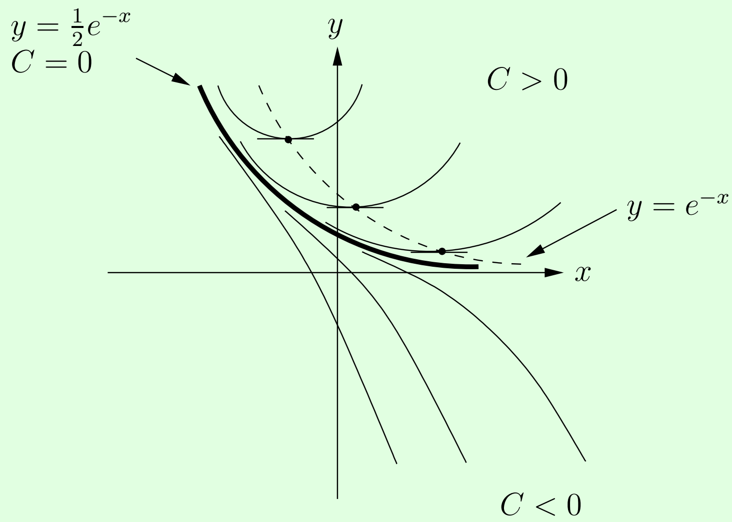
From Section 1.2.3, equation (1.30), the solutions are

$$y = \tfrac{1}{2}e^{-x} + Ce^{x}.$$

Solution: The DE implies that the *slope is zero* at all points on the curve

$$y = e^{-x} \qquad \text{(just set } \frac{dy}{dx} = 0 \text{ in the DE)}$$

(drawn as the dashed curve). The solution with $C = 0$ plays a special role (draw it as a dashed curve). As $x \to -\infty$, $y \approx \frac{1}{2}e^{-x}$, i.e. all solutions approach the $C = 0$ solution. As $x \to +\infty$, $y \approx Ce^{x}$, and so the shape of the solution curve depends on whether $C > 0$ or $C < 0$. □



Exceptional solutions:

The key to sketching the family of solution curves is to identify any *exceptional solutions*. In example 1, $y = 0$ is an exceptional solution, and in example 2, $y = \frac{1}{2}e^{-x}$ is exceptional. In each case the exceptional solution divides the whole family of solutions into two subclasses such that the members of a subclass have the same qualitative properties (i.e. the same overall shape). Another feature of exceptional solutions is that they often "attract" other solution curves, either as $x \to +\infty$ or as $x \to -\infty$.

Note that for a separable DE $\frac{dy}{dx} = A(x)B(y)$, any *equilibrium solution*, i.e. $y(x) = b$, with $B(b) = 0$, is also an exceptional solution (as in example 1). □

Remark: Solution Sketching Algorithm

In many cases, it is helpful to perform the following steps:

1. Find and sketch any *equilibrium solutions* and any other *exceptional solutions*. (Recall that equilibrium solutions are constant solutions, and exceptional solutions are normally found by setting $C = 0$ in the general solution.)

2. Analyze the *slope*, using the DE to determine where $\frac{dy}{dx} > 0$, < 0, and $= 0$. The latter gives the *horizontal isocline*.

3. Determine the *asymptotic behaviour*, normally as $x \to \infty$ and as $x \to -\infty$. (If there is a $\frac{1}{x}$ term, for example, you'll also want to check as $x \to 0^+$ and as $x \to 0^-$).

Example 3 (detailed)

The DE

$$\frac{dy}{dx} = y - x^2$$

has general solution

$$y(x) = Ce^x + x^2 + 2x + 2$$

(exercise). Give a qualitative sketch of the solution curves.

Solution:

Step 1: Equilibrium and Exceptional solutions.

To look for equilibrium solutions, we set $\frac{dy}{dx} = 0$ in the DE, since we assume that y is constant. But this leads to $y - x^2 = 0$, or $y = x^2$, so y is not constant! This is a contradiction of the constant assumption, so there are *no equilibrium solutions* to this DE.

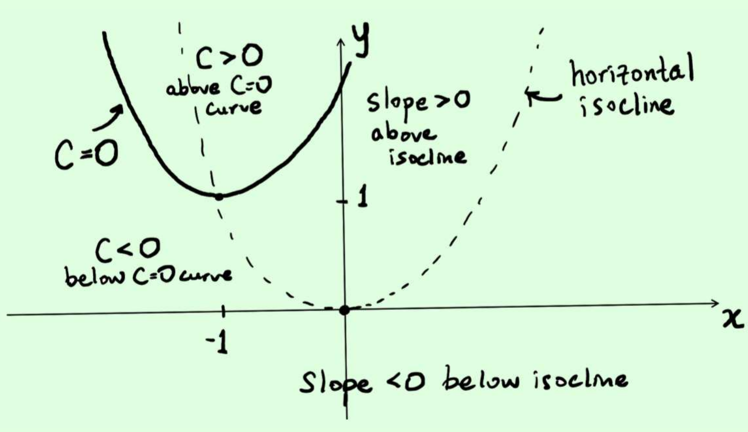
The exceptional solution is obtained by letting $C = 0$ in the general solution, giving $y = 0 + x^2 + 2x + 2 = (x + 1)^2 + 1$, a parabola with vertex $(-1, 1)$. We label this curve with $C = 0$ on the sketch. Note that, since $y = Ce^x + x^2 + 2x + 2$ and $e^x > 0$ for all x, then curves with $C > 0$ will be *above* the exceptional solution curve and curves with $C < 0$ will be *below* it.

Step 2: Slope analysis.

From the DE, we have

$$\frac{dy}{dx} = y - x^2 \begin{cases} > 0 & \text{if } y > x^2 \\ = 0 & \text{if } y = x^2 \\ < 0 & \text{if } y < x^2 \end{cases}$$

so the slope of any solution curve will be *zero* on the curve $y = x^2$, which is the horizontal isocline, *positive* above the isocline, and *negative* below it. We draw the isocline as a dotted curve, since it is not a solution curve, but simply a set of points at which the (intersecting) solution curve is horizontal. Here is what we have so far:

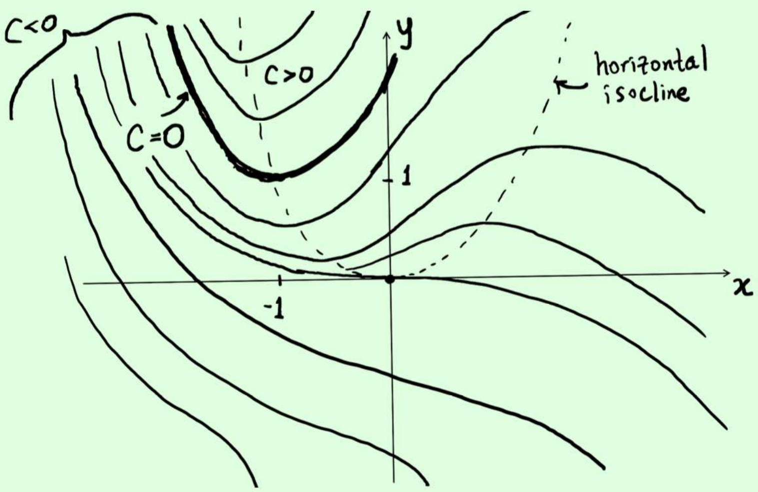


Step 3: Asymptotics.

By looking at the dominant term in each case, we have:

$$y = Ce^x + x^2 + 2x + 2 \approx \begin{cases} Ce^x & \text{as } x \to +\infty \\ x^2 & \text{as } x \to -\infty \end{cases}$$

since Ce^x is much larger in magnitude than the other terms for large positive x (assuming that $C \neq 0$, of course), i.e. Ce^x is the *dominant term*, and x^2 is the dominant term for large negative x. So, as we move to the right ($x \to +\infty$), the solution will behave like a growing exponential (if $C > 0$, or flipped upside down if $C < 0$), and as we move to the left ($x \to -\infty$), it will look like a quadratic function. Here is our final sketch of the family of solutions:



Remarks

In step 2, it is helpful to factor the DE as much as possible first. For example, $\frac{dy}{dx} = -7y + e^x = -7(y - \frac{1}{7}e^x)$, so the isocline is $y = \frac{1}{7}e^x$, the slope is negative above the isocline, etc.

Also, in some examples, it is not necessary to go to the trouble of following the above procedure. For example, if the general solution to a DE is $y = Ce^{-x^2}$, then one can simply sketch the curve $y = e^{-x^2}$ and then apply vertical scaling, with $C < 0$ flipping it upside down.

Finally, *why* should we learn to produce these sketches by hand? Not only does this strengthen our problem-solving ability and understanding of functions, but often *a computer-generated plot can miss—or at least allow the user to miss—important qualitative features.* For example, WolframAlpha, with $y' + y = e^{6x}$, gives a direction field which appears to include vertical slopes, and the "sample solution family" looks like one curve, as its initial conditions are too closely-spaced relative to the scale.

A fundamental property of families of solutions:

Consider the DE

$$\frac{dy}{dx} = f(x, y).$$

A fundamental theorem in the theory of DEs (the Existence-Uniqueness theorem, discussed in AMath 351) states that if the function $f(x, y)$ has continuous partial derivatives (i.e. f is of class C^1), then through a given point (x_0, y_0) there passes one and only one solution curve of the DE. This means that

if f is of class C^1, solution curves of the DE $\frac{dy}{dx} = f(x, y)$ cannot intersect.

This fact is very helpful when sketching families of solution curves.

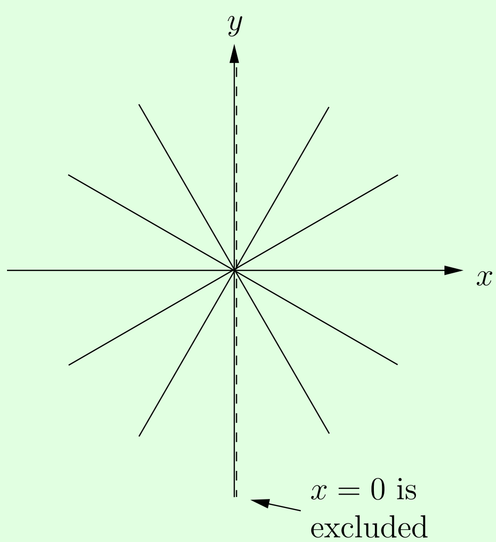
Here are two examples where f is not of class C^1 and intersections do occur.

Example 3:

Consider the DE

$$\frac{dy}{dx} = \frac{y}{x}$$

The solution curves are $y = Cx$. Here $f(x, y) = \frac{y}{x}$, which is not C^1 when $x = 0$ ($f(0, y)$ is not even defined). □



Example 4:

Consider the DE

$$\frac{dy}{dx} = 3y^{2/3}$$

The solution curves are

$$y = (x + C)^3,$$

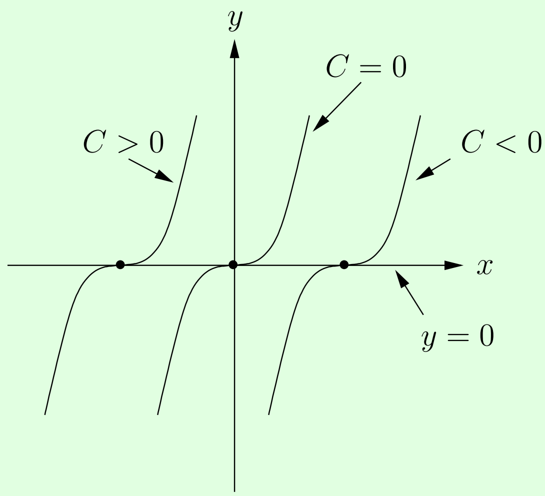
together with the equilibrium solution

$$y = 0.$$

Here $f(x, y) = 3y^{2/3}$, which is not C^1 when $y = 0$ since

$$\frac{\partial f}{\partial y} = \frac{2}{y^{1/3}}.$$

Intersections occur on the x-axis. □



1.2.5 First order linear DEs with constant coefficient

The general form of a linear DE with a constant coefficient is

$$\frac{dy}{dx} + ky = f(x), \tag{1.32}$$

where k *is a constant*. The general solution of such a DE can be found by obtaining an integrating factor as in Section 1.2.3. For certain functions $f(x)$ that arise commonly in applications, however, there is a quicker method for finding a particular solution (i.e. a single solution) of the DE (1.32), called the *method of undetermined coefficients*.[2] We first illustrate this method, and then show how to use it to find the *general* solution of (1.32) efficiently.

The method of undetermined coefficients:

Example 1:

Find a particular solution of

$$\frac{dy}{dx} + 2y = x. \tag{1.33}$$

Solution: Consider a trial function of the form

$$y = Ax + B, \tag{1.34}$$

[2] This method can also be applied to second order linear DEs with constant coefficients. See Section 3.2.4.

where A and B are constants (the "undetermined coefficients"). Substitute (1.34) in (1.33):

$$A + 2(Ax + B) = x.$$

Equate coefficients of x^1 and $x^0 = 1$, giving

$$\begin{aligned} 2A &= 1 \\ A + 2B &= 0. \end{aligned}$$

Solve for A and B: $A = \frac{1}{2}$, $B = -\frac{1}{4}$.

By (1.34), a particular solution of the DE (1.33) is

$$y_p = \tfrac{1}{2}x - \tfrac{1}{4}. \qquad \square$$

Remark

1) The trial function contains a number of constants, and the DE leads to a system of *linear algebraic* equations to be solved for these constants.

2) Choosing a trial function requires some experience. If one doesn't include enough terms and constants, the system of linear equations will be *incompatible*, and one has to try again. The method can be applied if $f(x)$ is composed of powers of x, $\sin \omega x$, $\cos \omega x$ and e^{rx}. The table below shows some simple cases for the DE (1.32).

$f(x)$	trial function y	
x	$Ax + B$	
x^2	$Ax^2 + Bx + C$	
$\sin \omega x$ or $\cos \omega x$	$A \sin \omega x + B \cos \omega x$	
$e^{rx} (r \neq -k)$	Ae^{rx}	here k is the coeff. in the DE
e^{-kx}	Axe^{-kx}	

A good rule of thumb to determine the correct trial function is to make a list of $f(x)$ and its derivatives: f, f', f'',... . Use the functions that appear in this list as terms in the trial function. For example, if $f(x) = x^2 e^x$, then the trial function would be $y_p = Ax^2e^x + Bxe^x + Ce^x$. Finally, if any of the terms in y_p are constant multiples of the homogeneous solution y_h (which is Ce^{-kx}), then we must "multiply by x" — this is explained in example 2 on page 24.

Exercise

Find a particular solution of

$$\frac{dy}{dx} + 2y = \cos x.$$

Answer: $y_p = \frac{1}{5}(2\cos x + \sin x)$. □

Suppose we have found a particular solution y_p of the DE (1.30). We can then find the general solution of this DE *without any extra work*. The following Proposition shows how. The proposition is more general than we need, since it does not depend on the coefficient k in the DE being constant.

Proposition

If y_p is a particular solution of the DE

$$\frac{dy}{dx} + k(x)y = f(x),$$

and y is *any* solution of this DE, then the difference

$$y_h = y - y_p \tag{1.35}$$

is a solution of the *homogeneous* DE

$$\frac{dy}{dx} + k(x)y = 0.$$

Proof: We are given that

$$\frac{dy}{dx} + k(x)y = f(x),$$

and

$$\frac{dy_p}{dx} + k(x)y_p = f(x).$$

Subtract these two equations to get

$$\frac{d}{dx}(y - y_p) + k(x)(y - y_p) = 0,$$

which is the desired result. □

It follows from (1.35) that the general solution y is the sum of two terms, $y = y_p + y_h$. This is the key result, which we now write out in full.

General solution of a linear DE:

The general solution of the linear DE

$$\frac{dy}{dx} + k(x)y = f(x) \tag{1.36}$$

has the form

$$y(x) = y_p(x) + y_h(x), \tag{1.37}$$

where $y_p(x)$ is *a particular* solution of the *inhomogeneous* DE (1.36), and $y_h(x)$ is the *general* solution of the *homogeneous* DE

$$\frac{dy}{dx} + k(x)y = 0. \qquad \square \tag{1.38}$$

Remark

The proposition depends in an essential way on the fact that the DE is *linear*.
In the *constant coefficient case*, the homogeneous DE (1.38) is

$$\frac{dy}{dx} + ky = 0,$$

(the "world's simplest") whose general solution we know to be

$$y = Ce^{-kx}.$$

Thus, once we have obtained a particular solution $y_p(x)$ of (1.32) using the method of undetermined coefficients, we can immediately write down the general solution using (1.37). □

Return to Example 1:

Find the general solution of

$$\frac{dy}{dx} + 2y = x. \tag{1.39}$$

Solution: We have found a particular solution

$$y_p(x) = \tfrac{1}{2}x - \tfrac{1}{4}.$$

The general solution of the homogeneous DE (the "world's simplest")

$$\frac{dy}{dx} + 2y = 0$$

is

$$y_h(x) = Ce^{-2x}.$$

Thus by (1.37), the general solution of the DE (1.39) is

$$y(x) = \left(\tfrac{1}{2}x - \tfrac{1}{4}\right) + Ce^{-2x}. \qquad \square$$

Exercise

Find the general solution of
$$\frac{dy}{dx} + 2y = \cos x.$$

Answer: $y = \frac{1}{5}(2\cos x + \sin x) + Ce^{-2x}$. □

Exercise

Find the general solution of
$$\frac{dy}{dx} - 3y = e^{2x}.$$

Answer: $y = -e^{2x} + Ce^{3x}$. □

Example 2: the "multiply by x rule"

Find the general solution to $\frac{dy}{dx} + 2y = e^{-2x}$.

Solution: As before, the general solution to the homogeneous DE $\frac{dy}{dx} + 2y = 0$ is $y = Ce^{-2x}$. Now, to find a particular solution, we might be inclined to try, based on the right-side function,
$$y_p = Ae^{-2x}$$
but this will lead to a contradiction ($0 = e^{-2x}$, try it). This is because our trial function is a constant multiple of the homogeneous solution, so we need to try a more general solution. It turns out (exercise) that trying a solution of the form $y_p = u(x)e^{-2x}$ leads to $u'(x) = \text{constant}$, so $u(x) = \text{constant} \cdot x$ will work. So, in the case when a trial function would be a multiple of the homogeneous solution, we try
$$y_p = Axe^{-2x}$$
which we can call the "multiply by x rule". Substituting this into the DE leads to
$$Ae^{-2x} - 2Axe^{-2x} + 2Axe^{-2x} = e^{-2x}$$
from which we see that $A = 1$, leading to $y_p = xe^{-2x}$ and general solution
$$y = Ce^{-2x} + xe^{-2x}.$$
□

Exercise

Find the general solution of
$$\frac{dy}{dx} - 3y = 4e^{3x}.$$

Answer: $y = Ce^{3x} + 4xe^{3x}$. □

Remark

In practice, it's best to find the homogeneous solution y_h *first*, because then you will know wheter or not the "multiply by x rule" is required when choosing the trial function for the particular solution y_p.

1.2.6 An important special case

Consider the DE

$$\frac{dy}{dx} = ky + b, \tag{1.40}$$

where k and b are constants, with $k \neq 0$. This DE is *linear* (since the only y-dependence on the right side is y itself), has a *constant coefficient* (since k, the coefficient of y, is a constant), and is also *separable* (since b is a constant).

This DE can thus be solved using each of the techniques that we have introduced so far, and we suggest that you do this as an exercise:

1) Solve (1.40) as a linear DE $\frac{dy}{dx} - ky = b$ by finding an integrating factor.

2) Solve as a separable DE.

3) Find a particular solution using the method of undetermined coefficients (use $y = A$, a constant as the trial function), and then write down the general solution.

The recommended method for solving (1.40), and the quickest method, is to convert the DE into the "world's simplest" form and solve by inspection as follows. Since b and k are constants and $k \neq 0$ we can rewrite (1.40) in the form

$$\frac{d}{dx}\left(y + \frac{b}{k}\right) = k\left(y + \frac{b}{k}\right). \tag{1.41}$$

If we define

$$u = y + \frac{b}{k} \tag{1.42}$$

this DE becomes

$$\frac{du}{dx} = ku,$$

whose solution is

$$u = Ce^{kx}, \tag{1.43}$$

where C is a constant. Using (1.42), we get

$$y + \frac{b}{k} = Ce^{kx}, \tag{1.44}$$

Thus

$$y = -\frac{b}{k} + Ce^{kx}$$

is the general solution of the DE (1.40).

In practice, there is no need to formally introduce u. Having rewritten the DE in the form (1.41) one can immediately write down the solution (1.44).

1.2.7 A common error

Consider a first order DE

$$\frac{dy}{dx} = f(x, y). \tag{1.45}$$

Knowing that

$$\int \frac{dy}{dx} dx = y + C,$$

a naive person might be tempted to try to solve (1.45) by taking the antiderivative of both sides with respect to x, obtaining

$$y + C = \int f(x, y) dx.$$

This attempt to solve (1.45) fails in general because the antiderivative on the right hand side contains y, which depends on x and hence *cannot be treated as a constant* (y is of course the unknown function which we are trying to find).

We note that this simple-minded approach only works in the very special case where the DE (1.45) has the form

$$\frac{dy}{dx} = f(x), \tag{1.46}$$

i.e. *the right hand side is independent of* y. We shall say that the DE (1.46) is *directly solvable*, because it can be solved simply by taking the antiderivative.

It is worth noting that in solving a linear DE by finding an integrating factor, one transforms the DE into a directly solvable form (see the Comment after equation (1.30)). In particular, in the example in Section 1.2.3, we converted the DE

$$\frac{dy}{dx} - y = -e^{-x}$$

into the directly solvable form

$$\frac{d}{dx}(e^{-x} y) = -e^{-2x}$$

[see equations (1.25) and (1.29)].

1.2.8 Initial value problems

The skydiver DE,

$$m\frac{dv}{dt} = mg - \alpha v,$$

determines the velocity of a skydiver as a function of time t. We have seen that a first order DE has a one-parameter family of solutions (the constant of integration is the parameter). When applying the skydiver DE, the physical process being described will begin at the time when the skydiver jumps from the plane. We label this time as $t = 0$, and so we have the *initial condition*

$$v(0) = 0.$$

This initial condition will determine the constant of integration, leading to a unique solution that gives the velocity of the skydiver at time t.

In general, for a DE

$$\frac{dy}{dx} = f(x, y), \tag{1.47}$$

the initial condition will be of the form

$$y(x_0) = y_0. \tag{1.48}$$

where x_0 and y_0 are given constants. The equations (1.47) and (1.48) are said to define an *initial value problem.*

Example

Find the unique solution of the initial value problem

$$\frac{dy}{dx} = -2y, \quad y(0) = 3.$$

Solution: By inspection the general solution of the DE is

$$y = Ce^{-2x}$$

Substituting $x = 0$ and $y = 3$ gives

$$3 = Ce^0,$$

i.e. $C = 3$. Thus the unique solution is

$$y = 3e^{-2x}. \qquad \square$$

Remark

The general solution of a first order DE (1.47) corresponds to a one-parameter family of solution curves. The initial condition (1.48) picks out a unique curve, namely the curve that passes through the point (x_0, y_0).

Exercise

Find the unique solution of the initial value problem

$$\frac{dy}{dx} = -2xy^2, \quad y(1) = \tfrac{1}{3}.$$

Answer: $y = \dfrac{1}{x^2 + 2}$. $\square$

Exercise

Find the unique solution of the initial value problem

$$x^2 \frac{dy}{dx} + 2xy = 1, \quad y(1) = 0.$$

Answer: $y = \frac{1}{x} - \frac{1}{x^2}$. □

1.3 Other applications of first order DEs

1.3.1 Mixing problems

Various problems in biology and engineering can be put in the following framework. Consider a tank containing a chemical solution. The contents are kept well-mixed, so that the concentration is uniform. There is an *inflow* of the chemical solution of specified concentration, and and *outflow* of chemical solution, whose concentration at time t equals the concentration of solution in the tank at time t.

The goal is to predict the amount of chemical in the tank at time t, or perhaps to adjust the inflow and outflow as as to achieve a desired concentration in the tank.

Let $m(t)$ denote the amount of chemical in the tank at time t. The rate of change of $m(t)$ equals the difference between the rate of inflow and rate of outflow:

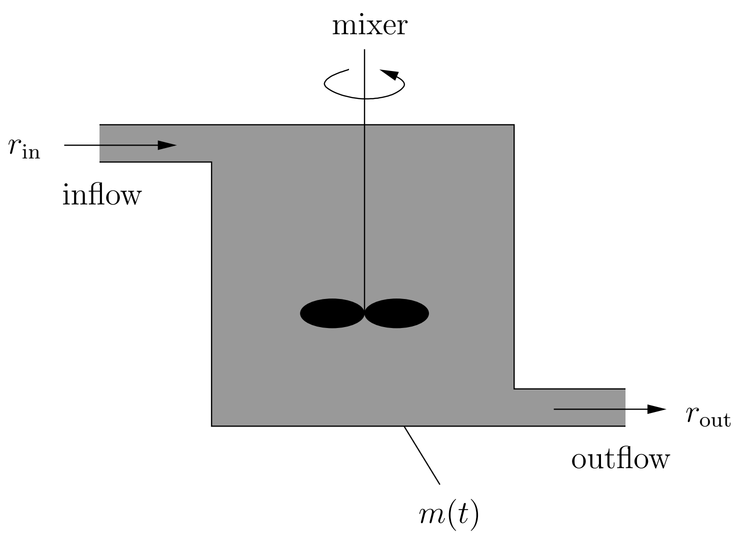


Figure 1.2: Mixing problem.

$$\frac{dm}{dt} = r_{\text{in}} - r_{\text{out}}, \tag{1.49}$$

where r_{in} is the rate at which chemical is added by the inflow and r_{out} is the rate at which chemical is removed by the outflow. By the Principle of Dimensional Homogeneity, each term in equation (1.49) will have the same dimensions, namely MT^{-1} (i.e. mass of chemical per unit time). We shall refer to equation (1.49) as the *mass balance equation* for a mixing tank.

Example:

A tank contains m_0 kg of salt dissolved in 100 litres of water. A salt solution containing $\frac{1}{4}$ kg per litre is added at 3 litre/min., and the well-stirred mixture leaves the tank at the same rate. Find the amount of salt in the tank at time t.

Solution: From the given data, the rate at which salt is added to the tank is

$$r_{\text{in}} = (3)\left(\tfrac{1}{4}\right) \text{ kg/min.}$$

Let $m(t)$ denote the amount of salt in the tank at time t. Then the concentration at time t is

$$\frac{m(t)}{100} \quad \text{kg/litre}$$

Thus, the rate at which salt leaves the tank is

$$r_{\text{out}} = (3)\left(\frac{m(t)}{100}\right) \text{kg/min.}$$

The mass balance equation (1.49) gives

$$\frac{dm}{dt} = \tfrac{3}{4} - \tfrac{3}{100}m = -\tfrac{3}{100}(m - 25). \tag{1.50}$$

The initial condition is

$$m(0) = m_0. \tag{1.51}$$

The DE (1.50) can be written

$$\frac{d}{dt}(m - 25) = -\tfrac{3}{100}(m - 25),$$

and hence can be solved by inspection:

$$m - 25 = Ce^{-\frac{3}{100}t}.$$

On setting $t = 0$, the initial condition (1.51) gives $C = m_0 - 25$, leading to the solution

$$m(t) = 25 + (m_0 - 25)e^{-\frac{3}{100}t}. \tag{1.52}$$

Interpretation: Based on the physical set-up, one expects that as time passes the concentration of the solution in the tank will approach the concentration of the inflow i.e. $\frac{1}{4}$ kg/litre. Thus the amount of salt in the tank will approach $\left(\frac{1}{4}\right)(100) = 25$ kg as $t \to \infty$, in agreement with equation (1.52). □

Remark

One can imagine problems such as the above arising in different contexts, e.g.

(1) nutrients flowing into and out of a cell (which plays the role of the tank),

(2) carbon monoxide seeping into a room and then being dispersed.

Overview:

In a mixing tank problem, the unknown function is the mass of chemical in the tank at time t, denoted by $m(t)$, with dimensions

$$[m(t)] = M.$$

There are two flow rates, the inflow rate f_{in} and the outflow rate f_{out}, with dimensions

$$[f_{\text{in}}] = [f_{\text{out}}] = L^3T^{-1}.$$

The flow rates f_{in} and f_{out} are given, and in general could be functions of time t, but in simple problems they will be constants, and may even be equal. If they are equal, then the volume V of solution in the tank will be constant in time.

There are two concentrations, the concentration of the inflow c_{in} and the concentration of the outflow c_{out}, with

$$[c_{\text{in}}] = [c_{\text{out}}] = ML^{-3}.$$

The inflow concentration is given, and will be constant in the simplest situation. The outflow concentration is the concentration of the solution in the tank at time t, and is hence given by the key relation

$$c_{\text{out}} = \frac{m(t)}{V(t)},$$

where $V(t)$ is the volume at time t. Finally, the rates of mass inflow r_{in} and mass outflow r_{out} that appear in the mass balance equation (1.49) are given by

$$r_{\text{in}} = c_{\text{in}}f_{\text{in}}, \quad r_{\text{out}} = c_{\text{out}}f_{\text{out}}.$$

Verify that these equations are dimensionally consistent.

References: Boyce & Diprima, pg. 51, #18-22.
Braun, pg. 56, #6-11.

1.3.2 Population growth

Let $x(t)$ be the population of some species (e.g. fish, bacteria, etc.) at time t. The simplest hypothesis is the Malthusian Law i.e. that the rate of change of x at time t is proportional to the population at time t:

$$\frac{dx}{dt} = rx, \tag{1.53}$$

where r is a constant with $[r] = T^{-1}$, called the *rate of growth* (*rate of decline*, if $r < 0$). By inspection the general solution of the DE (1.53) is

$$x(t) = Ce^{rt},$$

and the initial condition $x(0) = x_0$ leads to

$$x(t) = x_0 e^{rt},$$

describing *exponential growth* ($r > 0$) or *decay* ($r < 0$).

It is clear that exponential growth can only continue for a restricted time due to resource limitations. A more realistic model takes into account that a given environment can support at most a finite number of a particular species, denoted by K, and called the *carrying capacity*. A realistic model thus requires that the rate of change $\frac{dx}{dt}$ approach zero as x gets close to K. A simple way to accomplish this is to assume that the rate of growth is not simply a constant r as in (1.53), but depends on x according to

$$r\left(1 - \frac{x}{K}\right),$$

where K is the constant carrying capacity. The DE governing the population $x(t)$ then assumes the form

$$\frac{dx}{dt} = r\left(1 - \frac{x}{K}\right)x, \tag{1.54}$$

called the *logistic equation*, a separable, non-linear DE.

The variable x is dimensionless, $[x] = 1$, since it represents a number. There is, however, a choice of scale associated with x, since one can measure x in thousands (say) or millions (if the population is very large). The DEs (1.53) and (1.54) are in fact independent of the choice of scale, since if x (and K) are multiplied by the same constant factor, the DEs are unchanged.

Exercise

Solve the DE (1.54) using two different methods:

(i) by separation of variables (you will find that letting $u = \frac{x}{K}$, a simple rescaling of x, simplifies the algebra);

(ii) by making the change of variable $y = \frac{K}{x}$. This has the effect of changing (1.54) into the simple form $\frac{d}{dt}(y-1) = -r(y-1)$, which can be solved by inspection.

Answer: The solution can be written in the form

$$x(t) = \frac{x_0 e^{rt}}{1 - \frac{x_0}{K} + \frac{x_0}{K}e^{rt}},$$

after imposing the initial condition $x(0) = x_0$.

1.3.3 Epidemics

Consider a population of N individuals containing a number of individuals having an infectious disease. The goal is to determine how rapidly the disease will spread if no control measures are taken.

Let $x(t)$ denote the number of infectious individuals at time t, and let $N - x(t)$ denote the number of susceptible individuals. Assume that the rate of spread of the disease, $\frac{dx}{dt}$, is proportional to the number of contacts between infectious and susceptible individuals. Assume that both groups mingle freely so that the number of contacts is proportional to $x(N - x)$. Then $x(t)$ satisfies

$$\frac{dx}{dt} = \frac{\alpha}{N} x(N - x), \tag{1.55}$$

where α is a constant with $[\alpha] = T^{-1}$. We have written the constant of proportion as $\frac{\alpha}{N}$, in order that the DE remain unchanged if x and N are rescaled (see the comment in Section 1.3.2). Equation (1.55) is a first order DE for $x(t)$, with initial condition $x(0) = x_0$, which can be solved by separation of variables.

Reference: Boyce & DiPrima, page 65.

1.3.4 Cooling problems

Consider an object whose temperature at time t is $T(t)$, in surroundings whose temperature is T_A (called the *ambient temperature*). If $T(t)$ is larger than T_A the object will lose heat to its surroundings and its temperature will decrease. Newton's Law of Cooling states that the rate of decrease is proportional to the difference between $T(t)$ and the ambient temperature T_A:

$$\frac{dT}{dt} = -k(T - T_A), \tag{1.56}$$

where k is a constant with dimensions $[k] = T^{-1}$. (Newton's Law is an approximation which is reasonable provided the temperature differences are not too great.)

If the *ambient temperature is constant* (not necessary in general), the DE (1.56) can be written

$$\frac{d}{dt}(T - T_A) = -k(T - T_A),$$

which can be solved by inspection:

$$T(t) - T_A = Ce^{-kt},$$

giving an exponential rate of cooling.

Reference: Boyce & DiPrima, page 48.

1.3.5 Pursuit problems

This class of problems refers to the situation in which one participant tries to catch a second moving participant, e.g. a bull charging a running person, a hawk chasing a flying pigeon, a gunboat chasing a smuggler. These problems are difficult. Here is an example.

Example:

A dog swims across a river towards his master on the far bank, but is carried downstream by the current. The dog always paddles towards his master. What path will the dog follow? Assume that the speed of the river is w and of the dog in still water is u, with $w < u$.

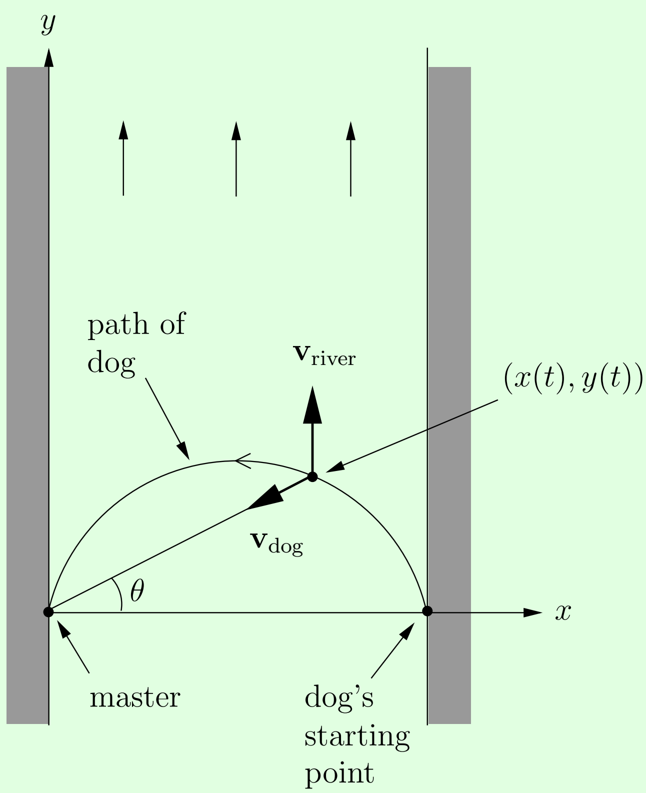
Solution: Since the dog always swims towards its master, its velocity is

$$\mathbf{v}_{\text{dog}} = (-u\cos\theta, -u\sin\theta).$$

The velocity of the river is

$$\mathbf{v}_{\text{river}} = (0, w).$$

The rate of change of the dog's position $\mathbf{x}(t) = (x(t), y(t))$ is the sum of the two velocities:



$$\frac{d\mathbf{x}}{dt} = \mathbf{v}_{\text{dog}} + \mathbf{v}_{\text{river}}.$$

In components,

$$\frac{dx}{dt} = -u\cos\theta, \quad \frac{dy}{dt} = -u\sin\theta + w. \tag{1.57}$$

But

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}. \tag{1.58}$$

We can write the dog's path in the form $y = y(x)$. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

By (1.57) and (1.58),

$$\frac{dy}{dx} = \frac{uy - w\sqrt{x^2 + y^2}}{ux}.$$

Let $k = \frac{w}{u}$, a dimensionless quantity. Since $x > 0$ we can rearrange the DE to get

$$\frac{dy}{dx} = \frac{y}{x} - k\sqrt{1 + \frac{y^2}{x^2}}, \tag{1.59}$$

a *first order DE for the path* $y = y(x)$ *of the dog.* The initial condition is $y(b) = 0$, where b is the width of the river.

Comment: $[x] = L$, $[y] = L$ and $[k] = 1$, making (1.59) dimensionally consistent.

Exercise: The form of the DE (1.59) suggests that we introduce a new dependent variable z defined by

$$z = \frac{y}{x}.$$

The DE (1.59) assumes the form

$$\frac{dz}{dx} = -k\frac{\sqrt{1 + z^2}}{x},$$

which is separable. The final solution for $y = y(x)$ is

$$y = \tfrac{1}{2}b\left[\left(\frac{x}{b}\right)^{1-k} - \left(\frac{x}{b}\right)^{1+k}\right].$$

The algebra involved in deriving this is somewhat tricky. An intermediate step is

$$z + \sqrt{1 + z^2} = \left(\frac{x}{b}\right)^{-k},$$

which can be solved for z giving

$$z = \tfrac{1}{2}\left[\left(\frac{x}{b}\right)^{-k} - \left(\frac{x}{b}\right)^{k}\right]. \qquad \square$$

Reference: Borelli & Coleman, pages 112-5.

1.3.6 Electrical circuits

We consider simple electrical circuits which are constructed from a voltage source and three basic elements, *resistors* (of resistance R), *inductors* (of inductance L) and *capacitors* (of capacitance C), indicated symbolically in figure 1.8.

$$V = RI \qquad V = L\tfrac{dI}{dt} \qquad Q = VC \tag{1.60}$$

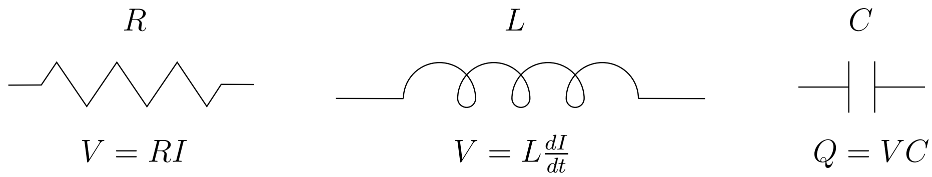


Figure 1.3: Resistor, inductor & capacitor.

Here I denotes the electric current through an element, V is the electrical potential difference across the element, and Q is the electric charge on the plate of the conductor onto which the current I is assumed to flow; I is therefore related to Q by

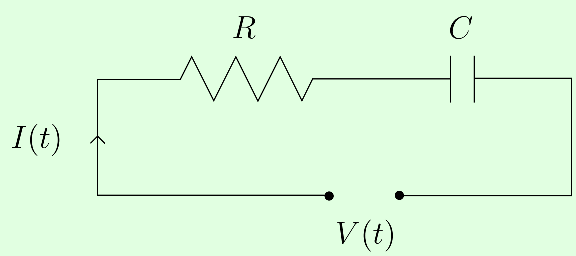
$$I = \frac{dQ}{dt}. \tag{1.61}$$

The three formulas in the figure may be regarded as defining R, L and C.

The current I in the electrical circuit varies with time t, and in simple cases is governed by a DE. The form of the DE is determined by one of *Kirchoff's Laws*, which states that the potential difference across the terminals (i.e. the voltage source) must equal the sum of the potential differences across the various elements:

$$V_{\text{terminals}} = \sum V_{\text{elements}}. \tag{1.62}$$

Example: the RC circuit.



By (1.62) and (1.60),

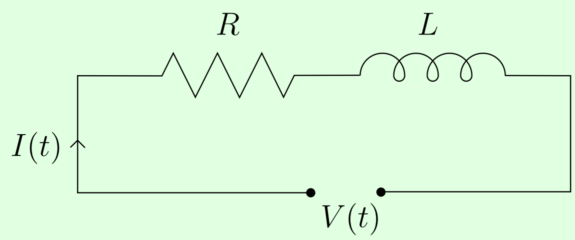
$$V(t) = V_R + V_C = RI + \frac{Q}{C}. \tag{1.63}$$

Differentiate (1.63) with respect to t and use (1.61) to eliminate $\frac{dQ}{dt}$, giving

$$\frac{dI}{dt} + \frac{1}{RC}I = \frac{1}{R}\frac{dV}{dt}. \tag{1.64}$$

Equation (1.64) is a first order linear DE for $I(t)$, the source voltage $V(t)$ being regarded as given. □

Example: the *RL* circuit.



By (1.62) and (1.60),

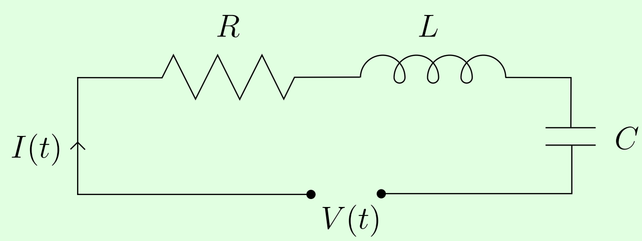
$$V(t) = V_R + V_L = RI + L\frac{dI}{dt},$$

that is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}V(t). \tag{1.65}$$

Equation (1.55) is a first order linear DE for $I(t)$. □

Example: the *RLC* circuit.



By (1.62) and (1.60),

$$V(t) = V_R + V_L + V_C = RI + L\frac{dI}{dt} + \frac{Q}{C}.$$

Differentiate with respect to t and use (1.61), as in passing from (1.63) to (1.64). We obtain

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}\frac{dV}{dt}, \tag{1.66}$$

a *second order* linear DE (to be studied later in the course).

References: Boyce & DiPrima, pages 180-1.
Borelli & Coleman, pages 176-86.

Chapter 2

Dimensional Analysis

2.1 Writing physical relations in dimensionless form

2.1.1 Characteristic scales and dimensionless variables

Problems in the sciences, arising for example in quantum mechanics, oceanography, the solar system and cosmology involve widely different time scales and length scales. Indeed, in studying any physical problem it is important to identify a typical or *characteristic time*, denoted t_c, which is the length of time over which the physical system changes in a significant way. It may also be necessary to define a *characteristic length* ℓ_c and a *characteristic mass* m_c.

The characteristic time is usually defined in terms of the physical parameters associated with the system in question, for example the physical parameters that appear in the DE that describes the evolution of the system. Here's the simplest example.

Example 1:

Consider exponential decay of a radioactive substance. The system is described by the DE

$$\frac{dm}{dt} = -km, \tag{2.1}$$

where k is a positive constant and m denotes the mass of the radioactive substance at time t. By dimensional homogeneity (see property D1 in Section 1.1.2),

$$[-km] = \left[\frac{dm}{dt}\right] = MT^{-1},$$

giving

$$[k] = \frac{MT^{-1}}{M} = T^{-1},$$

since $[m] = M$. It is thus reasonable to define a characteristic time by

$$t_c = \frac{1}{k}. \tag{2.2}$$

To get a physical interpretation of t_c, note that the solution of the DE (2.1) with initial condition $m(0) = m_0$ is

$$m(t) = m_0 e^{-kt}. \tag{2.3}$$

It follows that

$$m(t_c) = m_0 e^{-kt_c} = m_0 e^{-1},$$

by (2.2). Thus, t_c *is the length of time during which the mass decreases by a factor of* $\frac{1}{e}$. Having defined a characteristic time, one can use it to write the DE (2.1) in a simpler form. The idea is to define a dimensionless time variable τ by

$$\tau = \frac{t}{t_c}, \tag{2.4}$$

(τ is dimensionless, since $[\tau] = \frac{[t]}{[t_c]} = \frac{T}{T} = 1$). Then, regarding m as a function of τ,

$$m = m(\tau) = m(\tau(t)),$$

the Chain Rule gives

$$\frac{dm}{dt} = \frac{dm}{d\tau}\frac{d\tau}{dt} = \frac{1}{t_c}\frac{dm}{d\tau}.$$

Thus, using (2.2) the DE (2.1) becomes simply

$$\frac{dm}{d\tau} = -m.$$

There is no unique way of defining characteristic time. For a radioactive substance one can define the *half-life* $t_{1/2}$, which is the time taken for half the substance to decay;

$$m(t_{1/2}) = \tfrac{1}{2} m_0.$$

By (2.3), $\frac{1}{2} m_0 = m_0 e^{-kt_{1/2}}$, giving $e^{kt_{1/2}} = 2$, and hence

$$t_{1/2} = \frac{1}{k} \ln 2.$$

This time could equally well be used as the characteristic time.

In the next example, we write a *given* physical relation in dimensionless form, by defining a characteristic time and characteristic length.

Example 2:

Consider the motion of a baseball thrown vertically up with initial velocity v_0, ignoring air resistance. The height of the ball at time t is given by

$$h(t) = v_0 t - \tfrac{1}{2} g t^2, \tag{2.5}$$

where g is the acceleration due to gravity, assumed constant. [This equation can be derived by integrating $\frac{d^2h}{dt^2} = -g$ twice, assuming that $\frac{dh}{dt}(0) = v_0$, $h(0) = 0$.] In this situation there are two physical parameters v_0 and g, with

$$[v_0] = LT^{-1}, \quad [g] = LT^{-2}.$$

It follows that $\left[\frac{v_0}{g}\right] = T$, and so it is reasonable to define a characteristic time by

$$t_c = \frac{v_0}{g}. \tag{2.6}$$

Further, $\left[\frac{v_0^2}{g}\right] = L$, and so we define a characteristic length:

$$\ell_c = \frac{v_0^2}{g}.$$

As regards the physical interpretation, t_c is the time for a deceleration of g to reduce a velocity v_0 to zero, and since

$$\ell_c = v_0 t_c, \tag{2.7}$$

ℓ_c is the distance travelled in time t_c by an object moving with velocity v_0.
We now define a dimensionless time and height by

$$\tau = \frac{t}{t_c}, \quad H = \frac{h}{\ell_c}.$$

In terms of these variables the relation (2.5) becomes

$$H = \frac{1}{\ell_c}\left(v_0 t_c \tau - \tfrac{1}{2} g t_c^2 \tau^2\right),$$

which simplifies using (2.6) and (2.7) to

$$H = \tau - \tfrac{1}{2}\tau^2. \tag{2.8}$$

This is the dimensionless form of the physical relation (2.5), and captures its essential content in the simplest possible form. □

Overview:

These two examples illustrate the approach used to define a characteristic time/length/mass:

1) List the physical parameters in the DE or physical relation and give their dimensions.

2) By inspection find a combination using products and quotients that has the dimensions of time/length/mass, to play the role of characteristic quantity.

3) Deduce the physical interpretation of the characteristic quantity as a check that your definition is appropriate.

2.1.2 The mixing tank DE

Consider the DE describing the constant volume mixing tank as shown (see Section 1.3.1):

$$\frac{dm}{dt} = -\frac{f}{V}m(t) + fc_{\text{in}}. \tag{2.9}$$

Here $m(t)$ denotes the mass of chemical in the tank at time t, f denotes the constant flow rate and V the constant volume. The inflow concentration c_{in} may depend on time t.

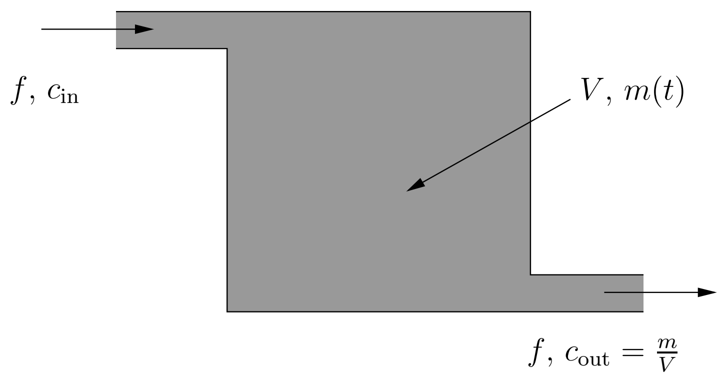


Figure 2.1: A mixing tank.

The two physical constants are f and V, with dimensions

$$[f] = L^3T^{-1}, \quad [V] = L^3. \tag{2.10}$$

It follows that

$$\left[\frac{V}{f}\right] = \frac{[V]}{[f]} = \frac{L^3}{L^3T^{-1}} = T.$$

So it is reasonable to define a characteristic time by

$$t_c = \frac{V}{f}. \tag{2.11}$$

Since $V = ft_c$, the interpretation is that t_c is *the time taken for an inflow at a given rate* f *to fill a tank of volume* V.

Introducing a dimensionless time τ by

$$\tau = \frac{t}{t_c}$$

as before, the DE (2.9) assumes the simpler form

$$\frac{dm}{d\tau} + m = Vc_{\text{in}}, \tag{2.12}$$

(verify the details; the calculation is similar to example 1 in Section 2.1.1).

In the special case when c_{in} *is a constant* we can simplify the DE (2.12) further by introducing a *characteristic mass*, as follows.

Since

$$[c_{\text{in}}] = ML^{-3},$$

it follows from (2.10) that

$$[Vc_{\text{in}}] = [V][c_{\text{in}}] = M.$$

It is thus reasonable to define a characteristic mass by

$$m_c = Vc_{\text{in}}.$$

The interpretation is that m_c is *the mass of chemical in a tank of volume* V *if the concentration of the chemical is* c_{in}.

We now define a *dimensionless mass variable* $\mathcal{M}$ by

$$\mathcal{M} = \frac{m}{m_c}. \tag{2.13}$$

On dividing (2.12) by $m_c = Vc_{\text{in}}$ and using (2.13), the DE assumes the simplest possible form

$$\frac{d\mathcal{M}}{d\tau} + \mathcal{M} = 1. \tag{2.14}$$

Remark

This DE can be solved by inspection by writing it in the form

$$\frac{d}{d\tau}(\mathcal{M} - 1) = -(\mathcal{M} - 1),$$

giving

$$\mathcal{M} - 1 = Ce^{-\tau}. \qquad \square$$

2.1.3 The skydiver DE

The DE is equation (1.4):

$$m\frac{dv}{dt} = mg - \alpha v. \tag{2.15}$$

The physical parameters are the mass m, the acceleration due to gravity g, and the drag coefficient α, having dimensions

$$[m] = M, \quad [g] = LT^{-2}, \quad [\alpha] = MT^{-1}. \tag{2.16}$$

By inspection

$$\left[\frac{m}{\alpha}\right] = T.$$

It is thus reasonable to define a characteristic time by

$$t_c = \frac{m}{\alpha}. \tag{2.17}$$

In order to describe the physical significance of t_c, we recall that the terminal velocity is

$$v_{\text{term}} = \frac{mg}{\alpha}, \tag{2.18}$$

as follows by setting $\frac{dv}{dt} = 0$ in (2.15). Using (2.17)

$$v_{\text{term}} = gt_c. \tag{2.19}$$

Thus t_c *is the time over which an acceleration of* g *must act in order to produce a velocity* v_{term}. Introducing a dimensionless time

$$\tau = \frac{t}{t_c} \tag{2.20}$$

in the usual way, the DE (2.15) assumes the simpler form

$$\frac{dv}{d\tau} + v = v_{\text{term}}, \tag{2.21}$$

(verify the details).

One can simplify the DE further by regarding v_{term} as a *characteristic velocity*, and using it to define a *dimensionless velocity* function

$$V = \frac{v}{v_{\text{term}}}. \tag{2.22}$$

The DE (2.21) then simplifies further to

$$\frac{dV}{d\tau} + V = 1. \tag{2.23}$$

This DE has *precisely the same form as the mixing tank DE* (2.14), showing that the mixing tank problem and the skydiver problem have the same mathematical content!

In the skydiver problem one can also define a characteristic length (distance). By inspection of (2.16) we observe that

$$\left[\frac{m^2 g}{\alpha^2}\right] = L.$$

So it is reasonable to define a characteristic length by

$$\ell_c = \frac{m^2 g}{\alpha^2}. \tag{2.24}$$

By (2.17), (2.18) and (2.24),

$$\ell_c = v_{\text{term}} t_c,$$

i.e. ℓ_c *is the distance fallen in time* t_c *when falling at the terminal velocity.*

The height fallen at time t, $h(t)$, is related to $v(t)$ in the usual way,

$$v(t) = \frac{dh(t)}{dt}. \tag{2.25}$$

One can now define a dimensionless height fallen H by

$$H = \frac{h}{\ell_c}. \tag{2.26}$$

It follows that (2.25) assumes the form

$$V(\tau) = \frac{dH(\tau)}{d\tau}$$

in terms of the dimensionless variables (2.20), (2.22) and (2.26) (Exercise). □

Typical values:

For a skydiver of mass 75kg, typical values of v_{term} are

$$\begin{array}{ll} v_{\text{term}} \approx 55\text{m/sec} & \text{without a parachute} \\ v_{\text{term}} \approx 5\text{m/sec} & \text{with a parachute.} \end{array}$$

These values lead to (using $g \approx 10\text{m}\, s^{-2}$)

$$t_c \approx 5.5s, \qquad \ell_c \approx 300m \qquad \text{without a parachute,}$$

and

$$t_c \approx 0.5s, \qquad \ell_c \approx 2.5m \qquad \text{with a parachute.}$$

A typical training jump begins at 1200m above ground level. After a free fall of approximately 10s, the chute is opened at approximately 760m above ground. The rest of the jump takes approximately 3 minutes, with a landing velocity of 5m/sec.

Reference:

Meade, D.B. 1998, DE models for the parachute problem, *SIAM Review* **40**, 327-32. □

Consistency check:

In setting up the DE (2.15), we made two simplifying assumptions (see Section 1.1.1):

(i) distances are small compared to the radius R of the earth, so that g can be regarded as constant,

(ii) velocities are small compared to the velocity of light c, so that Newton's Second Law can be used.

We can use the above characteristic values to give a consistency check i.e.

$$\frac{\ell_c}{R} \ll 1, \quad \frac{v_{\text{term}}}{c} \ll 1. \qquad \square$$

2.2 Deducing physical relations using dimensional analysis

The goal in this section is to use arguments based on dimensions to obtain useful information about the dependence of one physical quantity on other physical quantities, without giving a detailed analysis of the physical problem in question.

2.2.1 A motivating example

Consider the problem of determining a formula for the terminal velocity of a skydiver. This formula is by now quite familiar (see equation (2.18)), and, as we have seen, can be derived using the skydiver DE. However, the problem is an excellent one for illustrating the method of dimensional analysis,[1] which is based on the *Buckingham Pi Theorem*.

By thinking about the physical system, we convince ourselves that the terminal velocity v_{term} should depend only on the mass m of the skydiver, the gravitational acceleration g, and the drag coefficient α (we assume the force due to air drag is of the form αv, where v is the velocity):

$$v_{\text{term}} = \mathcal{F}(m, g, \alpha), \tag{2.27}$$

where $\mathcal{F}$ is an unknown function of 3 variables.

In order to apply the Buckingham Pi Theorem, we have to construct all possible independent dimensionless quantities from the physical quantities

$$\{v_{\text{term}}, m, g, \alpha\}, \tag{2.28}$$

and so we begin by discussing this procedure. The most general dimensionless quantity will be a product of integer powers of these quantities:[2]

$$\Pi = v^{P_v} m^{P_m} g^{P_g} \alpha^{P_\alpha}, \tag{2.29}$$

(see requirement D2 in Section 1.1.2.) The dimensions of the physical quantities are

$$[v] = LT^{-1}, \quad [m] = M, \quad [g] = LT^{-2}, \quad [\alpha] = MT^{-1}. \tag{2.30}$$

The dimensions of Π can be calculated as follows:

$$\begin{aligned} [\Pi] &= (LT^{-1})^{P_v}(M)^{P_m}(LT^{-2})^{P_g}(MT^{-1})^{P_\alpha} \\ &= M^{P_m+P_\alpha} L^{P_v+P_g} T^{-P_v-2P_g-P_\alpha}. \end{aligned}$$

(see requirement D2 in Section 1.1.2). Since we require Π to be dimensionless, i.e. $[\Pi] = 1$, the exponents of M, L and T must be zero, giving the following equations for the 4 unknowns P_v, P_m, P_g and P_α in equation (2.29):

$$\begin{array}{rrrrl} & P_m & & +P_\alpha & = 0 \\ P_v & & +P_g & & = 0 \\ -P_v & & -2P_g & -P_\alpha & = 0 \end{array} \tag{2.31}$$

[1] In practice one would do the dimensional analysis before setting up the DE.

[2] For brevity, we write v in place of v_{term}.

In matrix form

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} P_v \\ P_m \\ P_g \\ P_\alpha \end{pmatrix} = 0. \tag{2.32}$$

Since (2.31) is a linear system of 3 equations in 4 unknowns, the solution is not unique. In fact, one obtains

$$P_v = P_\alpha, \quad P_m = -P_\alpha, \quad P_g = -P_\alpha, \tag{2.33}$$

i.e. one can choose P_α arbitrarily and then P_v, P_m and P_g are determined. The dimensionless quantity (2.29) assumes the form

$$\Pi = v^{P_\alpha} m^{-P_\alpha} g^{-P_\alpha} \alpha^{P_\alpha} = \left(\frac{v\alpha}{mg}\right)^{P_\alpha}.$$

There is only one independent dimensionless quantity, obtained by choosing $P_\alpha = 1$ for simplicity, i.e.

$$\Pi = \frac{v\alpha}{mg}. \tag{2.34}$$

The Buckingham Pi Theorem asserts that the physical relation (2.27) is equivalent to a relation involving only the dimensionless quantities. Since in the present case there is only one dimensionless quantity, the conclusion is that *this quantity must equal a numerical constant*:

$$\Pi = C, \quad \text{i.e.} \quad \frac{v\alpha}{mg} = C. \tag{2.35}$$

This can be rearranged to give

$$v_{\text{term}} = C\frac{mg}{\alpha}. \tag{2.36}$$

Thus, *we have shown, purely by consideration of dimensions, that the formula* (2.27) *must have the form* (2.36). The only uncertainty is in the numerical constant C, which has to be determined by a more detailed mathematical analysis or by experiments.

2.2.2 Complete sets of dimensionless quantities

In this Section we discuss the process for constructing dimensionless quantities in more detail. The matrix in equation (2.32) is called the *dimensional matrix* associated with the physical quantities $\{v, m, g, \alpha\}$. Its entries are the dimensions of these quantities, which can be read off from equation (2.30), which we repeat here:

$$[v] = LT^{-1}, \quad [m] = M, \quad [g] = LT^{-2}, \quad [\alpha] = MT^{-1}.$$

The matrix is

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{cccc} v & m & g & \alpha \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -2 & -1 \end{pmatrix} \end{array}$$

It is a 3×4 matrix, corresponding to 3 distinct dimensions M, L and T, and 4 physical quantities. The equation (2.32) (or (2.31)) that determines the exponents can be written in the form

$$\mathcal{D}\mathbf{P} = \mathbf{0},$$

where $\mathcal{D}$ is the dimensional matrix, and

$$\mathbf{P} = (P_v, P_m, P_g, P_\alpha)^T.$$

We have seen that this equation has only one independent solution

$$\mathbf{P} = (1, -1, -1, 1)^T,$$

(see eq. (2.33) with $P_\alpha = 1$) giving one dimensionless quantity

$$\Pi = v^1 m^{-1} g^{-1} \alpha^1 = \frac{v\alpha}{mg}. \qquad \square$$

The key point is that *the form of the dimensional matrix $\mathcal{D}$ enables one to predict how many independent dimensionless quantities there will be.* In the previous example, the matrix $\mathcal{D}$ has rank 3 (since there are 3 linearly independent columns, as one can see by inspection) and hence the linear system

$$\mathcal{D}\mathbf{P} = \mathbf{0}$$

has $4 - 3 = 1$ independent solution, leading to 1 dimensionless scalar.

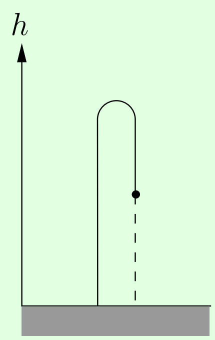
The general situation is this:

> if there are N physical quantities and the dimensional matrix $\mathcal{D}$ has rank r, then the equation $\mathcal{D}\mathbf{P} = \mathbf{0}$ will have $N - r$ linearly independent solutions, which will lead to $N - r$ independent dimensionless scalars.

Here is another example.

Example:

A baseball is thrown vertically upwards. We anticipate that subsequently its height h will depend on time t, the acceleration g due to gravity, its mass m, and initial velocity v_0:



$$h = \mathcal{F}(t, v_0, g, m). \tag{2.37}$$

Find the dimensional matrix and construct a complete set of independent dimensionless quantities.

Solution: List the physical quantities in the order

$$\{h, t, v_0, g, m\}. \tag{2.38}$$

Their dimensions are

$$[h] = L, \quad [t] = T, \quad [v_0] = LT^{-1}, \quad [g] = LT^{-2}, \quad [m] = M.$$

The dimensional matrix $\mathcal{D}$ is thus

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} h & t & v_0 & g & m \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -2 & 0 \end{pmatrix} \end{array}$$

By inspection there are 3 linearly independent columns, and hence the rank of $\mathcal{D}$ is 3. Thus the linear system

$$\mathcal{D}\mathbf{P} = \mathbf{0}, \tag{2.39}$$

where

$$\mathbf{P} = (P_h, P_t, P_v, P_g, P_m),$$

will have $5 - 3 = 2$ linearly independent solutions, implying that we can construct 2 independent dimensionless quantities of the form:

$$\Pi = h^{P_h} t^{P_t} v_0^{P_v} g^{P_g} m^{P_m} \tag{2.40}$$

from the physical quantities (2.38). The linear system (2.39) reads

$$\left.\begin{array}{lllll} & & & P_m & = 0 \\ P_h & & +P_v \quad +P_g & & = 0 \\ & P_t & -P_v \quad -2P_g & & = 0 \end{array}\right\}. \tag{2.41}$$

Regarding P_h and P_g as arbitrary, we can write the solution as

$$P_v = -P_h - P_g, \quad P_t = -P_h + P_g, \quad P_m = 0.$$

Choosing $P_g = 0$, $P_h = 1$ and substituting in (2.40), we get

$$\Pi_1 = h^1 t^{-1} v_0^{-1} g^0 m^0 = \frac{h}{v_0 t}$$

as one dimensionless quantity. A second linearly independent solution of (2.41) is obtained by choosing $P_g = 1$, $P_h = 0$. Substituting in (2.40) gives

$$\Pi_2 = h^0 t^1 v_0^{-1} g^1 m^0 = \frac{gt}{v_0}.$$

Thus, a complete set of independent dimensionless quantities is

$$\Pi_1 = \frac{h}{v_0 t}, \quad \Pi_2 = \frac{gt}{v_0}. \tag{2.42}$$

□

Remark

The above set of dimensionless quantities is not the only possible choice. If we regard P_h and P_t as arbitrary when solving (2.41) we get

$$P_g = P_h + P_t, \quad P_v = -2P_h - P_t, \quad P_m = 0.$$

Choosing $P_h = 1$, $P_t = 0$ and then $P_h = 0$, $P_t = 1$, and using (2.40) gives the two dimensionless quantities:

$$\hat{\Pi}_1 = \frac{gh}{v_0^2}, \quad \hat{\Pi}_2 = \frac{gt}{v_0}, \tag{2.43}$$

which also form a complete independent set.

Summary of the procedure:

1) List the N physical quantities and their dimensions.

2) Set up the dimensional matrix $\mathcal{D}$, and calculate its rank r.

3) There will be $N - r$ independent dimensionless quantities. These can be found either by using the solution procedure as illustrated in the example OR *by inspection i.e. by studying the dimensions of the set of physical quantities and forming obvious combinations*. This is always possible in simple cases and is the *recommended method*. □

Remark

The examples in the preceding pages are worked out "the long way" in order to explain why the method works. When *you* are solving these problems, we recommend solving them using the method in the following subsection, which makes use of an important theorem.

2.2.3 The Buckingham Pi Theorem

Loosely speaking, the *Buckingham Pi Theorem* asserts that

any physical relation involving (N)
physical quantities can be written

in terms of a complete set of $(N-r)$
independent dimensionless quantities,
where r is the rank of the dimensional matrix $\mathcal{D}$.

To see why this theorem is useful it is necessary to consider some examples.

Case 1: $N = 4$, $r = 3$

Suppose we have 4 physical quantities

$$Q_1, Q_2, Q_3, Q_4,$$

and we know that one of them, say Q_1, depends on the others. We write

$$Q_1 = \mathcal{F}(Q_2, Q_3, Q_4), \tag{2.44}$$

which represents the physical relation we wish to investigate.

If the rank of the dimensional matrix $\mathcal{D}$ is $r = 3$, then one can form only $N-r = 4-3 = 1$ dimensionless quantity from the Q's, which we denote by Π. The Buckingham Pi Theorem then implies that the physical relation (2.44) must have the simple form

$$\Pi = C, \tag{2.45}$$

where C is a *numerical constant.* □

Example 1:

Referring to the terminal velocity problem in Section 2.2.1, we write

$$Q_1 = v_{\text{term}}, \quad Q_2 = m, \quad Q_3 = g, \quad Q_4 = \alpha. \tag{2.46}$$

Equation (2.44) has the form

$$v_{\text{term}} = \mathcal{F}(m, g, \alpha). \tag{2.47}$$

As shown in Section 2.2.1 we can form only one dimensionless quantity from the physical quantities (2.46), namely

$$\Pi = \frac{v_{\text{term}}\alpha}{mg}. \tag{2.48}$$

The Buckingham Pi Theorem asserts that (2.47) can be written in the form (2.45) i.e.

$$\frac{v_{\text{term}}\alpha}{mg} = C.$$

Rearranging gives

$$v_{\text{term}} = C\frac{mg}{\alpha}. \tag{2.49}$$

What has been gained? Well, *the theorem has determined the function $\mathcal{F}(m, g, \alpha)$ in* (2.47) *almost completely*, i.e. up to the unknown numerical constant C. This represents a remarkable simplification and provides considerable information. For example, we can conclude that if m and g remain fixed, the terminal velocity varies inversely as the drag coefficient α. □

Case 2: $N = 5$, $r = 3$

We suppose that Q_1 depends on $Q_2, \ldots, Q_5$, and we write

$$Q_1 = \mathcal{F}(Q_2, Q_3, Q_4, Q_5). \tag{2.50}$$

If the rank of the dimensional matrix is $r = 3$, then we can form precisely $N - r = 5 - 3 = 2$ dimensionless quantities from the Q's, which we denote

$$\Pi_1, \Pi_2.$$

The Buckingham Pi Theorem asserts that (2.50) can be written in the form

$$\Pi_1 = f(\Pi_2), \tag{2.51}$$

where f is an unknown function of one variable. □

Here is an example to illustrate the above case.

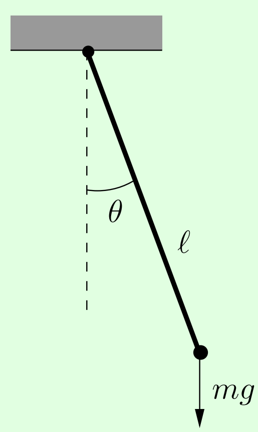
Example 2:

The period of a simple pendulum.

We wish to obtain information about the period P of a simple pendulum, ignoring all frictional effects.

The relevant physical quantities and their dimensions are

the period P,	$[P] = T$
the mass m,	$[m] = M$
the length ℓ,	$[\ell] = L$
the acceleration due to gravity g,	$[g] = LT^{-2}$
the amplitude of swing θ	$[\theta] = 1.$



We assume that

$$P = \mathcal{F}(m, \ell, g, \theta). \tag{2.52}$$

The dimensional matrix is

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} P & m & \ell & g & \theta \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{pmatrix} \end{array}.$$

By inspection, the rank is 3 (the matrix has 3 linearly independent columns). We can thus form precisely *two* independent dimensionless quantities from the physical quantities. By inspection of the list of dimensions, we see that $\frac{P^2 g}{\ell}$ and θ are dimensionless, and so we write

$$\Pi_1 = \frac{P^2 g}{\ell}, \quad \Pi_2 = \theta. \tag{2.53}$$

The Buckingham Pi Theorem asserts that the *physical relation* (2.52) *can be written in the form* (2.51), i.e.

$$\frac{P^2 g}{\ell} = f(\theta),$$

where $f(\theta)$ is an unknown function of one variable. Rearranging gives

$$P^2 = \frac{\ell}{g} f(\theta). \tag{2.54}$$

What has been gained? Well, *we know precisely how the period P depends on ℓ and g, and that it does not depend on m.* What is not determined is the dependence on θ, which is governed by the *unknown function f*. A detailed analysis of the pendulum is needed in order to determine $f(\theta)$. □

Case 3: $N = 6$, $r = 3$

In this case the Buckingham Pi Theorem will restrict a physical relation,

$$P_1 = \mathcal{F}(P_2, \ldots, P_6), \tag{2.55}$$

to have the form

$$\Pi_1 = f(\Pi_2, \Pi_3), \tag{2.56}$$

since there are $N - r = 6 - 3 = 3$ independent dimensionless quantities. Some examples of this case are in Problem Set 2. □

Example 3:

A baseball thrown vertically upwards ($N = 5, r = 3$).
Referring to equation (2.37) we have

$$h = \mathcal{F}(t, v_0, g, m). \tag{2.57}$$

We found two independent dimensionless quantities (see equation (2.42)):

$$\Pi_1 = \frac{h}{v_0 t}, \quad \Pi_2 = \frac{gt}{v_0}.$$

The Buckingham Pi Theorem asserts that (2.57) can be written in the form

$$\frac{h}{v_0 t} = f\left(\frac{gt}{v_0}\right), \tag{2.58}$$

where the function $f(\ )$ is unknown. □

Comments:

1) We in fact know the explicit formula for h, namely

$$h = v_0 t - \tfrac{1}{2} g t^2, \tag{2.59}$$

(see equation (2.5)). By comparing (2.59) and (2.58) we see that the function f in (2.58) is

$$f(z) = 1 - \tfrac{1}{2} z.$$

It is important to note that *this function cannot be obtained from the Buckingham Pi Theorem.*

2) We also gave a different set of dimensionless quantities for this problem (see equation (2.43)):

$$\hat{\Pi}_1 = \frac{gh}{v_0^2}, \quad \hat{\Pi}_2 = \frac{gt}{v_0}. \tag{2.60}$$

Observe that

$$\hat{\Pi}_1 = \Pi_1 \Pi_2, \quad \hat{\Pi}_2 = \Pi_2.$$

The theorem asserts that (2.57) can be written

$$\frac{gh}{v_0^2} = \hat{f}\left(\frac{gt}{v_0}\right), \tag{2.61}$$

where the function $\hat{f}$ is not specified. Comparison of (2.61) and (2.59) shows that

$$\hat{f}(z) = z - \tfrac{1}{2} z^2. \tag{2.62}$$

This example shows that when there is more than one dimensionless quantity, the information obtained from the Buckingham Pi Theorem can be written in more than one way, e.g. (2.58) and (2.61).

3) The dimensionless quantities $\hat{\Pi}_1, \hat{\Pi}_2$ in equation (2.62) in fact arose in the discussion of the baseball problem in Section 2.1.1 (see equations (2.6)-(2.8)). There we defined a dimensionless time τ and height H by

$$\tau = \frac{t}{t_c}, \quad H = \frac{h}{\ell_c},$$

where

$$t_c = \frac{v_0}{g}, \quad \ell_c = \frac{v_0^2}{g}.$$

It follows from equation (2.60) that

$$\hat{\Pi}_1 = H, \quad \hat{\Pi}_2 = \tau,$$

and the relation

$$\hat{\Pi}_1 = \hat{f}(\hat{\Pi}_2)$$

where $\hat{f}$ is given by equation (2.62), becomes

$$H = \tau - \tfrac{1}{2}\tau^2$$

which is equation (2.8). $\square$

Chapter 3

Second Order Linear DEs

3.1 Introduction

3.1.1 Oscillations and Second Order DEs

Oscillations are the single most important physical phenomenon described by second order DEs. As motivation, we consider a simple mechanical system consisting of a spring to which is attached a trolley that can run smoothly on a straight track.

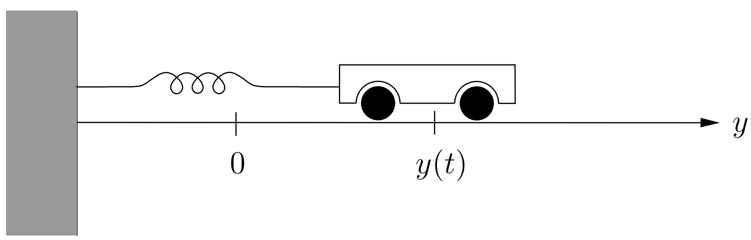


Figure 3.1: A mechanical oscillator.

If the trolley is displaced from its equilibrium position $y = 0$ and released, the spring will cause it to run to and fro on the track. It is this motion that we wish to describe.

Let $y(t)$ be the displacement of the trolley from its equilibrium position at time $t = 0$. When a spring is stretched or compressed it exerts a restoring force which, for small displacements can be assumed to be proportional to the displacement (Hooke's Law). The constant of proportionality is called the *stiffness constant*, and is denoted by k. The force exerted by the spring on the trolley is thus

$$F_{\text{spring}} = -ky. \tag{3.1}$$

We also assume a *damping force* (due to frictional effects, air resistance) that is proportional to the velocity, and acts so as to slow the motion:

$$F_{\text{damp}} = -c\frac{dy}{dt}, \tag{3.2}$$

where $c > 0$ is the *damping constant*.

Remark

$\frac{dy}{dt} > 0$, the trolley is moving to the right and the force acts to the left, while if $\frac{dy}{dt} < 0$ the trolley is moving to the left and the force acts to the right, in both cases opposing the motion, as required.

The motion of the trolley is governed by Newton's Second Law. Since its mass m is constant, we have

$$m\frac{d^2y}{dt^2} = F_{\text{spring}} + F_{\text{damp}}.$$

On account of (3.1) and (3.2) this equation gives

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0. \tag{3.3}$$

We shall show how to solve this DE (see Section 3.2.3) and shall find that if the damping constant c is non-zero, the solutions show two types of behaviour as $t \to \infty$:

i) exponential monotone decay to zero, or

ii) exponential oscillatory decay to zero.

We shall also consider the situation in which an external force $F(t)$ acts on the trolley, in which case (3.3) is replaced by

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t). \tag{3.4}$$

Of particular interest is the case in which $F(t)$ is *periodic*, which can lead to a variety of interesting behaviour (see Section 3.3).

Remark

The DEs (3.3) and (3.4) describe many different physical systems e.g. a pendulum with small amplitude, a buoy bobbing up and down in the ocean, electrical circuits, two connected mixing tanks, etc.

3.1.2 The world's simplest second order DE

On setting the damping constant c to zero in the DE (3.3) in Section 3.1.1, we obtain

$$m\frac{d^2y}{dt^2} + ky = 0.$$

On defining $\omega^2 = k/m$, this becomes

$$\frac{d^2y}{dt^2} + \omega^2 y = 0. \tag{3.5}$$

By dimensional homogeneity, the dimension of ω is $(\text{time})^{-1}$, i.e. $[\omega] = T^{-1}$, and so we can define a dimensionless time τ by

$$\tau = \omega t. \tag{3.6}$$

It follows from (3.6) and the Chain Rule that

$$\frac{dy}{dt} = \omega\frac{dy}{d\tau}, \quad \frac{d^2y}{dt^2} = \omega^2\frac{d^2y}{d\tau^2}.$$

The DE (3.5) thus becomes

$$\frac{d^2y}{d\tau^2} + y = 0, \tag{3.7}$$

which we'll consider *the world's simplest and most important second order DE.*[1] The solutions of this DE can be found by inspection, by recalling the differentiation property of sin and cos:

$$\frac{d^2}{d\tau^2}(\sin\tau) = -\sin\tau, \quad \frac{d^2}{d\tau^2}(\cos\tau) = -\cos\tau.$$

Thus $y_1 = \cos\tau$ and $y_2 = \sin\tau$ are solutions of (3.7). It is easy to verify that

$$y = c_1\cos\tau + c_2\sin\tau, \tag{3.8}$$

for any constants c_1 and c_2, is also a solution of (3.7). We shall see later (Section 3.2.1) that (3.8) is in fact the general solution of the DE (3.7). By (3.6) it follows that

$$y = c_1\cos(\omega t) + c_2\sin(\omega t)$$

is the general solution of (3.5). We summarize this most important result:

> the general solution of the DE
>
> $$\frac{d^2y}{dt^2} + \omega^2 y = 0$$
>
> is
>
> $$y = c_1\cos(\omega t) + c_2\sin(\omega t),$$
>
> where c_1 and c_2 are arbitrary constants. □

Remark

Since this solution is periodic of period $\frac{2\pi}{\omega}$, the solution describes the trolley moving to and fro with constant period. This type of motion is referred to as *simple harmonic motion.*

[1] One could argue that $y'' = 0$ is simpler. Integrating twice yields $y = c_1\tau + c_2$.

3.1.3 Types of Second Order DE

The *general form* of a second order DE is

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right). \tag{3.9}$$

The simplest class are the *linear DEs*, which are characterized by $f\left(x, y, \frac{dy}{dx}\right)$ being linear in y and $\frac{dy}{dx}$, i.e.

$$f\left(x, y, \frac{dy}{dx}\right) = F(x) + A(x)y + B(x)\frac{dy}{dx}.$$

A linear DE is usually written with the y and $\frac{dy}{dx}$ terms on the left side. Replacing $B(x)$ by $-P(x)$ and $A(x)$ by $-Q(x)$, the DE (3.9) becomes

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x),$$

or more concisely,

$$y'' + P(x)y' + Q(x)y = F(x). \tag{3.10}$$

This is the general form of a *non-homogeneous linear second order DE*.

If $F(x) = 0$, the DE is said to be homogeneous:

$$y'' + P(x)y' + Q(x)y = 0. \tag{3.11}$$

This is the general form of a *homogeneous linear second order DE*.

Finally, if $P(x) = p$ and $Q(x) = q$, where p and q are constants, (3.10) and (3.11) become

$$y'' + py' + qy = F(x), \tag{3.12}$$

and

$$y'' + py' + qy = 0. \tag{3.13}$$

This is the general form of a second order linear DE with *constant coefficients*, (3.12) being non-homogeneous and (3.13) being homogeneous. In this course *we shall be concerned almost exclusively with this type of second order DE*.

Note that the DE (3.4), namely

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t),$$

is of the form (3.12) – just divide by m to put this DE in standard form. Finally the world's simplest second order DE, namely

$$\frac{d^2y}{dt^2} + \omega^2 y = 0,$$

is of the form (3.13) with $p = 0$ and $q = \omega^2$.

3.1.4 The Initial Value Problem for Second Order DEs

Recall the spring and trolley problem from Section 3.1.1, described by the DE

$$my'' + cy' + ky = 0, \tag{3.14}$$

where $y(t)$ is the displacement of the trolley at time t, from the equilibrium position and $'$ denotes $\frac{d}{dt}$.

What are the different ways of setting the trolley in motion? The simplest way is to pull the trolley from its equilibrium position, hold it at rest, and then release it at time $t = 0$. This procedure corresponds to the initial conditions

$$y(0) = y_0, \quad y'(0) = 0, \tag{3.15}$$

i.e. the initial velocity is zero. Another possibility is to give the trolley a tap with a hammer while it is at rest in its equilibrium position. The impact will set the trolley in motion, and the initial condition will be

$$y(0) = 0, \quad y'(0) = v_0, \tag{3.16}$$

i.e. the initial velocity is non-zero. The most general possibility is to pull the trolley from equilibrium and give it a little jerk as you release it, thereby imparting an initial displacement and initial velocity to it. The initial conditions will be

$$y(0) = y_0, \quad y'(0) = v_0. \tag{3.17}$$

With initial conditions such as (3.15)-(3.17) we certainly expect the physical system to move in a uniquely determined way, and hence we expect the DE (3.14) to have a unique solution.

This simple example illustrates what holds in general for a linear second order DE:

$$y'' + P(x)y' + Q(x)y = F(x), \tag{3.18}$$

namely, that the appropriate initial conditions are

$$y(x_0) = y_0, \quad y'(x_0) = v_0, \tag{3.19}$$

and that these initial conditions determine a unique solution of the DE (3.18) (subject to appropriate restrictions on P, Q and F). This existence-uniqueness result will be discussed in AMath 351. □

3.2 Solving Second Order Linear DEs with constant coefficients

3.2.1 A fundamental property of homogeneous second order linear DEs

Consider the homogeneous DE

$$y'' + P(x)y' + Q(x)y = 0. \tag{3.20}$$

The fact that this DE is linear AND homogeneous has an immediate and important consequence.

Proposition

If $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous linear DE (3.20), then

$$c_1 y_1(x) + c_2 y_2(x)$$

is also a solution, for any constants c_1 and c_2.

Proof: Since y_1 and y_2 are solutions of (3.20),

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \tag{3.21}$$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0. \tag{3.22}$$

Multiply (3.21) by c_1 and (3.22) by c_2 and add, to get

$$(c_1 y_1 + c_2 y_2)'' + P(x)(c_1 y_1 + c_2 y_2)' + Q(x)(c_1 y_1 + c_2 y_2) = 0,$$

as required. □

The *fundamental property* referred to in the title is this:

> Any solution of the homogeneous linear DE (3.20) has the form
>
> $$y = c_1 y_1(x) + c_2 y_2(x), \tag{3.23}$$
>
> where $y_1(x)$ and $y_2(x)$ are *linearly independent*[2] solutions of (3.20), and c_1, c_2 are constants.

Example

Consider the world's simplest second order DE

$$y'' + \omega^2 y = 0. \tag{3.24}$$

Two solutions are

$$y_1 = \sin \omega t, \qquad y_2 = \cos \omega t,$$

and these are linearly independent, since cos( ) is not a multiple of sin( ). The general solution of (3.24) is thus

$$y = c_1 \sin \omega t + c_2 \cos \omega t, \tag{3.25}$$

where c_1 and c_2 are constants. □

The fundamental property is a consequence of the existence and uniqueness theorem for second order linear DEs (AMath 351). The general solution (3.23) contains TWO arbitrary constants because the initial conditions are TWO in number:

$$y(x_0) = y_0, \qquad y'(x_0) = v_0.$$

[2] $y_1(x)$ and $y_2(x)$ are linearly independent means that no linear combination of $y_1(x)$ and $y_2(x)$ equals zero, for all x.

When one solves an initial value problem, c_1 and c_2 are determined in terms of y_0 and v_0.

Exercise

Show that the unique solution of the initial value problem

$$y'' + \omega^2 y = 0; \quad y(0) = y_0, \quad y'(0) = v_0$$

is

$$y = \frac{v_0}{\omega} \sin \omega t + y_0 \cos \omega t. \qquad \square$$

3.2.2 General form of the solution

Consider a non-homogeneous second order linear DE

$$y'' + P(x)y' + Q(x)y = F(x). \tag{3.26}$$

The linearity of this DE has an important consequence.

Proposition

If y_1 and y_2 are solutions of the non-homogeneous DE (3.26), then the difference

$$y_1(x) - y_2(x)$$

is a solution of the associated homogeneous DE

$$y'' + P(x)y' + Q(x)y = 0. \tag{3.27}$$

Proof: Same as the proposition on page 18 for first order DEs. $\square$

Suppose $y_p(x)$ is a particular solution of (3.26) and $y(x)$ is *any* solution of (3.26), i.e. it represents the general solution. Then by the Proposition,

$$y_h(x) = y(x) - y_p(x)$$

is a solution of the homogeneous DE (3.27). We can rewrite this equation as

$$y(x) = y_h(x) + y_p(x),$$

giving the general solution. $\square$

General solution of a second order linear DE:

The general solution of the DE (3.26) is of the form

$$y(x) = y_h(x) + y_p(x),$$

where $y_h(x)$ is the general solution of the homogeneous DE (3.27), and $y_p(x)$ is a particular solution of the non-homogeneous DE (3.26). (compare with page 18).

It follows from Section 3.2.1 that

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

where y_1 and y_2 are two linearly independent solutions of the homogeneous DE (3.27).

We thus need to develop two algorithms:

(1) An algorithm to give the *general solution* of a *homogeneous* linear DE. This is possible only for the case of *constant coefficients*

$$y'' + py' + qy = 0,$$

where p and q are constants. We develop this algorithm in Section 3.2.3.

(2) An algorithm to find a *particular solution* of a *non-homogeneous* linear DE

$$y'' + py' + qy = f(x).$$

Here we simply extend the method of undetermined coefficients that we used in the first order case (see Section 1.2.5). This extension is discussed in Section 3.2.4. □

3.2.3 General Solution of the Homogeneous DE

A homogeneous second order linear DE *with constant coefficients* is of the form

$$y'' + py' + qy = 0, \tag{3.28}$$

where p and q are constants. Our goal is to find the *general solution* of any such DE.

We begin by considering a trial function of the form

$$y = e^{mx}, \tag{3.29}$$

where m is a constant. Since $y' = me^{mx}$ and $y'' = m^2 e^{mx}$, equation (3.28) yields

$$(m^2 + pm + q)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, (3.29) is a solution of (3.28) if and only if m is a solution of

$$m^2 + pm + q = 0. \tag{3.30}$$

This quadratic equation is called *the characteristic equation of the DE* (3.28). Its roots are

$$m_{1,2} = \tfrac{1}{2}\left[-p \pm \sqrt{p^2 - 4q}\right]. \tag{3.31}$$

There are three distinct cases, each of which has to be treated separately:

A) Distinct real roots ($p^2 > 4q$)

B) Distinct complex roots ($p^2 < 4q$)

C) Equal real roots ($p^2 = 4q$).

Case A: Distinct real roots

In this case, substituting the two roots (3.31) into (3.29), we get two solutions

$$e^{m_1x} \quad \text{and} \quad e^{m_2x}.$$

Since the ratio $\dfrac{e^{m_1x}}{e^{m_2x}} = e^{(m_1-m_1)x}$ is not constant, these solutions are linearly independent. By Section 3.2.1, the *general solution* of the DE (3.28) is thus

$$y = c_1e^{m_1x} + c_2e^{m_2x}, \tag{3.32}$$

where c_1 and c_2 are arbitrary constants and m_1, m_2 are the real roots of the characteristic equation (3.30). □

Example 1:

Find the general solution of the DE

$$y'' + 5y' + 6y = 0. \tag{3.33}$$

Solution: Substituting the trial function $y = e^{mx}$ in (3.33) gives the characteristic equation

$$m^2 + 5m + 6 = 0,$$

which factors as

$$(m + 2)(m + 3) = 0.$$

There two distinct real roots $m = -2$ and $m = -3$, give two independent solutions

$$e^{-2x} \quad \text{and} \quad e^{-3x}.$$

The general solution is thus

$$y = c_1e^{-2x} + c_2e^{-3x}. \qquad \square$$

Case B: Distinct complex roots

The roots m_1 and m_2, as given by (3.31), are distinct and complex if and only if

$$p^2 - 4q < 0.$$

In this case it is convenient to write m_1 and m_2 in the form

$$m_1 = a + ib, \quad m_2 = a - ib,$$

where a and b are real. We substitute m_1 and m_2 into (3.29) to obtain two complex solutions which we can decompose into real and imaginary parts using Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The two solutions of the DE (3.28) are

$$e^{m_1 x} = e^{(a+ib)x} = e^{ax} e^{ibx} = e^{ax}(\cos bx + i \sin bx) \tag{3.34}$$

and

$$e^{m_2 x} = e^{(a-ib)x} = e^{ax} e^{-ibx} = e^{ax}(\cos bx - i \sin bx). \tag{3.35}$$

Since we want real solutions we consider the linear combinations

$$\begin{aligned} \tfrac{1}{2}(e^{m_1 x} + e^{m_2 x}) &= e^{ax} \cos bx \\ \tfrac{1}{2i}(e^{m_1 x} - e^{m_2 x}) &= e^{ax} \sin bx, \end{aligned}$$

as follows from (3.34) and (3.35). These solutions are linearly independent. Thus, *the general solution of the DE* (3.28) is

$$y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx, \tag{3.36}$$

where c_1 and c_2 are arbitrary constants, and

$$a \pm ib$$

are the roots of the characteristic equation. □

Example 2:

Find the general solution of the DE

$$y'' + 2y' + 5y = 0. \tag{3.37}$$

Solution: Substituting the trial function $y = e^{mx}$ into (3.37) gives

$$m^2 + 2m + 5 = 0.$$

Completing the square leads to

$$(m + 1)^2 + 2^2 = 0,$$

giving the complex roots

$$m = -1 \pm 2i.$$

A complex solution is

$$e^{(-1+2i)x} = e^{-x}(\cos 2x + i \sin 2x),$$

(by Euler's formula), from which one can read off the two independent real solutions

$$e^{-x} \cos 2x \quad \text{and} \quad e^{-x} \sin 2x.$$

Thus, the general solution of the DE (3.37) is

$$y = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x. \qquad \square$$

Case C: Equal real roots

The roots m_1 and m_2, as given by (3.31), are real and equal if and only if

$$p^2 - 4q = 0.$$

In this case, we obtain only one solution from (3.29) and (3.31), namely

$$y_1 = e^{mx}, \quad m = -\frac{1}{2}p$$

To find a second linearly independent solution we consider a trial function of the form

$$y = v(x)e^{mx}. \tag{3.38}$$

We substitute (3.38) into the DE (3.28) to obtain

$$v'' + (2m + p)v' + (m^2 + pm + q)v = 0, \tag{3.39}$$

after collecting terms and cancelling a factor of e^{mx} (fill in the details). Now a miracle happens ... since $m = -\frac{1}{2}p$ and m is a solution of the characteristic equation $m^2+pm+q = 0$, equation (3.39) reduces to

$$v'' = 0.$$

Choose $v(x) = x$ as a particular solution, and then (3.38) gives

$$y_2 = xe^{mx} \tag{3.40}$$

as a second linearly independent solution. Thus, *the general solution of the DE* (3.28) is

$$y = c_1e^{mx} + c_2xe^{mx},$$

where c_1 and c_2 are arbitrary constants, and m is the single solution of the characteristic equation. □

Example 3:

Find the general solution of the DE

$$y'' + 6y' + 9y = 0. \tag{3.41}$$

Solution: Substituting the trial function $y = e^{mx}$ in (3.41) gives

$$m^2 + 6m + 9 = 0,$$

which is a perfect square,

$$(m + 3)^2 = 0$$

giving a single root $m = -3$, and one solution $y_1 = e^{-3x}$. A second solution is obtained from (3.40), which gives

$$y_2 = xe^{-3x}.$$

(There is no need to repeat the whole derivation – the second solution is simply x times the first one.) Thus, the general solution of (3.41) is

$$y = c_1 e^{-3x} + c_2 x e^{-3x}. \qquad \square$$

3.2.4 The method of undetermined coefficients

We now show how to find a *particular solution* of the *non-homogeneous* linear DE with constant coefficients, whose general form is

$$y'' + py' + qy = f(x),$$

where p and q are constants, and f is a given function.

We use the method of undetermined coefficients, as introduced in Section 1.2.5, for first order DEs. This means that we restrict our considerations to the case where f is

- an exponential e^{bx}
- a sine or cosine
- a polynomial

or

- sums of such functions.

Example 1:

Find the general solution of the DE

$$y'' + 5y' + 6y = e^{2x}. \tag{3.42}$$

Solution: We consider a trial function

$$y = Ae^{2x}, \tag{3.43}$$

where A is a constant. Since $y' = 2Ae^{2x}$ and $y'' = 4Ae^{2x}$, substituting (3.43) in (3.42) yields

$$[4A + 5(2A) + 6(A)]e^{2x} = e^{2x}.$$

The e^{2x} cancels, and solving for A gives $A = \frac{1}{20}$. Thus (3.43) gives the particular solution

$$y_p = \tfrac{1}{20}e^{2x}. \tag{3.44}$$

The general solution of (3.42) has the form

$$y = y_h(x) + y_p(x), \tag{3.45}$$

where y_h is the general solution of the homogeneous DE $y'' + 5y' + 6y = 0$. We solved this problem in Example 1 of Section 3.2.3:

$$y_h = c_1 e^{-2x} + c_2 e^{-3x}. \tag{3.46}$$

By (3.44), (3.45) and (3.46), the general solution of the DE (3.42) is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \tfrac{1}{20} e^{2x}. \qquad \square$$

It is of interest to consider the previous example with an exponential driving term, in greater generality. Consider

$$y'' + py' + qy = e^{ax} \tag{3.47}$$

where p, q and a are constants. As in the previous example, we consider a trial function

$$y = Ae^{ax}. \tag{3.48}$$

Substituting (3.48) in (3.47) yields

$$A(a^2 + pa + q)e^{ax} = e^{ax}$$

(fill in the details). Thus

$$A = \frac{1}{a^2 + pa + q},$$

giving a particular solution (3.48). However, it is clear that A is undefined for certain values of the constant a, namely, those values which satisfy

$$a^2 + pa + q = 0. \tag{3.49}$$

This is precisely the characteristic equation of the homogeneous DE associated with (3.47). Thus, *the trial function* (3.48) *does not give a solution if a is a root of the characteristic equation.*

How do we find a particular solution in this special case? Based on our experience in case C in the previous section when we had to multiply by x to find a second solution, we consider the trial function

$$y = Axe^{ax}. \tag{3.50}$$

On substituting (3.50) in (3.47) and simplifying, we get

$$A(a^2 + pa + q)xe^{ax} + A(2a + p)e^{ax} = e^{ax}.$$

The first term in brackets is zero, because we are assuming that a satisfies (3.49). Thus

$$A = \frac{1}{2a + p}.$$

Using (3.50), this value of A gives a particular solution unless

$$a = -\tfrac{1}{2}p. \tag{3.51}$$

What is the significance of this condition? Well, if the characteristic equation (3.49) has a double real root, i.e. if $p^2 = 4q$, then (3.49) becomes

$$\left(a + \tfrac{1}{2}p\right)^2 = 0.$$

Thus, (3.51) states that *a is a double root of the characteristic equation.* In this case we generalize (3.50) and use

$$y = Ax^2e^{ax} \tag{3.52}$$

as a trial function. On substituting (3.52) in (3.47) we get

$$A(a^2 + pa + q)x^2e^{ax} + 2A(2a + p)xe^{ax} + 2Ae^{ax} = e^{ax}.$$

Since we are assuming a is a double root of the characteristic equation, equations (3.49) and (3.51) hold, and hence the terms in brackets are zero, leaving

$$A = \tfrac{1}{2}.$$

Thus, in this case equation (3.52) gives a particular solution of the DE (3.47).

Summary:

Consider the DE

$$y'' + py' + qy = e^{ax}, \tag{3.53}$$

with characteristic equation

$$a^2 + pa + q = 0. \tag{3.54}$$

- If a is not a root of (3.54), (3.53) has a particular solution of the form $y = Ae^{ax}$.
- If a is a single root of (3.54), (3.53) has a particular solution of the form $y = Axe^{ax}$.
- If a is a double root of (3.54), (3.53) has a particular solution of the form $y = Ax^2e^{ax}$. □

We next consider the case of a *polynomial driving term*:

$$y'' + py' + qy = a_0 + a_1x + \cdots + a_nx^n. \tag{3.55}$$

Since the derivative of a polynomial is a polynomial, we consider a trial function of the form

$$y_p = A_0 + A_1x + \cdots + A_nx^n, \tag{3.56}$$

where $A_0, A_1, \ldots, A_n$ are the undetermined coefficients. This choice works unless $q = 0$ in (3.55), in which case one multiplies (3.56) by x. □

Exercise 1:

Find a particular solution of

$$y'' + 5y' + 6y = 6x.$$

Answer: $y_p = -\frac{5}{6} + x$

Exercise 2:

Find a particular solution of

$$y'' + y' = 2x.$$

Answer: $y_p = -2x + x^2$. $\square$

We finally consider a *sine or cosine driving term*:

$$y'' + py' + qy = \sin \omega x. \tag{3.57}$$

Since the derivatives of $\sin \omega x$ are multiples of $\sin \omega x$ and $\cos \omega x$, we take a trial function of the form

$$y_p = A \sin \omega x + B \cos \omega x. \tag{3.58}$$

This choice works unless $p = 0$ and $\omega^2 = q$. This special case is

$$y'' + \omega^2 y = \sin \omega x. \tag{3.59}$$

To get a suitable trial function we multiply (3.58) by x:

$$y_p = x(A \sin \omega x + B \cos \omega x).$$

Exercise 3:

Find a particular solution of

$$y'' - 3y' - 4y = 2 \sin x.$$

Answer: $y_p = -\frac{5}{17} \sin x + \frac{3}{17} \cos x$.

Exercise 4:

Find a particular solution of

$$y'' + 4y = \sin 2x.$$

Answer: $y_p = -\frac{1}{4} x \cos 2x$. $\square$

If the *driving term is a sum* one can use the following proposition to find a particular solution.

Proposition

If y_1 is a solution of $y'' + py' + qy = f_1(x)$,
and y_2 is a solution of $y'' + py' + qy = f_2(x)$,
then $y = y_1 + y_2$ is a solution of

$$y'' + py' + qy = f_1(x) + f_2(x).$$

Proof: Exercise (a consequence of linearity...). □

3.3 Analysis of the oscillator DE

3.3.1 Introductory remarks

In this section we describe the properties and physical interpretation of the solutions of the DE that describes the mass-spring system with damping and periodic driving force:

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{3.60}$$

We divide by m and write the DE in the form

$$y'' + 2\lambda y' + \omega_0^2 y = f_0 \cos \omega t, \tag{3.61}$$

where

$$\lambda = \frac{c}{2m}, \quad \omega_0^2 = \frac{k}{m}, \quad f_0 = \frac{F_0}{m}. \tag{3.62}$$

Since the DE (3.61) describes a variety of oscillating systems, both mechanical and electrical, we shall refer to it as *the oscillator DE.*

The parameter λ and ω_0 characterize the system itself (while f_0 and ω characterize the driving force), and are thus called *the system parameters.* λ is called the *damping parameter* and ω_0 is called the *natural frequency* (the latter name will be justified in Section 3.3.2). Both system parameters have dimensions T^{-1}, as follows from (3.61) and can thus be used to define two characteristic times $\frac{1}{\omega_0}$ and $\frac{1}{\lambda}$. We shall see that the ratio $\frac{\lambda}{\omega_0}$ of these two parameters, which is dimensionless, plays an important role in determining the behaviour of the solutions of the DE.

The DE (3.61) also describes the current in an RLC electrical circuit. In Section 1.3.6 we saw that the current in such a circuit satisfies the DE

$$LI'' + RI' + \frac{1}{C}I = V'(t),$$

where $V(t)$ is the applied voltage. With

$$V(t) = V_0 \sin(\omega t)$$

this DE becomes

$$LI'' + RI' + \frac{1}{C}I = V_0\omega \cos \omega t. \tag{3.63}$$

Dividing by L gives

$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{V_0\omega}{L}\cos\omega t,$$

which is of the form (3.61) with

$$\lambda = \frac{R}{2L}, \quad \omega_0^2 = \frac{1}{LC}, \quad f_0 = \frac{V_0\omega}{L}. \tag{3.64}$$

3.3.2 Zero driving force

In this section we discuss the oscillator DE (3.61) with zero force, i.e.

$$y'' + 2\lambda y' + \omega_0^2 y = 0. \tag{3.65}$$

There are four qualitatively different cases, depending on the *dimensionless ratio*

$$\zeta = \frac{\lambda}{\omega_0}.$$

A) The undamped case: $\zeta = 0$

B) The underdamped case: $0 < \zeta < 1$

C) The critically damped case: $\zeta = 1$

D) The overdamped case: $\zeta > 1$

These cases correspond to the different possibilities for the roots of the characteristic equation of the DE (3.65).

Case A: The undamped case ($\zeta = 0$)

The DE (3.65) specializes to

$$y'' + \omega_0^2 y = 0,$$

the "world's simplest". By inspection the general solution is (see Section 3.1.2)

$$y = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t). \tag{3.66}$$

In order to interpret the solution it is necessary to use the trig identity

$$\cos(A - B) = \cos A\cos B + \sin A\sin B \tag{3.67}$$

to rewrite (3.66) as single cosine.

Think of the constants c_1, c_2 in (3.66) as defining a point (c_1, c_2) on a circle of radius $R = \sqrt{c_1^2 + c_2^2}$, and let ϕ be the radian measure of the angle defined by (c_1, c_2) (see Figure 3.2).

It follows that

$$c_1 = R\cos\phi, \qquad c_2 = R\sin\phi. \tag{3.68}$$

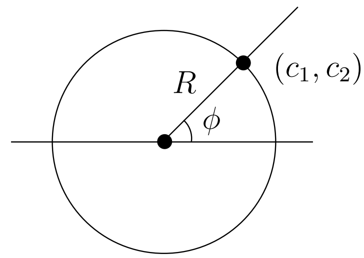


Figure 3.2: Definition of the phase angle ϕ.

The solution (3.66) becomes

$$y = R[\cos(\omega_0 t)\cos\phi + \sin(\omega_0 t)\sin\phi],$$

which, using (3.67), can be written as

$$y = R\cos(\omega_0 t - \phi). \tag{3.69}$$

Thus, the system (e.g. the trolley attached to the spring) oscillates with *period*

$$P = \frac{2\pi}{\omega_0}$$

(recall that $\cos t$ is periodic of period 2π and hence $\cos(\omega_0 t - \phi)$ is periodic of period $\frac{2\pi}{\omega_0}$). We say that the system undergoes *simple harmonic motion* (SHM). The constant R in the solution (3.69), which represents the maximum displacement of the system, is called the *amplitude* of the SHM. The *frequency* of the SHM is

$$\nu = \frac{1}{P} = \frac{\omega_0}{2\pi},$$

but we often just refer to ω_0 as the frequency. ($\frac{\omega_0}{2\pi}$ is a cyclic frequency measured in oscillations per second, while ω_0 is an angular frequency, measured in radians per second. It is usually more convenient to use the latter.) The dimensionless constant ϕ is called the *phase shift*.

It should be kept in mind that the frequency ω_0, which is one of the system parameters, is an intrinsic property of the system. Since the initial conditions

$$y(0) = y_0, \qquad y'(0) = v_0 \tag{3.70}$$

enter into the solution (3.69) through the constants c_1 and c_2, *the frequency ω_0 is independent of the initial conditions.*

On the other hand, the amplitude R and phase ϕ do depend on the initial conditions. It follows from equations (3.69) and (3.70) that

$$R = \sqrt{y_0^2 + \frac{v_0^2}{\omega_0^2}}, \tag{3.71}$$

and that ϕ is determined by

$$\cos\phi = \frac{y_0}{R}, \qquad \sin\phi = \frac{v_0}{\omega_0 R}, \tag{3.72}$$

(exercise on Problem Set 3).

The special case of "release from rest", i.e. $y_0 \neq 0$, $v_0 = 0$, gives

$$R = \mid y_0 \mid \quad \text{and} \quad \phi = \begin{cases} 0, & \text{if} \quad y_0 > 0 \\ \pi, & \text{if} \quad y_0 < 0 \end{cases}.$$

The other special case of "a kick while in equilibrium", i.e. $y_0 = 0$, $v_0 \neq 0$, gives

$$R = \frac{\mid v_0 \mid}{\omega_0} \quad \text{and} \quad \phi = \begin{cases} \frac{\pi}{2}, & \text{if} \quad v_0 > 0 \\ \frac{3\pi}{2}, & \text{if} \quad v_0 < 0 \end{cases}.$$

Case B: The underdamped case ($0 < \zeta < 1$)

The characteristic equation for the DE (3.65) is

$$m^2 + 2\lambda m + \omega_0^2 = 0.$$

The roots are

$$m = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2},$$

which we rewrite as

$$m = \omega_0 \left[-\zeta \pm i\sqrt{1-\zeta^2} \right]$$

using $\zeta = \frac{\lambda}{\omega_0}$. The general solution of the DE is thus

$$y = e^{-\zeta\omega_0 t} \left[c_1 \cos\left(\sqrt{1-\zeta^2}\omega_0 t \right) + c_2 \sin\left(\sqrt{1-\zeta^2}\omega_0 t \right) \right],$$

(as in Section 3.2.3). As in Case A, we can write the expression in square brackets as a single cosine, giving

$$y = Re^{-\zeta\omega_0 t} \cos\left[\sqrt{1-\zeta^2}\omega_0 t - \phi \right]. \tag{3.73}$$

The constants R and ϕ are related to c_1 and c_2 by equation (3.68) and are thus determined by the initial conditions via equations analogous to, but more complicated than, equations (3.71) and (3.72). We do not need the specific form of these equations.

Equation (3.73) gives the general solution of the oscillator DE (3.65) in the underdamped case. The function $y(t)$ satisfies

$$-Re^{-\zeta\omega_0 t} \leq y \leq Re^{-\zeta\omega_0 t}.$$

The graph of $y(t)$ thus oscillates between these two exponential curves. The zeros are equally spaced, with the spacing given by

$$\frac{\pi}{\omega_0\sqrt{1-\zeta^2}}.$$

which is half the period of the cosine.

Of course $y(t)$ itself is *not* periodic. We say that $y(t)$ describes a system that is performing *damped oscillations*. The graph of $y(t)$ is shown below.

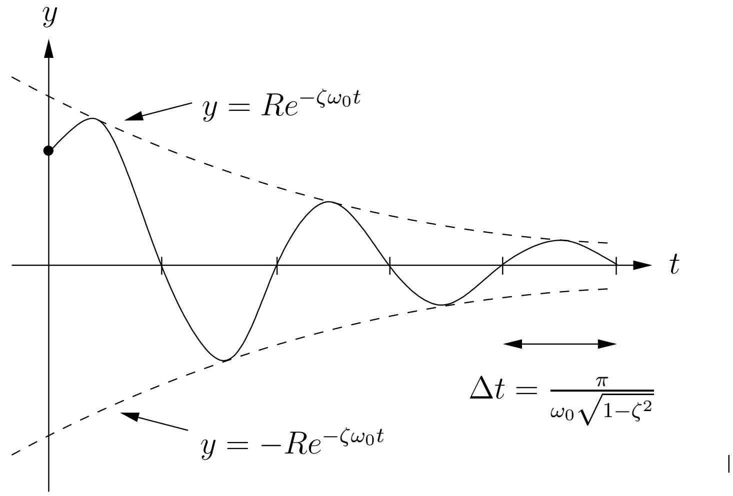


Figure 3.3: Graph of $y = Re^{-\zeta\omega_0 t}\cos\left[\sqrt{1-\zeta^2}\,\omega_0 t - \phi\right]$.

Remark

Damped oscillations can approximate SHM under the following conditions. If the dimensionless constant

$$\zeta = \frac{\lambda}{\omega_0}$$

is very small, the system will perform a significant number of oscillations before the damping has had sufficient time to decrease the amplitude of the oscillations appreciably. This behaviour can be explained heuristically in terms of the characteristic times associated with the damping and with the oscillations, namely

$$t_{\text{damp}} = \frac{1}{\lambda}, \qquad t_{\text{osc}} = \frac{1}{\omega_0}. \tag{3.74}$$

The restriction $\zeta \ll 1$ is equivalent to

$$t_{\text{osc}} \ll t_{\text{damp}}.$$

For example, let $\Delta t_{1\%}$ be the time for the amplitude factor $e^{-\lambda t}$ to decay by 1%. Show that if $\zeta = 10^{-5}$, there will be approximately 160 oscillations during this time interval.

Case C: The critically damped case ($\zeta = 1$)

The characteristic equation for the DE (3.65) is

$$m^2 + 2\omega_0 m + \omega_0^2 = 0,$$

with a single repeated root $m = -\omega_0$. As in Section 3.2.3, the general solution is

$$y = (c_1 + c_2 t)e^{-\omega_0 t}.$$

It follows (Exercise) that the unique solution which satisfies the initial conditions $y(0) = y_0$ and $y'(0) = v_0$, is

$$y = [y_0 + (v_0 + \omega_0 y_0)t]\, e^{-\omega_0 t}. \tag{3.75}$$

For all initial conditions, $\lim_{t\to\infty} y = 0$, i.e. the system returns to a state of equilibrium. The shape of the graph of $y(t)$ (i.e. the qualitative behaviour) depends on y_0 and v_0. Figure 3.4 shows the different possibilities for $y_0 > 0$.

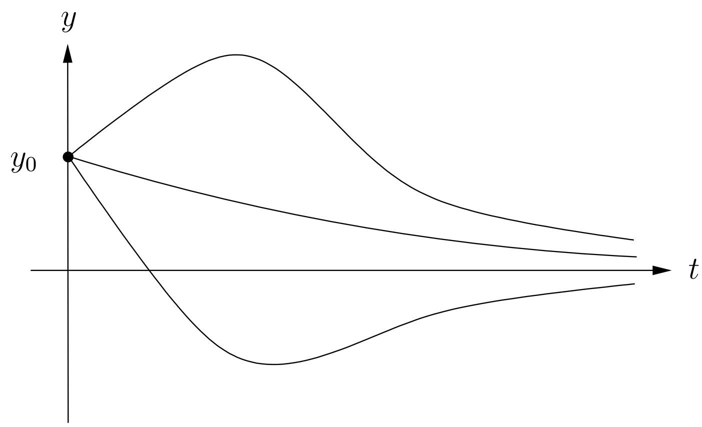


Figure 3.4: Displacement of a critically damped oscillator.

Case D: The overdamped case ($\zeta > 1$)

The characteristic equation for the DE (3.65) is

$$m^2 + 2\lambda m + \omega_0^2 = 0,$$

with distinct real roots

$$m_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}.$$

In terms of $\zeta = \lambda/\omega_0$,

$$m_{1,2} = \omega_0 \left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]. \tag{3.76}$$

The general solution of the DE (3.65) is

$$y = e^{-\zeta\omega_0 t}\left[c_1 e^{\sqrt{\zeta^2-1}\,\omega_0 t} + c_2 e^{-\sqrt{\zeta^2-1}\,\omega_0 t}\right]. \tag{3.77}$$

The constants c_1 and c_2 are determined by the initial conditions. The expressions are a bit complicated, and are not important. The essential point is that for all initial conditions, i.e. for all c_1 and c_2,

$$\lim_{t\to+\infty} y = 0,$$

i.e. the system eventually returns to a state of equilibrium.

Summary:

We have analyzed the solutions of the DE

$$y'' + 2\lambda y' + \omega_0^2 y = 0, \tag{3.78}$$

which governs the displacement of a mechanical oscillator and also the current in an electrical circuit. The first essential result is a consequence of the form of the general solutions (3.73), (3.75) and (3.77).

Proposition

If $\lambda > 0$ and $\omega_0 > 0$ then all solutions of the DE

$$y'' + 2\lambda y' + \omega_0^2 y = 0$$

satisfy

$$\lim_{t\to+\infty} y(t) = 0. \qquad \square$$

The interpretation is as follows. The DE (3.78) admits the *equilibrium solution* $y(t) = 0$ for all t, corresponding to the physical system being in a state of equilibrium. The proposition implies that if the system starts at time $t = 0$ with the initial conditions $y(0) = y_0$ and $y'(0) = v_0$, then *it will eventually return arbitrarily closely to the equilibrium state* $y = 0$, *no matter what the initial conditions are.* The physical cause of this behaviour is the *damping*, either the mechanical damping or the electrical damping (the resistor R in the electrical circuit).

The second essential result is this: *the system undergoes (damped) oscillations while returning to equilibrium if an only if the dimensionless system parameter* $\zeta = \frac{\lambda}{\omega_0}$ *satisfies* $0 < \zeta < 1$.

A final question of practical concern arises from the Proposition, namely, *how rapidly does the system approach equilibrium*, i.e. how rapidly does y tend to zero? The decay rate of y is governed by the exponential term in equations (3.73), (3.75) and (3.77):

$$y(t) \sim \begin{cases} e^{-\zeta\omega_0 t}, & \text{if} \quad 0 < \zeta < 1 \\ e^{-\omega_0 t}, & \text{if} \quad \zeta = 1 \\ e^{-(\zeta-\sqrt{\zeta^2-1})\omega_0 t}, & \text{if} \quad \zeta > 1 \end{cases} \tag{3.79}$$

It follows (see problem set 3) that *the displacement* y *decays most rapidly for* $\zeta = 1$, *i.e. in the critically damped case.* Thus, if one wishes to design a mechanical or electrical system that will return rapidly to a state of equilibrium after being disturbed, he/she should arrange that the damping is close to critical i.e. $\zeta = \frac{\lambda}{\omega_0} \approx 1$.

3.3.3 Non-zero driving force

The oscillator DE with non-zero driving force is

$$y'' + 2\lambda y' + \omega_0^2 y = f_0 \cos(\omega t), \tag{3.80}$$

[see equation (3.61) in Section 3.3.1]. We know from Section 3.2.2 that the general solution of this DE is of the form

$$y(t) = y_h(t; c_1, c_2) + y_p(t), \tag{3.81}$$

where y_h is the general solution of the homogeneous DE

$$y'' + 2\lambda y + \omega_0^2 y = 0,$$

(and hence depends on two arbitrary constants c_1 and c_2), and y_p is a particular solution of (3.80). We also know from Section 3.2.4 that $y_p(t)$ is of the form

$$y_p(t) = a_1 \cos \omega t + a_2 \sin \omega t,$$

(use the method of undetermined coefficients). This solution can also be written in the form

$$y_p(t) = A \cos(\omega t - \delta), \tag{3.82}$$

where the constants a_1 and a_2 have been replaced by the amplitude A and phase δ (see equations (3.66) and (3.69) in Section 3.3.2).

One of the main results from Section 3.3.2 is that

$$\lim_{t\to\infty} y_h(t; c_1, c_2) = 0.$$

Thus, by (3.81) and (3.82), for sufficiently large t

$$y(t) \approx A \cos(\omega t - \delta),$$

i.e. *for sufficiently large t the response $y(t)$ of the system is periodic, having the same period as the driving force*. Heuristically, one can say that the driving force "overcomes" the effects of the system parameters, namely the natural frequency ω_0 and the damping constant λ, and compels the system to oscillate at the driving force's frequency, namely ω.

Referring to equation (3.81), the term y_h, which dies away, is called the *transient part* of the solution, while the term y_p, which is periodic and hence persists indefinitely, is called the *steady state* part of the solution. In many applications, the transient part will die out rapidly (depending on $\zeta = \lambda/\omega_0$ and ω_0; see equation (3.79) in Section 3.3.2), in which case the behaviour of the physical system is essentially described by the steady state part, which we refer to as *the response of the system to the driving (i.e. external) force*.

The essential question then is: what is the amplitude A of the steady state term (3.82)? Or more precisely, *how does the amplitude A depend on the system parameters λ and ω_0, and on the driving force parameters, namely the frequency ω and the amplitude f_0?*

We shall show, by deriving the particular solution y_p, that A *depends in a critical way on the dimensionless ratio $\frac{\omega}{\omega_0}$ and on the dimensionless damping parameter $\zeta = \frac{\lambda}{\omega_0}$*. This dependence has important implications for the design of many physical systems.

1. Transforming the DE to dimensionless form

In order to find the steady state amplitude A we have to find the explicit form of the particular solution (3.82) of the DE (3.80). Since there are 4 parameters in this DE it makes sense to first simplify the DE by introducing a dimensionless time τ and a dimensionless displacement y. It is natural to use

$$t_c = \frac{1}{\omega_0}$$

as the characteristic time, and to define the dimensionless time by

$$\tau = \frac{t}{t_c} = \omega_0 t. \tag{3.83}$$

To make the change in the DE (3.80) divide through by ω_0^2, obtaining

$$\frac{1}{\omega_0^2}\frac{d^2y}{dt^2} + 2\left(\frac{\lambda}{\omega_0}\right)\frac{1}{\omega_0}\frac{dy}{dt} + y = \frac{f_0}{\omega_0^2}\cos\left[\frac{\omega}{\omega_0}(\omega_0 t)\right].$$

In the usual way the chain rule and equation (3.83) lead to

$$\frac{d^2y}{d\tau^2} + 2\zeta\frac{dy}{d\tau} + y = \frac{f_0}{\omega_0^2}\cos(\Omega\tau) \tag{3.84}$$

where we have introduced the dimensionless parameters

$$\zeta = \frac{\lambda}{\omega_0}, \qquad \Omega = \frac{\omega}{\omega_0}. \tag{3.85}$$

We note for future reference that

$$\Omega\tau = \omega t, \tag{3.86}$$

as follows from (3.83) and (3.85).

The next step is to note that the requirement of dimensional homogeneity applied to equation (3.84) implies

$$\left[\frac{f_0}{\omega_0^2}\right] = [y].$$

We do not specify the dimensions of y, since it could represent a displacement, an electric current, charge, etc. It now follows that *the new variable Y defined by*

$$Y = \frac{y}{\left(\frac{f_0}{\omega_0^2}\right)} = \frac{\omega_0^2}{f_0}y \tag{3.87}$$

is a dimensionless variable. Thus, on dividing equation (3.84) by $\frac{f_0}{\omega_0^2}$ this equation assumes the simpler form

$$\frac{d^2Y}{d\tau^2} + 2\zeta\frac{dY}{d\tau} + Y = \cos(\Omega\tau),$$

involving only two dimensionless parameters ζ and Ω. Finally, we write $\frac{d}{d\tau}$ as $'$ to obtain:

$$Y'' + 2\zeta Y' + Y = \cos(\Omega\tau). \tag{3.88}$$

2. Derivation of the steady state solution

A particular solution of the DE (3.88) can be found in the usual way, using "undetermined coefficients". Here we take the opportunity to show that the solution can be found more quickly by using *complex numbers.*

Consider the complex DE
$$z'' + 2\zeta z' + z = e^{i\Omega\tau}. \tag{3.89}$$
Since $e^{i\Omega\tau} = \cos\Omega\tau + i\sin\Omega\tau$, the DE (3.88) is the real part of (3.89), with $Y = Re(z)$. Assume a trial function of the form

$$z(\tau) = \mathcal{A}e^{i(\Omega\tau-\delta)} \tag{3.90}$$

where $\mathcal{A}$ and δ are the undetermined coefficients (they are assumed to be real). When (3.90) is substituted in (3.89) one finds after a non-trivial calculation that

$$\mathcal{A} = \left[(1-\Omega^2)^2 + 4\zeta^2\Omega^2\right]^{-\frac{1}{2}}, \tag{3.91}$$

and
$$\cos\delta = \mathcal{A}(1-\Omega^2), \quad \sin\delta = 2\mathcal{A}\zeta\Omega.$$
[The details are an exercise on Problem Set 3.]

Taking the real part of (3.90) gives the particular solution of (3.88)

$$Y(\tau) = \mathcal{A}\cos(\Omega\tau - \delta), \tag{3.92}$$

where $\mathcal{A}$ and δ are given by equations (3.91). We can use (3.86) and (3.87) to transform the particular solution (3.92) to a particular solution of the DE (3.80) in terms of the physical variables y and t, obtaining
$$y(t) = \mathcal{A}\frac{f_0}{\omega_0^2}\cos(\omega t - \delta). \tag{3.93}$$
In summary, *the steady state solution of the oscillator DE*

$$y'' + 2\lambda y' + \omega_0^2 y = f_0\cos(\omega t),$$

is given by (3.93)*, with*

$$\begin{gathered}\mathcal{A} = \left[(1-\Omega^2)^2 + 4\zeta^2\Omega^2\right]^{-\frac{1}{2}}, \\ \cos\delta = \mathcal{A}(1-\Omega^2), \quad \sin\delta = 2\mathcal{A}\zeta\Omega.\end{gathered} \tag{3.94}$$

Thus the steady state response is *simple harmonic motion* with *amplitude*

$$A = \frac{f_0}{\omega_0^2}\mathcal{A}, \tag{3.95}$$

angular frequency ω (the same as the driving force), and *phase* δ.

3. Analysis of the amplitude of the steady state solution

Consider a fixed system, with system parameters λ and ω_0, governed by the oscillator DE (3.80). We are interested in the response of the system to a driving force of frequency ω and amplitude f_0. The amplitude A of the response is given by (3.95) and (3.94):

$$A = \frac{f_0}{\omega_0^2}\mathcal{A}(\Omega, \zeta) = \frac{f_0}{\omega_0^2}\mathcal{A}\left(\frac{\omega}{\omega_0}, \frac{\lambda}{\omega_0}\right). \tag{3.96}$$

For a given system ω_0 and λ, and hence $\zeta = \frac{\lambda}{\omega_0}$, are fixed. The dependence of A on the amplitude f_0 of the driving force is simple: A is proportional to f_0, by (3.96). The dependence of A on the frequency ω of driving force is governed by the function $\mathcal{A}(\Omega, \zeta)$ in (3.96), and is complicated. One can imagine changing ω. How does A change? We will be able to answer this question at a glance if we sketch the graph of $A(\Omega, \zeta)$ versus Ω, for fixed ζ. This graph is called the *frequency response curve of the system.*

The graph can be drawn using the following information:

(i) $\mathcal{A}(0, \zeta) = 1$, for all $\zeta \geq 0$

(ii) $\lim_{\Omega\to\infty} \mathcal{A}(\Omega, \zeta) = 0$, for all $\zeta \geq 0$.

These results follow immediately from (3.94).

(iii) If $2\zeta^2 \geq 1$, then $\mathcal{A} < 1$ for all $\Omega > 0$. This follows by writing (3.94) in the form

$$\mathcal{A} = \left[1 + \Omega^4 + 2(2\zeta^2 - 1)\Omega^2\right]^{-\frac{1}{2}}. \tag{3.97}$$

Thus, if $2\zeta^2 \geq 1$, the maximum of $\mathcal{A}$ on the interval $0 \leq \Omega < \infty$ occurs at the endpoint $\Omega = 0$.

(iv) If $2\zeta^2 < 1$, then $\mathcal{A}$ equals 1 for $\Omega = 0$ and $\Omega = \sqrt{2(1 - 2\zeta^2)}$, and is greater than 1 between these values. This result follows by writing (3.97) in the form

$$\mathcal{A} = \left[1 + \Omega^2\{\Omega^2 - 2(1 - 2\zeta^2)\}\right]^{-\frac{1}{2}}. \tag{3.98}$$

Thus, if $0 < 2\zeta^2 < 1$, $\mathcal{A}$ must have a maximum value greater than 1.

(v) If $0 < 2\zeta^2 < 1$ the maximum value of $\mathcal{A}$ is

$$\mathcal{A} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}, \tag{3.99}$$

and occurs for

$$\Omega = \sqrt{1 - 2\zeta^2}. \tag{3.100}$$

This result is obtained by showing that the derivative of $\mathcal{A}$ with respect to Ω has the form

$$\frac{\partial \mathcal{A}}{\partial \Omega} = -2\Omega[\Omega^2 - (1 - 2\zeta^2)]\mathcal{A}^3,$$

so that

$$\frac{\partial \mathcal{A}}{\partial \Omega} = 0 \quad \text{implies} \quad \Omega = 0 \quad \text{or} \quad \Omega = \sqrt{1 - 2\zeta^2}.$$

Then substitute $\Omega = \sqrt{1 - 2\zeta^2}$ into (3.98) to get (3.99).

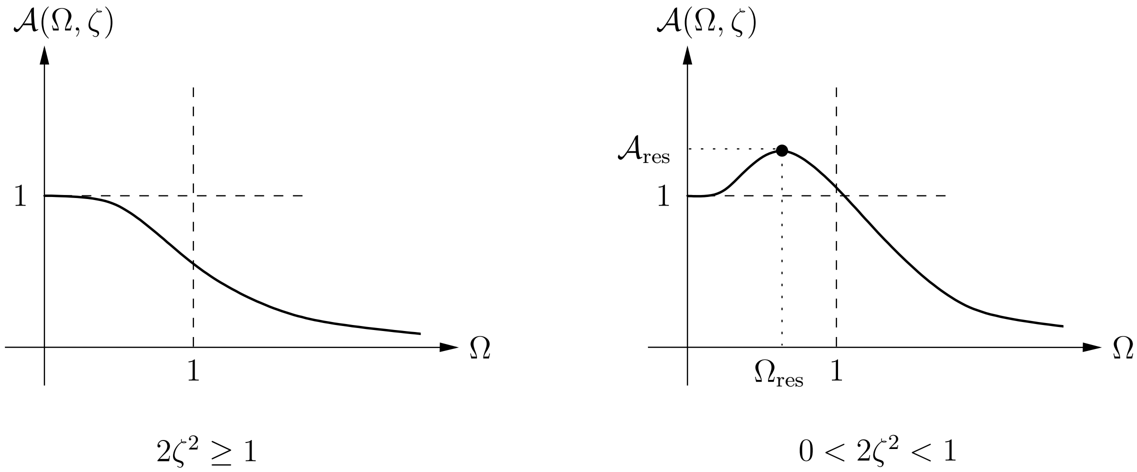


Figure 3.5: Typical frequency response curves in the cases $2\zeta^2 \geq 1$ and $0 < 2\zeta^2 < 1$.

When $\mathcal{A}$ attains a maximum greater than 1, one says the *the system undergoes resonance.* We thus label the values of $\mathcal{A}$ and Ω with a subscript "res", i.e. we write (3.99) and (3.100) as

$$\mathcal{A}_{\text{res}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, \quad \Omega_{\text{res}} = \sqrt{1-2\zeta^2}. \tag{3.101}$$

By (3.95) and (3.85), the physical amplitude and frequency at resonance are

$$A_{\text{res}} = \frac{f_0}{2\omega_0^2\zeta\sqrt{1-\zeta^2}}, \quad \omega_{\text{res}} = \omega_0\sqrt{1-2\zeta^2}, \tag{3.102}$$

with $\zeta = \lambda/\omega_0$. Observe that

$$\omega_{\text{res}} < \omega_0 \quad \text{for all} \quad \zeta \quad \text{with} \quad 2\zeta^2 < 1,$$

i.e. *the resonant frequency is less than the natural frequency.* The second result is that A_{res} *can be arbitrarily large if the dimensionless damping parameter* ζ *is sufficiently close to* 0. Indeed, if $0 < \zeta \ll 1$, equation (3.102) gives the approximations

$$A_{\text{res}} \approx \frac{f_0}{2\omega_0^2\zeta}, \qquad \omega_{\text{res}} \approx \omega_0. \tag{3.103}$$

It is instructive to draw the family of frequency response curves for different values of ζ, in the $\Omega\mathcal{A}$-plane. For this purpose it is useful to note that

$$\mathcal{A}_{\text{res}} = \frac{1}{\sqrt{1-\Omega_{\text{res}}^4}}, \tag{3.104}$$

as follows from (3.101).

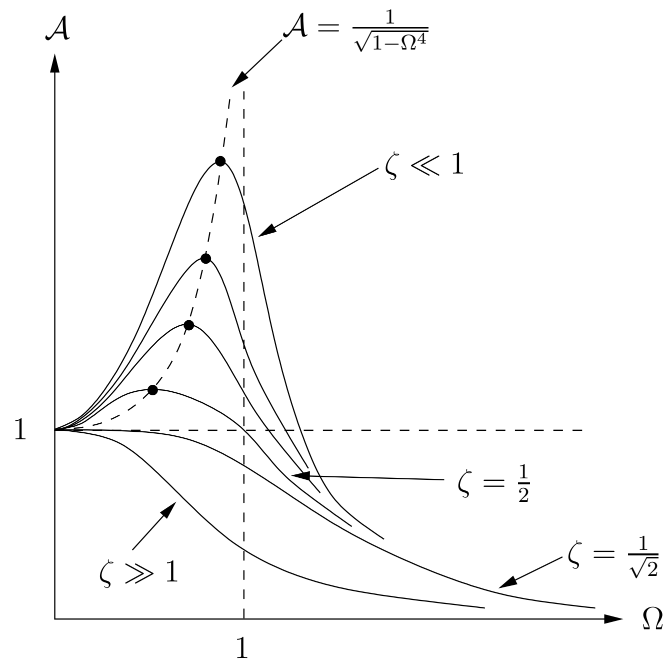


Figure 3.6: The frequency response diagram. Note that the maxima lie on the curve (3.104).

Remark

If $0 < \zeta \ll 1$, the frequency response curve has a steep and narrow peak, with

$$\mathcal{A}_{\text{res}} \approx \frac{1}{2\zeta}, \quad \omega_{\text{res}} \approx \omega_0$$

showing how *resonance provides amplification.* Engineers refer to the quantity

$$Q = \frac{1}{2\zeta}$$

as the *Q-factor of the system* (Q for Quality). In some situations, a high Q-factor is needed, as in a radio tuning circuit (an RLC circuit, see Section 3.3.1). In this situation, the driving frequency ω would be the frequency of the station you wish to receive, and you would "tune in" the station by varying $\omega_0 = \frac{1}{\sqrt{LC}}$ to obtain resonance i.e. the circuit selectively amplifies the signal of the station. In other situations, it is desirable to have a *flat* frequency response curve. The value $\zeta = \frac{1}{\sqrt{2}}$, i.e. $A = \frac{1}{\sqrt{2}}$, which is the value of ζ which just avoids resonance, gives the flattest response curve, i.e. very flat for $0 < \Omega < 1$ ($0 < \omega < \omega_0$). Such a system would act as a *filter*, excluding frequencies with $\omega \gg \omega_0$.

4. Analysis of the phase of the steady state solution

Recall from equation (3.94) that the phase δ of the steady state solution is given by

$$\cos\delta = \mathcal{A}(1-\Omega^2), \quad \sin\delta = 2\mathcal{A}\zeta\Omega, \tag{3.105}$$

where

$$\mathcal{A} = \left[(1-\Omega^2)^2 + 4\zeta^2\Omega^2\right]^{-\frac{1}{2}}.$$

We are going to sketch the graph of δ as a function of Ω, for various values of ζ.

We need the following properties of the function $\delta = \delta(\Omega, \zeta)$, which are a consequence of equation (3.105):

(i) $\delta(0,\zeta) = 0$ and $\delta(1,\zeta) = \frac{\pi}{2}$ for all $\zeta > 0$,

(ii) $\lim_{\Omega\to\infty} \delta(\Omega,\zeta) = \pi$,

(iii) $\dfrac{\partial\delta}{\partial\Omega}(0,\zeta) = 2\zeta$ and $\dfrac{\partial\delta}{\partial\Omega}(1,\zeta) = \frac{1}{\zeta}$,

obtained by differentiating $\tan\delta = \dfrac{2\zeta\Omega}{1-\Omega^2}$ with respect to Ω.

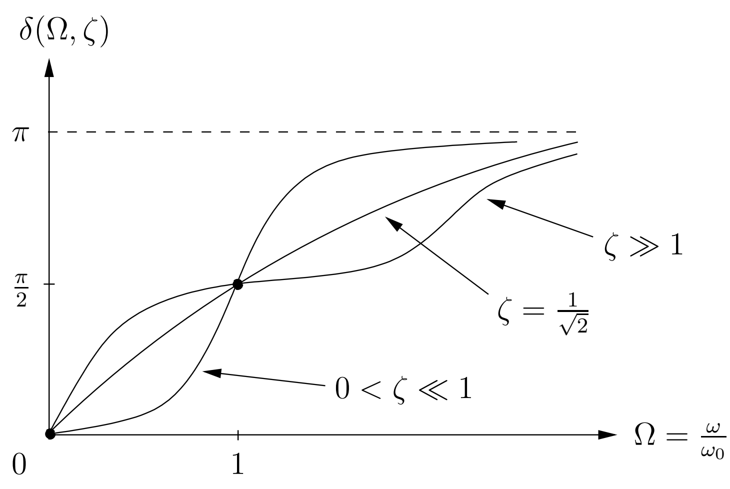


Figure 3.7: The phase response diagram.

The main results are:

(1) the response of the system always lags the driving force, since $\delta > 0$,

(2) if $\zeta \ll 1$, resonance occurs for $\Omega \approx 1$ and then the phase is close to $\frac{\pi}{2}$.

5. The case of zero damping

We finally consider the idealized case of zero damping ($\lambda = 0$), so that the oscillator DE (3.80) reduces to

$$y'' + \omega_0^2 y = f_0 \cos(\omega t). \tag{3.106}$$

If $\omega \neq \omega_0$, the general solution is

$$y = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{f_0}{\omega_0^2 - \omega^2} \cos(\omega t), \tag{3.107}$$

and if $\omega = \omega_0$, the general solution is

$$y = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{f_0}{2\omega_0} t \sin(\omega_0 t). \tag{3.108}$$

(exercise using undetermined coefficients, preferably in complex form for efficiency).

The first significant difference from the damped case is that when $\omega = \omega_0$, i.e. the driving frequency equals the natural frequency, the response y grows without bound as $t \to +\infty$, due to the $t\cos(\omega_0 t)$ term in (3.108). This is an extreme but idealized form of resonance.

The second significant difference is that the general solution does not separate into a transient term and a steady state term, since there are no (decaying) exponential terms in the general solution. This implies that *the long term behaviour depends on the initial conditions.*

We consider in detail the case where the system starts in equilibrium, i.e. the initial conditions are

$$y(0) = 0, \qquad y'(0) = 0.$$

It follows from equations (3.107) and (3.108) that the unique solutions are

$$y = \frac{f_0}{\omega_0^2 - \omega^2}(\cos \omega t - \cos \omega_0 t), \quad \omega \neq \omega_0, \tag{3.109}$$

and

$$y = \frac{f_0}{2\omega_0} t \sin(\omega_0 t), \quad \omega = \omega_0. \tag{3.110}$$

In the second case, the response is a linearly growing oscillation, as shown in Figure 3.8.

In the first case, we can use the trig identity

$$\cos A - \cos B = -2 \sin\left(\frac{A-B}{2}\right) \sin\left(\frac{A+B}{2}\right),$$

to write the solution (3.109) in the form

$$y = \left[\frac{2f_0}{\omega_0^2 - \omega^2} \sin \tfrac{1}{2}(\omega_0 - \omega)t\right] \sin \tfrac{1}{2}(\omega_0 + \omega)t. \tag{3.111}$$

If $\mid \omega_0 - \omega \mid \ll 1$ and $\omega_0 + \omega \gg \mid \omega_0 - \omega \mid$, then $\sin \frac{1}{2}(\omega_0 + \omega)t$ is a rapidly oscillating function compared to $\sin \frac{1}{2}(\omega_0 - \omega)t$. Thus, *the solution* (3.111) *represents a rapid oscillation with frequency* $\frac{1}{2}(\omega_0 + \omega)$, *but with a slowly varying sinusoidal amplitude with frequency* $\frac{1}{2}(\omega_0 - \omega)$ [see Figure 3.9]. This type of response, with a periodic variation of amplitude, exhibits what are called *beats.* This phenomenon can be heard when two tuning forks of nearly equal frequency are sounded simultaneously. In electronics, the periodic variation of amplitude with time is called *amplitude modulation.*

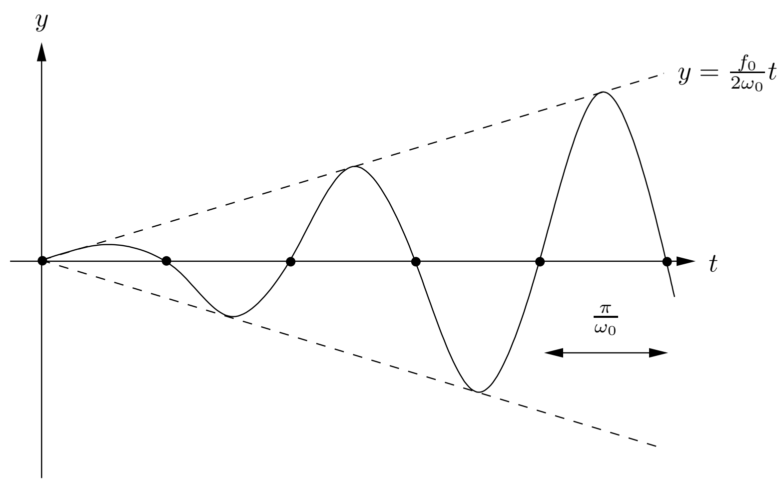


Figure 3.8: Undamped response when the driving frequency equals the natural frequency.

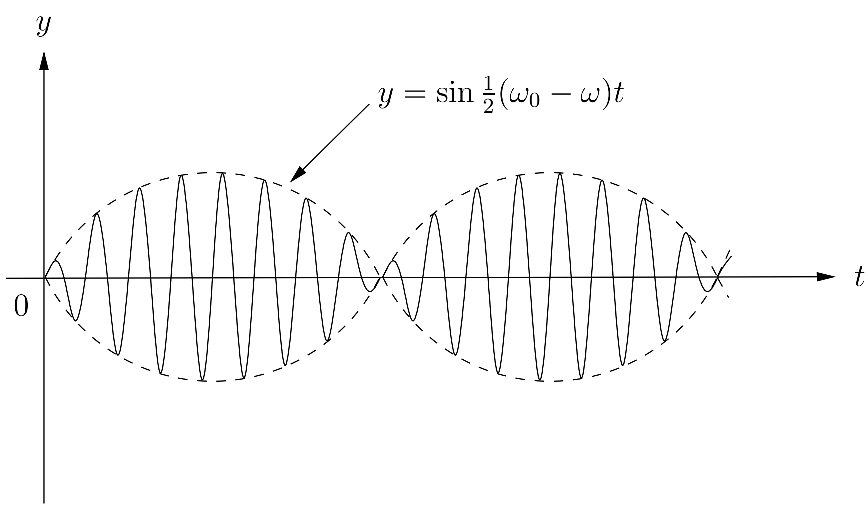


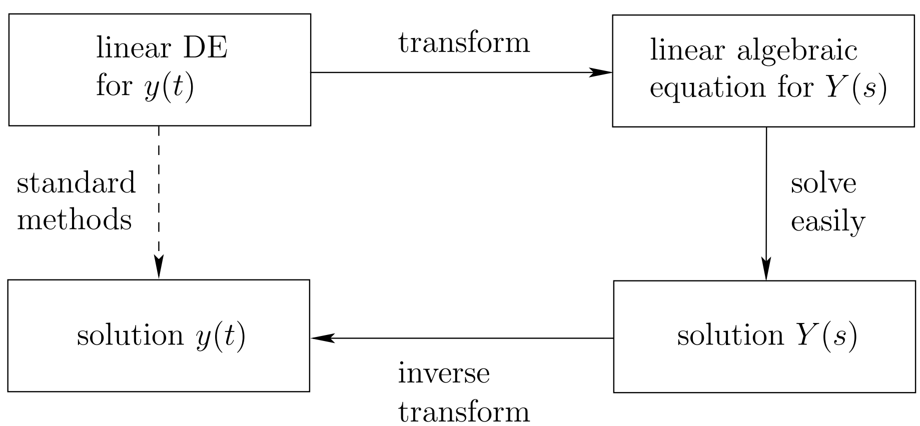
Figure 3.9: Graph of $y = \left[\sin\frac{1}{2}(\omega_0 - \omega)t\right]\sin\frac{1}{2}(\omega_0 + \omega)t$, showing the phenomenon of beats.

Chapter 4

The Laplace Transform and DEs

The idea behind the Laplace transform is this:

> Transform a linear *differential* equation for $y(t)$ into a linear *algebraic* equation for $Y(s)$, thereby making it easier to solve.



The function $Y(s)$ is called the *Laplace transform* of the function $y(t)$.

Remark

In its simplest form, the method is restricted to *linear differential equations with constant coefficients*. It provides an alternative to the standard methods that we have developed so far. We'll mention its advantages as we proceed.

4.1 Elementary properties of the Laplace transform

4.1.1 Definition of the Laplace transform

The Laplace transform is one of several "transforms" that are used in engineering mathematics. They are defined using the integral.

Definition

Given a real-valued (or complex-valued) function $y(t)$ *the Laplace transform of* y is defined to be

$$Y(s) = \int_0^\infty e^{-st} y(t) dt, \tag{4.1}$$

for all values of s for which the improper integral converges.

Recall the definition of "converges":

The improper integral $\int_a^\infty g(t)dt$ *converges* means that $\lim_{r\to\infty} \int_a^r g(t)dt$ exists. □

Before discussing the significance of this definition we work out the simplest and most important example.

Example

Find the Laplace transform of $y(t) = e^{\alpha t}$, where α is a constant.

Solution: Referring to equation (4.1), consider

$$\begin{aligned} \int_0^r e^{-st} e^{\alpha t} dt &= \int_0^r e^{-(s-\alpha)t} dt, \\ &= -\frac{1}{s-\alpha} e^{-(s-\alpha)t}\Big|_0^r, \quad \text{assuming} \quad s \neq \alpha, \\ &= \frac{1}{s-\alpha}[1 - e^{-(s-\alpha)r}]. \end{aligned} \tag{4.2}$$

If $s > \alpha$, then $\lim_{r\to\infty} e^{-(s-\alpha)r} = 0$, which by (4.2) implies that

$$\lim_{r\to\infty} \int_0^r e^{-st} e^{\alpha t} dt = \frac{1}{s-\alpha} \quad \text{(exists)}.$$

Thus, if $s > \alpha$, the Laplace transform of $y(t) = e^{\alpha t}$ is

$$Y(s) = \frac{1}{s-\alpha},$$

by the definition. □

We think of equation (4.1) as defining an *operator* $\mathcal{L}$ which acts on a function $y(t)$ to give a new function $Y(s)$. We write

$$\mathcal{L}[y] = Y,$$

or, if we wish to indicate the arguments,

$$\mathcal{L}[y(t)] = Y(s),$$

where $Y(s)$ is given by (4.1). We shall refer to $\mathcal{L}$ as the *Laplace transform operator.*

Referring to the Example, we can use the operator notation to write

$$\mathcal{L}[e^{\alpha t}] = \frac{1}{s-\alpha}, \quad s > \alpha. \tag{4.3}$$

This equation is read as "the Laplace transform of $e^{\alpha t}$ is $\frac{1}{s-\alpha}$".

The idea of an "operator" which acts on functions is not unfamiliar. One can think of the process of differentiation as defining an operator D that acts on functions, i.e. D acts on a differentiable function f to give its derivative:

$$D[f] = f'.$$

The operator D and $\mathcal{L}$ have one important property in common. The operator D satisfies

$$D[f_1 + f_2] = Df + D[f_2],$$

and

$$D[cf] = cD[f],$$

where c is a constant. These equations are simply the sum property and "multiplication by a constant" property of derivatives. These equations mean that D is a *linear operator.*[1]

Proposition

The Laplace transform operator $\mathcal{L}$ is a linear operator:

$$\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2],$$

and

$$\mathcal{L}[cy] = c\mathcal{L}[y],$$

where c is a constant.

Proof: These results follow from the linearity of the integral:

$$\int_a^b [y_1(t) + y_2(t)]dt = \int_a^b y_1(t)dt + \int_a^b y_2(t)dt,$$

and

$$\int_a^b cy(t)dt = c\int_a^b y(t)dt,$$

by taking limits to obtain the improper integrals. We skip the details. □

This proposition is used extensively when working with the Laplace transform.

Before continuing, we pause briefly to consider a theoretical question: what conditions on f are needed to ensure that $\mathcal{L}[f]$ exists?

Speaking intuitively, we can say that $\mathcal{L}[f]$ exists provided that

[1] This is the same concept as in linear algebra, where a 3×3 matrix A acting on vectors $\mathbf{x}$ in $\mathbb{R}^3$ (for example) satisfies $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2$, and $A(\lambda\mathbf{x}) = \lambda A\mathbf{x}$, thereby defining a *linear transformation.*

(i) $f(t)$ does not grow too rapidly as $t \to +\infty$,

and

(ii) the only discontinuities of f for $t \geq 0$ are jump discontinuities.

First, to describe the growth condition it is convenient to use the *big-O notation*. Recall the definition:

$$f(t) = O(g(t)) \quad \text{as} \quad t \to \infty,$$

means that there exist constants B and b such that

$$| f(t) | \leq B \, | g(t) | \quad \text{for all} \quad t \geq b.$$

The required growth condition is that

$$f(t) = O(e^{at}) \quad \text{as} \quad t \to \infty, \tag{4.4}$$

for *some constant a*. Intuitively (4.4) means that $f(t)$ does not grow more rapidly than e^{at} as $t \to \infty$.

Second, f has a *jump discontinuity at c* means that the 1-sided limits

$$\lim_{t \to c^+} f(t) \quad \text{and} \quad \lim_{t \to c^-} f(t)$$

exist and are unequal. The value of f at c is unimportant. The statement "*f is piecewise continuous on a finite interval I*" means that f is continuous on I except for a *finite* number of jump discontinuities.

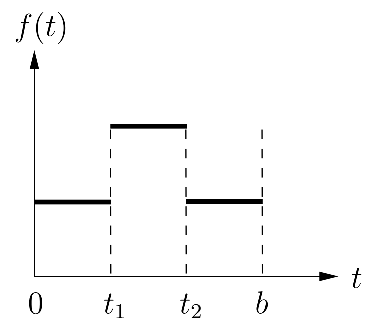


Figure 4.1: A piecewise continuous function with two jump discontinuities.

A function such as this (a pulse function) could act as the driving force of an oscillator.

Proposition: Existence of the Laplace transform

If H_1 : f is piecewise continuous on each interval $0 \le t \le r$,
H_2 : there is a constant a such that

$$f(t) = O(e^{at}) \quad \text{as} \quad t \to \infty,$$

then the Laplace transform $\mathcal{L}[f]$ exists for $s > a$.

Proof: Outline.

By H_1, $\int_0^r f(t)dt$ exists (just divide the interval into subintervals on which f is continuous).

By H_2, $\mid f(t) \mid \le Be^{at}$ for $t \ge b$. Hence

$$\mid e^{-st} f(t) \mid \le Be^{-(s-a)t}, \quad \text{for} \quad t \ge b.$$

If $s > a$, $\int_0^\infty Be^{-(s-a)t}dt$ converges. Hence by the Comparison Test for improper integrals, $\int_0^\infty e^{-st} f(t)dt$ converges i.e. $\mathcal{L}[f]$ exists by definition. □

Examples:

(i) $f(t) = (1+t)^t$ does not satisfy the growth condition H_2, and $\mathcal{L}[f]$ may not (and it turns out, does not) exist.

(ii) $f(t) = \frac{1}{t}$ is not piecewise continuous on any interval $[0, r]$, and $\mathcal{L}[f]$ may not (and it turns out, does not) exist. □

4.1.2 Calculating Laplace transforms

Our goal is to build up a table of Laplace transforms of elementary functions, which we can use to solve differential equations.

So far we have shown that

$$\mathcal{L}[e^{\alpha t}] = \frac{1}{s-\alpha}, \quad \text{for} \quad s > \alpha. \tag{4.5}$$

This formula includes the important special case of the constant function $f(t) = 1$, i.e. choose $\alpha = 0$:

$$\mathcal{L}[1] = \frac{1}{s}, \quad \text{for} \quad s > 0. \tag{4.6}$$

Another useful example is the Laplace transform of t^n.

Example

Show that

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad \text{for} \quad s > 0, \tag{4.7}$$

where $n \geq 0$ is an integer.

Solution: Outline (fill in the details as an exercise).

Use induction on n, with (4.6) giving the result for $n = 0$. Apply integration by parts to $\displaystyle\int_0^b e^{-st} t^{k+1} dt$ to reduce the power of t by 1, and then let $b \to \infty$. Then the case $n = k$ will imply the case $n = k + 1$. You will also need the result

$$\lim_{b\to\infty} \frac{b^{k+1}}{e^{sb}} = 0 \quad \text{for} \quad s > 0,$$

i.e. an exponential dominates a power as $b \to \infty$. □

The direct evaluation of Laplace transforms using the definition can be time consuming, and so one seeks shortcuts, for example

(i) by using complex functions,

(ii) by developing theorems to give new Laplace transforms from old, without extra calculation.

We first illustrate (i). The derivation leading to (4.5) is valid if α is complex, provided that we make one small change:

$$\lim_{r\to\infty} e^{-(s-\alpha)r} = 0 \quad \text{provided that} \quad s > Re(\alpha),$$

where $Re(\alpha)$ is the real part of α. We can now use Euler's formula

$$e^{ibt} = \cos bt + i \sin bt \tag{4.8}$$

and the linearity of $\mathcal{L}$ to calculate $\mathcal{L}[\cos bt]$ and $\mathcal{L}[\sin bt]$ directly from (4.5).

Example

Show that

$$\mathcal{L}[\cos bt] = \frac{s}{s^2 + b^2}, \qquad \mathcal{L}[\sin bt] = \frac{b}{s^2 + b^2}. \tag{4.9}$$

Solution: By linearity of $\mathcal{L}$ and (4.8):

$$\mathcal{L}[e^{ibt}] = \mathcal{L}[\cos bt] + i\mathcal{L}[\sin bt]. \tag{4.10}$$

The right hand side of (4.5), with $\alpha = ib$ is

$$\frac{1}{s-ib} = \frac{s+ib}{(s-ib)(s+ib)} = \frac{s+ib}{s^2+b^2}. \tag{4.11}$$

Choosing $\alpha = ib$ in (4.5) and substituting (4.10) and (4.11) gives

$$\mathcal{L}[\cos bt + i \sin bt] = \frac{s}{s^2+b^2} + i\frac{b}{s^2+b^2}.$$

Equating real and imaginary parts leads to (4.9). □

We now give a theorem which provides a quick way of calculating $\mathcal{L}[e^{ct}f(t)]$ if $\mathcal{L}[f(t)]$ is known.

Definition: First Shift Theorem

If $\mathcal{L}[f(t)] = F(s)$ exists for $s > a$, then

$$\mathcal{L}[e^{ct}f(t)] = F(s-c), \quad \text{for} \quad s > a+c, \tag{4.12}$$

where c is a constant.

Proof: Consider

$$\int_0^r e^{-st}e^{ct}f(t)dt = \int_0^r e^{-(s-c)t}f(t)dt. \tag{4.13}$$

By the hypothesis, $\lim_{r\to\infty}\int_0^r e^{-(s-c)t}f(t)dt$ exists for $s-c > a$, and equals $F(s-c)$. The result follows by letting $r \to \infty$ in (4.13). □

Example

Calculate the Laplace transform of $g(t) = te^{ct}$.

Solution: By (4.7), $\mathcal{L}[t] = \frac{1}{s^2}$ for $s > 0$. Thus by the Shift Theorem,

$$\mathcal{L}[te^{ct}] = \frac{1}{(s-c)^2} \quad \text{for} \quad s > c. \qquad \square$$

Exercise

Calculate $\mathcal{L}[e^{-2t}\sin \pi t]$.

Answer: $\dfrac{\pi}{(s+2)^2+\pi^2}$. □

4.1.3 The Laplace transform of a derivative

In using the Laplace transform to solve a linear DE,

$$y' + ky = f(t),$$

where k is a constant, we begin by applying the operator $\mathcal{L}$ to the DE:

$$\mathcal{L}[y' + ky] = \mathcal{L}[f].$$

By linearity of $\mathcal{L}$, we obtain

$$\mathcal{L}[y'] + k\mathcal{L}[y] = \mathcal{L}[f].$$

To proceed further we need to relate $\mathcal{L}[y']$ to $\mathcal{L}[y]$. The next proposition does just that.

Proposition 1

If H_1 : $f(t) = O(e^{at})$ as $t \to \infty$, for some a,

H_2 : f' is piecewise continuous and f is continuous, on any interval $[0, r]$,

then $\mathcal{L}[f'(t)]$ exists, and

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0), \quad \text{for} \quad s > a. \tag{4.14}$$

Proof: For simplicity, we assume that f' is continuous.
Using integration by parts:

$$\int_0^r e^{-st} f'(t)dt = \left[e^{-sr} f(r) - f(0)\right] - \int_0^r (-s)e^{-st} f(t)dt. \tag{4.15}$$

By H_1, $| e^{-sr} f(r) | \leq Be^{-(s-a)r}$. Thus, if $s > a$, $\lim_{r\to\infty} e^{-sr} f(r) = 0$.
The result now follows by letting $r \to \infty$ in (4.15). □

Proposition 1 extends in a natural way to the second derivative.

Proposition 2

If H_1 : $f(t) = O(e^{at})$ and $f'(t) = O(e^{at})$ as $t \to \infty$,

H_2 : f'' is piecewise continuous and f', f are continuous, on any interval $[0, r]$,

then $\mathcal{L}[f''(t)]$ exists, and

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0), \quad \text{for} \quad s > a. \tag{4.16}$$

Proof: Apply Proposition 1, with f replaced by f', obtaining

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0).$$

Now apply Proposition 1 to the right side directly to get

$$\mathcal{L}[f''(t)] = s\left\{s\mathcal{L}[f(t)] - f(0)\right\} - f'(0). \qquad \square$$

4.1.4 The Inverse Laplace Transform

Referring to the diagram on the first page of this Chapter, the final (and most difficult) step in using the Laplace transform to solve a DE is this:

given a Laplace transform $Y(s)$, find the function $y(t)$.

One should first ask: given $Y(s)$, is there a *unique* $y(t)$?

The next proposition gives the answer YES.

Proposition

If y_1 and y_2 are continuous functions of order e^{at} as $t \to \infty$, then

$$y_1 \neq y_2 \quad \text{implies} \quad \mathcal{L}[y_1] \neq \mathcal{L}[y_2]. \qquad \square$$

Proof: The proof is beyond the scope of this course. See for example, Brauer & Nohel, page 391-2. □

This proposition states that the Laplace transform operator $\mathcal{L}$ is a *one-to-one operator*, and hence has an *inverse operator* $\mathcal{L}^{-1}$ which maps a Laplace transform $Y(s)$ onto the original function $y(t)$:

$$\mathcal{L}^{-1}[F(s)] = f(t) \quad \text{means that} \quad \mathcal{L}[f(t)] = F(s).$$

Equivalently we can write

$$\mathcal{L}^{-1}[\mathcal{L}[f(t)]] = f(t) \quad \text{and} \quad \mathcal{L}[\mathcal{L}^{-1}[F(s)]] = F(s).$$

We shall call $\mathcal{L}^{-1}$ the *inverse Laplace transform operator*, and shall call $f(t) = \mathcal{L}^{-1}[F(s)]$ the *inverse Laplace transform of* $F(s)$. Note that $\mathcal{L}^{-1}$ is also linear. □

In elementary discussions, inverse Laplace transforms are found by referring to a *table of Laplace transforms* – calculating them directly is difficult (and sometimes impossible). Based on the results of section 4.1.2 we can construct the following Table:

$f(t)$	$F(s) = \mathcal{L}[f(t)]$
$e^{\alpha t}$	$\dfrac{1}{s-\alpha}$
$\cos bt$	$\dfrac{s}{s^2+b^2}$
$\sin bt$	$\dfrac{b}{s^2+b^2}$
t^n	$\dfrac{n!}{s^{n+1}}$
$e^{ct}f(t)$	$F(s-c)$

For example, we can conclude directly from the Table that

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}, \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+\pi^2}\right] = \cos \pi t.$$

In the examples we shall encounter, the Laplace transform $F(s)$ will mostly be a *rational function*, i.e.

$$F(s) = \frac{p(s)}{q(s)},$$

where $p(s)$ and $q(s)$ are polynomial functions. It will be necessary to use *partial fraction methods* to write $F(s)$ as a sum of the simple terms appearing in the Table. The first shift theorem (summarized in the last line of the Table) will also be useful.

Example 1

Find $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+5s+6}\right]$.

Solution: Using partial fractions

$$\frac{s}{s^2+5s+6} = \frac{s}{(s+2)(s+3)} = \frac{3}{s+3} - \frac{2}{s+2}.$$

Using the Table and linearity of $\mathcal{L}^{-1}$:

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s}{s^2+5s+6}\right] &= \mathcal{L}^{-1}\left[\frac{3}{s+3}\right] - \mathcal{L}^{-1}\left[\frac{2}{s+2}\right]\\ &= 3e^{-3t} - 2e^{-2t}. \qquad \square\end{aligned}$$

Example 2

Find $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+4s+5}\right]$.

Solution: Since the denominator does not factor, we complete the square:

$$\frac{s}{s^2+4s+5} = \frac{s}{(s+2)^2+1} = \frac{(s+2)}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}.$$

From the Table (using the Shift Theorem),

$$\mathcal{L}[e^{-2t}\cos t] = \frac{s+2}{(s+2)^2+1}, \quad \mathcal{L}[e^{-2t}\sin t] = \frac{1}{(s+2)^2+1}.$$

It thus follows that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+4s+5}\right] = e^{-2t}\cos t - 2e^{-2t}\sin t. \qquad \square$$

In view of this example it is convenient to restate the First Shift Theorem in terms of the inverse Laplace transform operator $\mathcal{L}^{-1}$.

First Shift Theorem (Inverse form)

$$\text{If} \quad \mathcal{L}^{-1}[F(s)] = f(t), \quad \text{then} \quad \mathcal{L}^{-1}[F(s-c)] = e^{ct}f(t).$$

Exercise

Find (i) $\mathcal{L}^{-1}\left[\dfrac{s-3}{s^2-2s+5}\right]$ (ii) $\mathcal{L}^{-1}\left[\dfrac{s+4}{s^2+2s}\right]$

Answer: (i) $e^t(\cos 2t - \sin 2t)$ (ii) $2 - e^{-2t}$.

4.1.5 The Heaviside Step Function

A driving force $f(t)$ [or input function] that changes abruptly with time can be modelled by using a *discontinuous function*, and one that does not change smoothly can be modelled by a *non-differentiable function*.

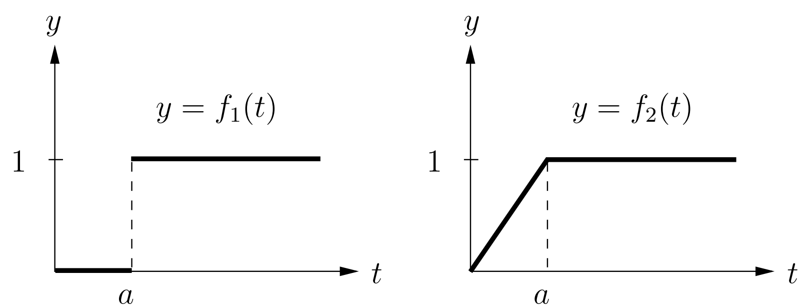


Figure 4.2: A step function $f_1(t)$ and a ramp function $f_2(t)$.

When working with the Laplace transform of functions such as these it is convenient to introduce the *Heaviside* or *unit step* function $H(t)$, defined by

$$H(t) = \begin{cases} 0, & \text{if} \quad t < 0 \\ 1, & \text{if} \quad t \geq 0. \end{cases} \tag{4.17}$$

Then the above step function $f_1(t)$ is given by

$$f_1(t) = H(t-a), \tag{4.18}$$

(translation to the right).

Example

Express the ramp function $f_2(t)$ above in terms of $H(t)$. *Solution:* We have

$$f_2(t) = \tfrac{1}{a}t, \quad H(t-a) = 0 \quad \text{for} \quad 0 \le t < a,$$
$$f_2(t) = 1, \quad H(t-a) = 1, \quad \text{for} \quad t \ge a.$$

Thus

$$f_2(t) = \frac{1}{a}t + \left(-\frac{1}{a}t + 1\right) H(t-a), \quad \text{for} \quad t \ge 0.$$

Thus the *ramp function* $f_2(t)$ can be written

$$f_2(t) = \tfrac{1}{a}[t - (t-a)H(t-a)]. \tag{4.19}$$

□

The next step is to calculate the Laplace transform of $H(t-a)$.

Example

Show that

$$\mathcal{L}[H(t-a)] = \frac{e^{-as}}{s}, \tag{4.20}$$

for $s > 0$, where $a > 0$ is a constant.

Solution: Consider, with $r > a$,

$$\int_0^r e^{-st} H(t-a)dt = \int_a^r e^{-st}dt, \quad \text{since } H(t-a) = 0 \text{ if } t < a,$$
$$= -\frac{1}{s}(e^{-sr} - e^{-sa}),$$

by the Fundamental Theorem of Calculus. Thus, if $s > 0$,

$$\lim_{r\to\infty} \int_0^r e^{-st} H(t-a)dt$$

exists and equals $\frac{1}{s}e^{-sa}$. The result follows by definition of the Laplace transform. □

We now give a theorem which provides a quick way of calculating $\mathcal{L}[H(t-c)f(t-c)]$ if $\mathcal{L}[f(t)]$ is known.

Second Shift Theorem

If $\mathcal{L}[f(t)] = F(s)$ exists for $s > a \ge 0$, and c is a positive constant, then

$$\mathcal{L}[H(t-c)f(t-c)] = e^{-cs}F(s), \quad s > a. \tag{4.21}$$

Proof: Consider

$$\begin{aligned}\int_0^r e^{-st}H(t-c)f(t-c)dt &= \int_c^r e^{-st}f(t-c)dt, \quad \text{by definition of} \quad H,\\ &= \int_0^{r-c} e^{-s(u+c)}f(u)du, \quad \text{by making a change of variable} \quad u=t-c,\\ &= e^{-cs}\int_0^{r-c} e^{-su}f(u)du.\end{aligned} \tag{4.22}$$

Since $\mathcal{L}[f(t)]$ exists, $\lim_{r\to\infty}\int_0^{r-c} e^{-su}f(u)du$ exists and equals $F(s)$ (note that $r-c\to\infty$, and the integration variable u can be relabelled t). The result follows by letting $r\to\infty$ in (4.22). □

Example

Calculate the Laplace transform of the ramp function (see Figure 4.2)

$$f_2(t) = \tfrac{1}{a}[t-(t-a)H(t-a)].$$

Solution: By linearity of $\mathcal{L}$,

$$\mathcal{L}[f_2(t)] = \tfrac{1}{a}\left\{\mathcal{L}[t] - \mathcal{L}[(t-a)H(t-a)]\right\}.$$

Since $\mathcal{L}[t] = \frac{1}{s^2}$ (see the Table in Section 4.1.4), the Second Shift Theorem gives

$$\mathcal{L}[f_2(t)] = \frac{1}{a}\left[\frac{1}{s^2} - \frac{e^{-as}}{s^2}\right] = \frac{1}{as^2}(1-e^{-as}). \qquad \square$$

Another classic example is the *saw-tooth function*: (see figure 4.3)

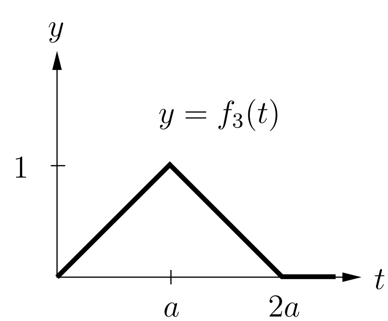


Figure 4.3: The saw-tooth function.

Exercise

Show that

$$f_3(t) = \tfrac{1}{a}[t - 2(t-a)H(t-a) + (t-2a)H(t-2a)].$$

Exercise

Show that

$$\mathcal{L}[f_3(t)] = \frac{1}{a}\left(\frac{1-e^{-as}}{s}\right)^2.$$

The Second Shift Theorem can be restated in terms of the inverse operator $\mathcal{L}^{-1}$, thereby providing a useful tool for calculating inverse Laplace transforms.

Second Shift Theorem (inverse form)

If $\mathcal{L}^{-1}[F(s)] = f(t)$, then $\mathcal{L}^{-1}[e^{-cs}F(s)] = H(t-c)f(t-c)$.

Example

Find $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s+3}\right]$.

Solution: We know $\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$, from the Table. Thus, by the Second Shift Theorem,

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s+3}\right] = H(t-2)e^{-3(t-2)} = \begin{cases} 0, & 0 \le t < 2 \\ e^{-3(t-2)}, & t \ge 2. \end{cases} \qquad \square$$

4.2 Solving DEs using the Laplace Transform

We have now developed the machinery needed to implement the algorithm outlined on the first page of this Chapter. In this Section we give examples to illustrate its use.

4.2.1 First order DEs

Example

Solve the first order initial value problem

$$y' + 3y = 13\sin(2t); \quad y(0) = y_0. \tag{4.23}$$

Solution: Apply the Laplace transform operator $\mathcal{L}$ to the DE and use linearity to obtain

$$\mathcal{L}[y'] + 3\mathcal{L}[y] = 13\mathcal{L}[\sin(2t)]. \tag{4.24}$$

By Proposition 1 in Section 4.1.3,

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0),$$

and by the Table in Section 4.1.4,

$$\mathcal{L}[\sin(2t)] = \frac{2}{s^2+4}.$$

Writing $\mathcal{L}[y(t)] = Y(s)$ as usual, and using the initial condition (4.23), equation (4.24) becomes

$$sY - y_0 + 3Y = \frac{26}{s^2+4}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{y_0}{s+3} + \frac{26}{(s+3)(s^2+4)}. \tag{4.25}$$

The next step is to calculate $y(t) = \mathcal{L}^{-1}[Y(s)]$, the inverse Laplace transform of $Y(s)$. Before applying $\mathcal{L}^{-1}$ to (4.25), we expand the second term in partial fractions (exercise):

$$\frac{26}{(s+3)(s^2+4)} = 2\left[\frac{1}{s+3} - \frac{s}{s^2+4} + \frac{3}{s^2+4}\right]. \tag{4.26}$$

Thus, applying $\mathcal{L}^{-1}$ to (4.25) and using (4.26) gives

$$\mathcal{L}^{-1}[Y(s)] = y_0\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] - 2\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] + 3\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right].$$

Referring to the Table, we get

$$y(t) = (y_0+2)e^{-3t} - 2\cos(2t) + 3\sin(2t).$$

as the solution of the initial value problem (4.23). □

Exercise

Solve the first order initial value problem

$$y' - y = 2e^t, \quad y(0) = y_0.$$

Answer: $y(t) = y_0e^t + 2te^t$. HINT: You will need to use the First Shift Theorem with $\mathcal{L}[t] = \frac{1}{s^2}$. □

Here's an example that involves arbitrary parameters.

Exercise

Solve the initial value problem:

$$y' + ky = kA\cos(\omega t); \quad y(0) = y_0.$$

Answer:

$$y(t) = \frac{kA}{k^2+\omega^2}\left[k(\cos\omega t - e^{-kt}) + \omega\sin\omega t\right] + y_0 e^{-kt}.$$

HINT: You will need the following partial fraction expansion:

$$\frac{s}{(s+k)(s^2+\omega^2)} = \frac{1}{k^2+\omega^2}\left[-\frac{k}{s+k} + \frac{ks}{s^2+\omega^2} + \frac{\omega^2}{s^2+\omega^2}\right].$$

Technical comment:

In applying $\mathcal{L}$ to the DE we used Proposition 1 in Section 4.1.3. You may ask: are the hypotheses of the Proposition satisfied in this application? We can be sure that H_2 is satisfied, since the solution of any constant coefficient linear DE with continuous input function is continuous and hence has a continuous derivative ($y' = -ky + f(t)$, both continuous). What about H_1: $y(t) = O(e^{at})$ as $t \to \infty$ for some a? We have no way of verifying this in advance, since $y(t)$ is the unknown function. So we proceed "on faith", and having found the solution, verify after the fact that the hypothesis is satisfied. □

4.2.2 Second order DEs

Example

Solve the second order initial value problem:

$$y'' + 2y' + 2y = 3e^{-t}; \quad y(0) = y_0, \quad y'(0) = 0. \tag{4.27}$$

Solution: Apply the operator $\mathcal{L}$ to the DE and use linearity to obtain

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] + 2\mathcal{L}[y] = 3\mathcal{L}[e^{-t}]. \tag{4.28}$$

As in Section 4.1.3 the transforms of the derivatives are

$$\mathcal{L}[y'] = sY(s) - y(0),$$
$$\mathcal{L}[y''] = s^2Y(s) - sy(0) - y'(0),$$

where $Y(s) = \mathcal{L}[y(t)]$, and the Table gives

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1}, \quad s > -1.$$

Thus (4.28) becomes

$$s^2Y - sy_0 + 2[sY - y_0] + 2Y = \frac{3}{s+1},$$

which is a *linear algebraic equation* for the unknown $Y(s)$. Collecting like terms gives

$$(s^2 + 2s + 2)Y(s) = (s+2)y_0 + \frac{3}{s+1}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{s+2}{s^2+2s+2}y_0 + \frac{3}{(s+1)(s^2+2s+2)}. \tag{4.29}$$

Since $s^2 + 2s + 2 = (s+1)^2 + 1$, we express both terms in terms of $(s+1)$. First,

$$\frac{s+2}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}. \tag{4.30}$$

Second,

$$\frac{3}{(s+1)(s^2+2s+2)} = \frac{3}{s+1} - \frac{3(s+1)}{(s+1)^2+1}. \tag{4.31}$$

Aside: The simple way to get (4.31) is to observe that

$$\frac{1}{u(u^2+1)} = \frac{1}{u} - \frac{u}{u^2+1},$$

and choose $u = (s+1)$.
We now use the First Shift Theorem, knowing

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \cos t, \quad \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t.$$

It follows from (4.30) and (4.31) that

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{s+2}{s^2+2s+2}\right] &= e^{-t}\cos t + e^{-t}\sin t, \\ \mathcal{L}^{-1}\left[\frac{3}{(s+1)(s^2+2s+2)}\right] &= 3e^{-t} - 3e^{-t}\cos t. \end{aligned} \tag{4.32}$$

We now apply the operator $\mathcal{L}^{-1}$ to (4.29) and use (4.32) to obtain

$$y(t) = \mathcal{L}^{-1}[Y(s)] = y_0e^{-t}(\cos t + \sin t) + 3e^{-t}(1 - \cos t),$$

as the solution of the initial value problem (4.27). □

Exercise

Solve

$$y'' + 2y' + y = t; \quad y'(0) = 1, \quad y(0) = 0.$$

Answer: $y = -2 + t + 2e^{-t}(1 + t)$. □

The examples in this Section illustrate one advantage of the Laplace transform method, namely that the initial conditions are automatically incorporated.

4.2.3 Discontinuous and Non-differentiable Inputs

A second advantage of the Laplace transform method is that non-smooth input functions e.g. step functions and saw-tooths, can be dealt with easily.

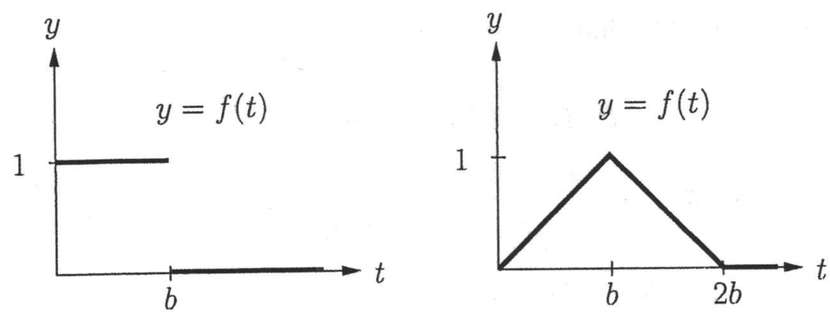


Figure 4.4: A step function and a saw-tooth function.

Consider a system (e.g. a mixing tank or electrical circuit) whose state $y(t)$ is governed by the DE

$$y' + y = f(t),$$

where $f(t)$ is the input function (assume that dimensionless variables have been introduced). If the input function is $f(t) = 0$, the solution is

$$y(t) = y(0)e^{-t},$$

showing that the system decays exponentially to its natural equilibrium state $y(t) = 0$. If the input function is a constant, say $f(t) = 1$, the solution is

$$y(t) = 1 + [y(0) - 1]e^{-t},$$

i.e. the system exponentially approaches a new equilibrium state $y(t) = 1$. The question we now ask is: what is the response $y(t)$ of the system if the input function is a step function:

$$f(t) = \begin{cases} 1, & 0 \le t < b \\ 0, & t \ge b, \end{cases} \tag{4.33}$$

i.e. a constant input of 1 for a finite time?

Example

Solve the initial value problem

$$y' + y = f(t); \quad y(0) = y_0, \tag{4.34}$$

where $f(t)$ is the step function (4.33).

Solution: We begin by writing the input function in terms of the Heaviside step function

$$f(t) = 1 - H(t - b). \tag{4.35}$$

Apply the operator $\mathcal{L}$ to the DE (4.34) with (4.35), using linearity to obtain

$$\mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[1] - \mathcal{L}[H(t - b)]. \tag{4.36}$$

We know from equation (4.20) that

$$\mathcal{L}[H(t - b)] = \frac{e^{-bs}}{s}, \quad s > 0$$

and from Section 4.1.3 that

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0).$$

Also, by equation (4.6)

$$\mathcal{L}[1] = \frac{1}{s}.$$

Thus, writing $\mathcal{L}[y] = Y(s)$ as usual, equation (4.36) becomes

$$sY - y_0 + Y = \frac{1}{s} - \frac{e^{-bs}}{s}.$$

Solving for $Y(s)$ leads to

$$Y(s) = \frac{y_0}{s + 1} + \frac{1 - e^{-bs}}{s(s + 1)}. \tag{4.37}$$

In order to find the inverse transform, we first use the partial fraction expansion

$$\frac{1}{s(s + 1)} = \frac{1}{s} - \frac{1}{s + 1}$$

to write (4.37) in the form

$$Y(s) = \frac{y_0 - 1}{s + 1} + \frac{1}{s} - e^{-bs}\left(\frac{1}{s} - \frac{1}{s + 1}\right). \tag{4.38}$$

We now use the Second Shift Theorem (Section 4.1.5). We know that

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s + 1}\right] = 1 - e^{-t},$$

from which follows

$$\mathcal{L}^{-1}\left[e^{-bs}\left(\frac{1}{s} - \frac{1}{s + 1}\right)\right] = H(t - b)(1 - e^{-t+b}).$$

Thus, applying $\mathcal{L}^{-1}$ to (4.38) gives

$$y(t) = \mathcal{L}^{-1}[Y(s)] = (y_0 - 1)e^{-t} + 1 - H(t - b)(1 - e^{-t+b}). \tag{4.39}$$

as the solution of the initial value problem (4.34). □

Analysis of the solution:

Without further analysis the solution (4.39) does not give one much insight. The solution can be written in the form

$$y(t) = \begin{cases} 1 + (y_0 - 1)e^{-t}, & 0 \le t < b \\ (y_0 - 1 + e^b)e^{-t}, & t \ge b. \end{cases} \tag{4.40}$$

The derivative is

$$y'(t) = \begin{cases} -(y_0 - 1)e^{-t}, & 0 \le t < b \\ -(y_0 - 1 + e^b)e^{-t}, & t > b. \end{cases} \tag{4.41}$$

Note that $y'(b)$ does not exist, but that $y(t)$ is continuous at $t = b$.

It follows from (4.40) that

$$\lim_{t\to\infty} y(t) = 0 \quad \text{for all values of} \quad y_0,$$

i.e. the system approaches its natural equilibrium state $y(t) = 0$ once the input is switched off at $t = b$. Equation (4.41) shows that the intermediate behaviour depends on y_0, and that there are two special values of y_0, namely

$$y_0 = 1 \qquad \text{and} \qquad y_0 = -e^b + 1.$$

By considering the sign of $y'(t)$, we reach the following conclusions.

- If $y_0 - 1 > 0$, then $y_0 - 1 + e^b > 0$ and $y(t)$ is *decreasing* for $t \ge 0$.
- If $y_0 - 1 + e^b < 0$, then $y_0 - 1 < 0$ and $y(t)$ is *increasing* for $t \ge 0$.
- If $-e^b + 1 < y_0 < 1$, then the response attains a maximum value

  $$y_{\max} = 1 + (y_0 - 1)e^{-b} > 0$$

  at $t = b$.

These results enable us to sketch the graphs in Figure 4.5.

Exercise

Solve $y' + y = f(t)$; $y(0) = y_0$, where f is defined in Figure 4.5. *Answer:* $y(t) = (y_0 - 1)e^{-t} + 1 - 2H(t-b)(1 - e^{-t+b}) + H(t - 2b)(1 - e^{-t+2b})$ □

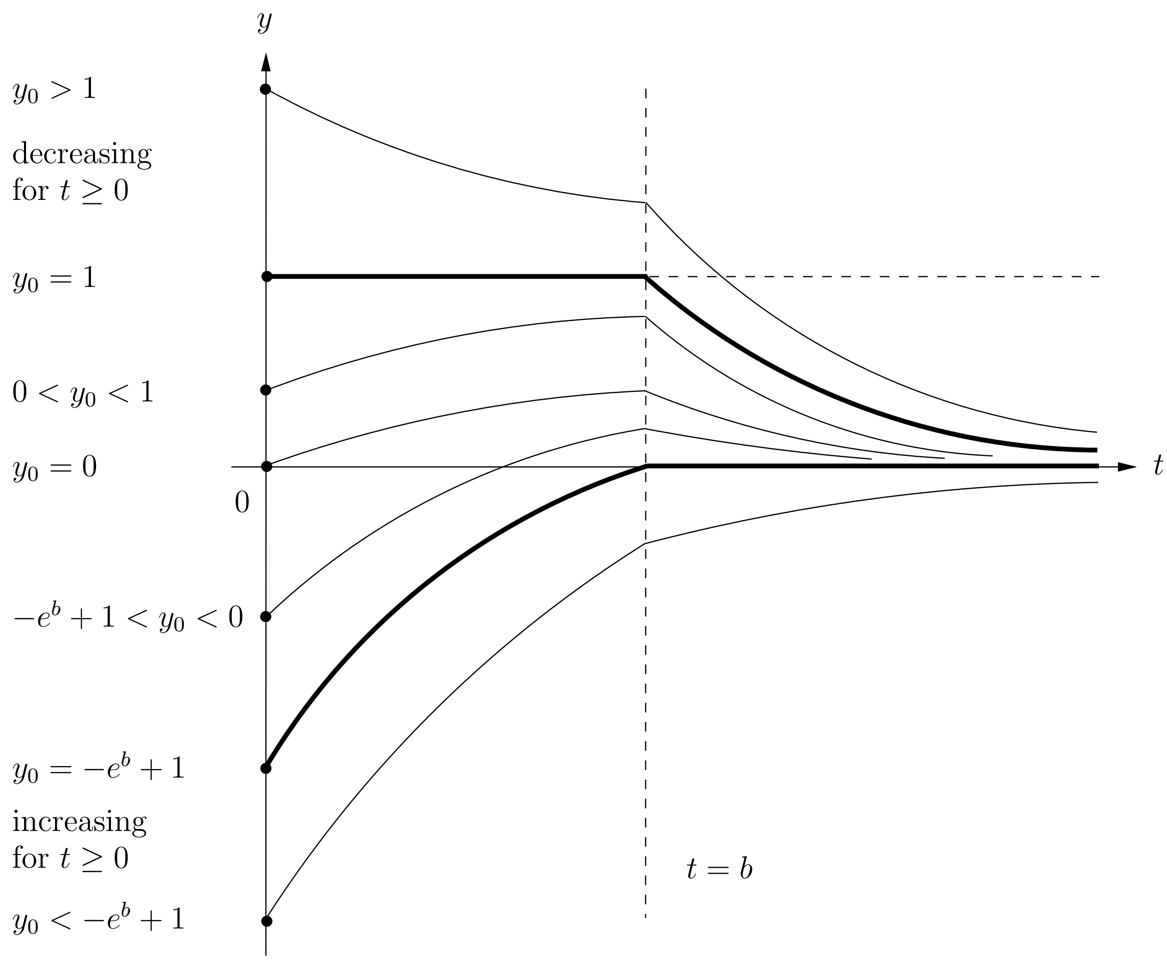


Figure 4.5: The family of solutions (4.40) with initial state y_0 as parameter. The exceptional solutions given by $y_0 = 1$ and $y_0 = -e^b + 1$ are shown in bold.

4.3 Convolution of functions

Note: Optional sections are covered at the instructor's discretion (see the preface). However, any students who are interested in programs such as Applied Mathematics or Mathematical Physics which require or prefer the advanced version of this course (AMath 251) should study this material.

In this section we introduce the *convolution operation* and discuss the so-called *Convolution Theorem.*

4.3.1 Motivation and definition

In order to motivate the need for the convolution operation we consider the second order DE

$$y'' + a_1 y' + a_0 y = u(t), \tag{4.42}$$

and assume zero initial conditions,

$$y(0) = 0, \qquad y'(0) = 0.$$

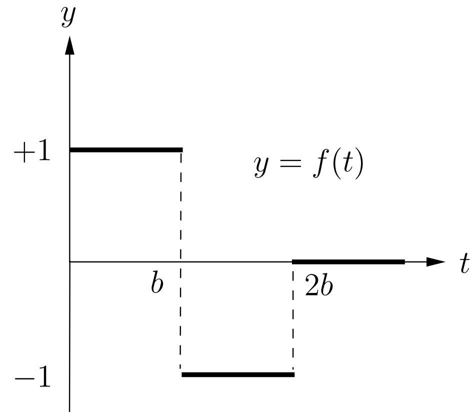


Figure 4.6: Input function for the exercise.

Apply the Laplace transform operator $\mathcal{L}$ to the DE and use linearity and equations (4.14) and (4.16). One obtains

$$Y(s)(s^2 + a_1 s + a_0) = U(s)$$

where $Y(s) = \mathcal{L}[y]$ and $U(s) = \mathcal{L}[u]$. It follows that

$$Y(s) = G(s)U(s), \tag{4.43}$$

where

$$G(s) = \frac{1}{s^2 + a_1 s + a_0}. \tag{4.44}$$

The solution of the DE is obtained by taking the inverse Laplace transform of $Y(s)$ in (4.43),

$$y(t) = \mathcal{L}^{-1}[G(s)U(s)]. \tag{4.45}$$

In order to proceed further, we have to answer the question: how is $\mathcal{L}^{-1}[G(s)U(s)]$ related to $\mathcal{L}^{-1}[G(s)]$ and $\mathcal{L}^{-1}[U(s)]$? An answer to this question will give a formula for the solution $y(t)$, since we know $\mathcal{L}^{-1}[U(s)] = u(t)$ and can calculate $\mathcal{L}^{-1}[G(s)]$.

The preceding question can be formulated in the following equivalent form:

$$\text{if} \qquad \mathcal{L}[f] = F(s) \quad \text{and} \quad \mathcal{L}[g] = G(s),$$

what function $h(t)$ satisfies

$$\mathcal{L}[h] = F(s)G(s) \quad ?$$

Consideration of simple functions shows that $h(t) \neq f(t)g(t)$, as one might initially hope. For example, if

$$f(t) = 1, \qquad g(t) = t,$$

then

$$F(s) = \frac{1}{s}, \qquad G(s) = \frac{1}{s^2},$$

but

$$\mathcal{L}[fg] = \mathcal{L}[t] = \frac{1}{s^2} \neq F(s)G(s).$$

In other words

$$\mathcal{L}[fg] \neq \mathcal{L}[f]\mathcal{L}[g].$$

In order to obtain the correct "product formula" for $\mathcal{L}$, one has to define a new type of product of functions, namely the convolution operation.

Definition

Let f, g be piecewise continuous functions on any interval $0 \leq t \leq r$. The *convolution of f and g*, denoted by $f * g$, is defined by

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau, \tag{4.46}$$

for $t \geq 0$.

We illustrate this idea with a simple example.

Example 1:

Calculate the convolution $t * 1$, and show that

$$\mathcal{L}[t * 1] = \mathcal{L}[t]\mathcal{L}[1]. \tag{4.47}$$

Solution: By the definition (4.46),

$$\begin{aligned} t * 1 &= \int_0^t (t-\tau)(1)d\tau \\ &= \left(t\tau - \tfrac{1}{2}\tau^2\right)\Big|_{t=0}^{t=\tau} \\ &= \tfrac{1}{2}t^2, \end{aligned}$$

after simplifying. Using the formula

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}},$$

we have

$$\mathcal{L}[t * 1] = \mathcal{L}\left[\tfrac{1}{2}t^2\right] = \frac{1}{s^3},$$

and

$$\mathcal{L}[t]\mathcal{L}[1] = \left(\frac{1}{s^2}\right)\left(\frac{1}{s}\right) = \frac{1}{s^3},$$

which gives the desired result (4.47).
This result is a special case of the Convolution Theorem, which we discuss in Section 4.3.2. □

Exercise

Calculate the convolution $e^t * 1$, and verify directly that

$$\mathcal{L}[e^t * 1] = \mathcal{L}[e^t]\mathcal{L}[1].$$

Properties of the convolution $f * g$:

As suggested by the notation $f * g$, the convolution of f and g should be regarded as a product operation on functions. It is, of course, completely different from the usual product of f and g, defined by

$$(fg)(t) = f(t)g(t).$$

Nevertheless it does satisfy the same properties as the usual product:

i) $f * g = g * f$ (commutativity)

ii) $f * (g + h) = f * g + f * h$ (distributivity)

iii) $f * (g * h) = (f * g) * h$ (associativity).

Here the operation $+$ is the usual addition of functions. We leave the proofs of i)-iii) as exercises.

4.3.2 The Convolution Theorem

In this Section we discuss the main theorem concerning the convolution operation.

Convolution Theorem:

If $\mathcal{L}[f] = F(s)$ and $\mathcal{L}[g] = G(s)$ exist for $Re(s) > a$, then

$$\mathcal{L}[f * g] = \mathcal{L}[f]\mathcal{L}[g]. \tag{4.48}$$

Discussion:

Thinking of the Laplace transform $\mathcal{L}$ as an operator that maps a function $f(t)$ on the half-line $0 \leq t < \infty$ (the time domain) onto a function $F(s)$ on the complex s-plane (the frequency domain), one can express the convolution theorem symbolically as follows:

$$\begin{array}{ll} \text{if} & f(t) \xrightarrow{\mathcal{L}} F(s) \quad \text{and} \quad g(t) \xrightarrow{\mathcal{L}} G(s), \\ \text{then} & (f * g)(t) \xrightarrow{\mathcal{L}} F(s)G(s). \end{array}$$

One can describe the theorem colloquially by saying that "convolution in the time domain corresponds to multiplication in the frequency domain", giving another example of the way in which an operation in the frequency domain is simpler than the corresponding operation in the time domain.

The convolution theorem can also be expressed in terms of the inverse Laplace transform.

Inverse form of the Convolution Theorem:

$$\mathcal{L}^{-1}[FG] = \mathcal{L}^{-1}[F] * \mathcal{L}^{-1}[G]. \tag{4.49}$$

We now give the proof of the Convolution Theorem.

Proof: By definition of the convolution operation and of the Laplace transform,

$$\begin{aligned}\mathcal{L}[f*g] &= \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right]dt \\ &= \int_0^\infty \left[\int_0^t e^{-st}f(t-\tau)g(\tau)d\tau\right]dt.\end{aligned}$$

Reversing the order of integration[2] as indicated in Figure 4.7 gives

$$\mathcal{L}[f*g] = \int_0^\infty \left[\int_\tau^\infty e^{-st}f(t-\tau)g(\tau)dt\right]d\tau.$$

Making the change of variable $t = \tau + r$ in the bracketed integral leads to

$$\begin{aligned}\mathcal{L}[f*g] &= \int_0^\infty \left[\int_0^\infty e^{-s(\tau+r)}f(r)g(\tau)dr\right]d\tau \\ &= \int_0^\infty e^{-s\tau}g(\tau)\left[\int_0^\infty e^{-sr}f(r)dr\right]d\tau \\ &= \mathcal{L}[f]\mathcal{L}[g],\end{aligned}$$

the last step following by definition of the Laplace transform. □

We now return to the problem of solving the DE (4.42), i.e.

$$y'' + a_1y' + a_0y = u(t),$$

with zero initial conditions, that we considered as motivation in Section 3.2.1. We obtained the solution in the form (4.45), i.e.

$$y(t) = \mathcal{L}^{-1}[G(s)U(s)],$$

where

$$G(s) = \frac{1}{s^2 + a_1s + a_0}, \tag{4.50}$$

and $U(s) = \mathcal{L}[u(t)]$. It now follows immediately from the inverse form of the Convolution Theorem (4.49) that

$$y(t) = (g*u)(t), \tag{4.51}$$

where

$$g(t) = \mathcal{L}^{-1}[G(s)]. \tag{4.52}$$

[2]Proving that the limits of integration can be reversed for improper double integrals requires a discussion of uniform convergence for improper integrals, and is beyond the scope of this course. We refer to Churchill, Operational Mathematics.

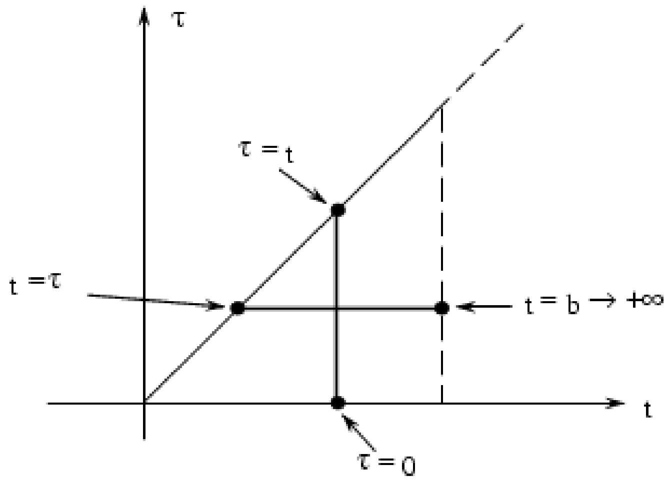


Figure 4.7: The inequalities $0 \leq \tau \leq t$ and $0 \leq t < +\infty$ describe the same region as the inequalities $\tau \leq t < +\infty$ and $0 \leq \tau < +\infty$.

Using the definition (4.46) of the convolution, (4.51) assumes the form

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau. \tag{4.53}$$

We now summarize the result.

Summary:

The unique solution of the initial value problem

$$y'' + a_1 y' + a_0 y = u(t),$$

$$y(0) = 0, \quad y'(0) = 0,$$

is

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau,$$

where

$$g(t) = \mathcal{L}^{-1}[G(s)],$$

and

$$G(s) = \frac{1}{s^2 + a_1 s + a_0}. \qquad \square$$

The form of $g(t)$ depends on the constants a_1 and a_0, as shown in the following example. It is worth noting that $G(s)$ can be written in the form

$$G(s) = \frac{1}{p(s)}, \tag{4.54}$$

where

$$p(\lambda) = \lambda^2 + a_1\lambda + a_0, \tag{4.55}$$

is the characteristic polynomial of the given DE.

Example 2:

Express the solution of the initial value problem

$$y'' + \omega^2 y = u(t),$$

$$y(0) = 0, \quad y'(0) = 0,$$

where the input function $u(t)$ is (piecewise) continuous, as a convolution.

Solution:
We know from (4.43) that the Laplace transform of $y(t)$ has the form

$$Y(s) = G(s)U(s),$$

where $U(s) = \mathcal{L}[u(t)]$ and

$$G(s) = \frac{1}{s^2 + \omega^2}.$$

Using the fact that

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2},$$

it follows that

$$g(t) = \mathcal{L}^{-1}[G(s)] = \frac{1}{\omega} \sin \omega t.$$

By the inverse form of the Convolution Theorem, the solution of the initial value problem is

$$y(t) = (g * u)(t),$$

i.e.

$$y(t) = \frac{1}{\omega} \int_0^t \sin \omega (t - \tau) u(\tau) d\tau. \qquad \square$$

4.4 Application to linear time-invariant systems: The Transfer Function

A *linear time-invariant system* (LTI) is a system whose evolution in time is described by a linear DE with constant coefficients.

In Section 4.3.1 we showed that for a scalar DE

$$y'' + a_1 y' + a_0 y = u(t),$$

with initial conditions

$$y(0) = 0, \quad y'(0) = 0,$$

the Laplace transform $Y(s) = \mathcal{L}[y(t)]$ of the solution, i.e., the response of the physical system, is given by

$$Y(s) = G(s)U(s), \tag{4.56}$$

where $U(s) = \mathcal{L}[u(t)]$ is the Laplace transform of the input, and

$$G(s) = \frac{1}{s^2 + a_1 s + a_0}. \tag{4.57}$$

The solution itself is given as a convolution

$$y(t) = (g * u)(t),$$

where

$$g(t) = \mathcal{L}^{-1}[G(s)],$$

(see (4.51)). Written in full using the definition of convolution the solution is

$$y(t) = \int_0^t g(t - \tau)u(\tau)d\tau. \tag{4.58}$$

Equation (4.58) gives the dependence of the response on the input in the t-domain (the *time-domain*), while (4.56) gives this dependence in the s-domain (called the *frequency domain*). The key point is that this dependence is particularly simple in the s-domain, i.e. *multiplication by the function* $G(s)$, which is directly determined by the coefficients of the DE through equation (4.57). This function is called the *transfer function* of the physical system.

More generally, the right hand side of the DE will be a linear combination of $u(t)$ and its derivatives; the transfer function is defined using (4.56) as

$$G(s) = \frac{Y(s)}{U(s)}$$

which will be a rational function.

The importance of the transfer function, in addition to the simplicity of the relation (4.56), lies in the fact that knowing $G(s)$ one can obtain information about the behaviour of the physical system. An important example is to obtain the response of the system to a sinusoidal input

$$u(t) = Re[\alpha e^{i\omega t}], \tag{4.59}$$

of angular frequency ω and amplitude α. We consider a trial solution

$$y(t) = Re[\mathcal{A} e^{i\omega t}], \tag{4.60}$$

where $\mathcal{A}$ is a complex number which will depend on the frequency ω. On substituting the complex input $\alpha e^{i\omega t}$ and complex trial solution $\mathcal{A} e^{i\omega t}$ into the DE, we obtain

$$\mathcal{A}(\omega)\left[(i\omega)^2 + a_1(i\omega) + a_0\right] = \alpha,$$

after cancelling a common factor $e^{i\omega t}$. On replacing s by $i\omega$ in (4.57) we see that the preceding equation can be written in the form

$$\mathcal{A}(\omega) = G(i\omega)\alpha,$$

where

$$G(i\omega) = \frac{1}{(i\omega)^2 + a_1(i\omega) + a_0}. \tag{4.61}$$

We can express $G(i\omega)$ in terms of its magnitude and argument

$$G(i\omega) = \rho(\omega)e^{i\delta(\omega)}. \tag{4.62}$$

Substituting (4.62) into (4.60) gives the solution

$$y(t) = \alpha\rho(\omega)\cos\left[\omega t + \delta(\omega)\right]. \tag{4.63}$$

This equation, with (4.62) and (4.61), shows how the amplitude $\rho(\omega)$ and phase shift $\delta(\omega)$ of the response depends on the frequency ω.

The classic example is the DE

$$y'' + 2\lambda y' + \omega_0^2 y = u(t), \tag{4.64}$$

where λ and ω are positive constants, which can represent a mechanical oscillator or an electrical RLC circuit. The transfer function is thus

$$G(s) = \frac{1}{s^2 + 2\lambda s + \omega_0^2}. \tag{4.65}$$

It is customary to plot $20\log_{10}\rho$ versus $\log_{10}\omega$, and δ versus $\log_{10}\omega$. The quantity

$$dB = 20\log_{10}\rho$$

gives a decibel measure of the amplitude. These graphs are shown in Figure 4.8.

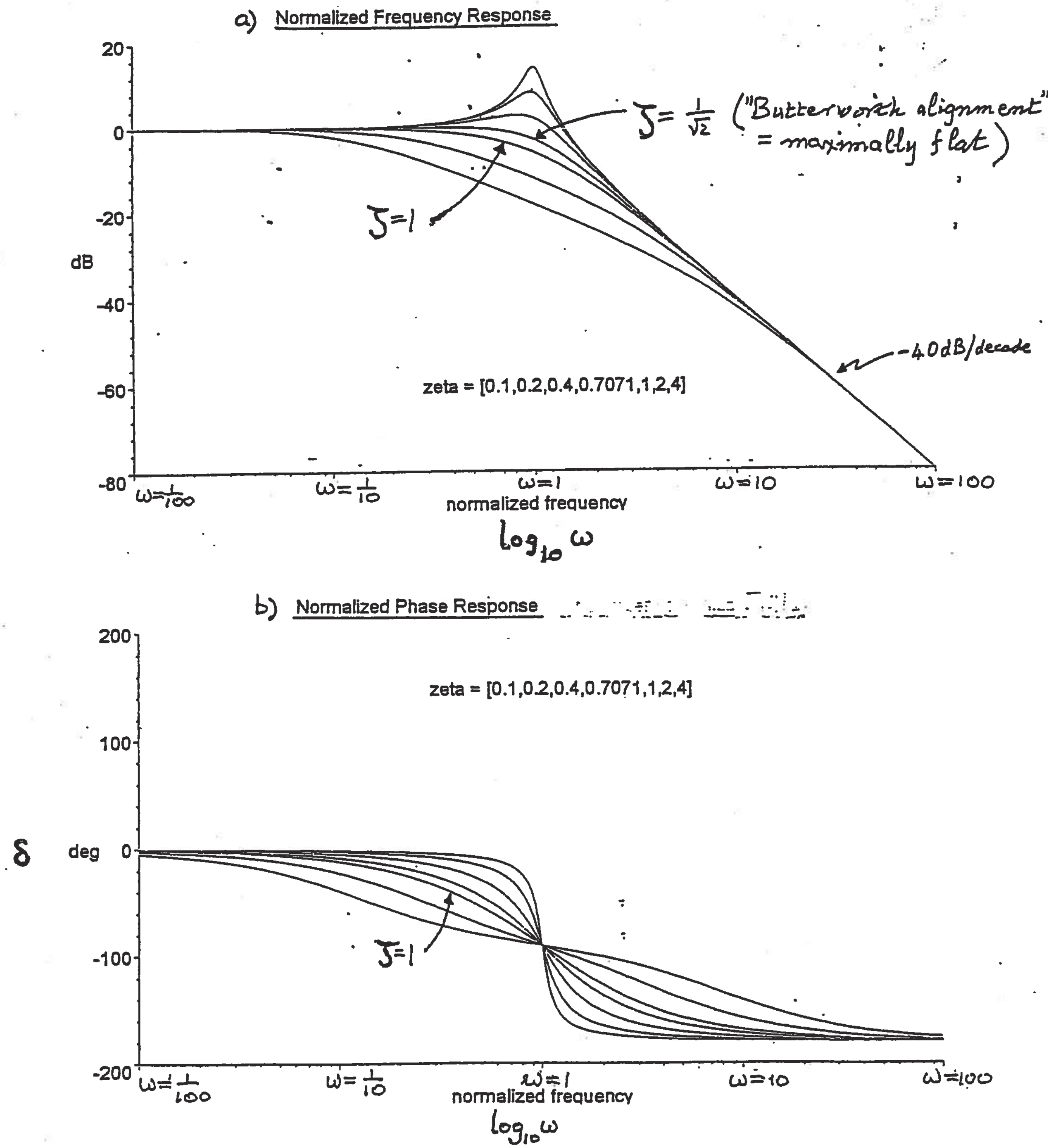


Figure 4.8: Frequency response showing $dB = 20\log_{10}\rho$ versus $\log_{10}\omega$ and phase response showing δ versus $\log_{10}\omega$.

4.5 Response to an impulsive input

In this section we give an introduction to the notion of an *impulsive force*, which can be thought as a very large force that acts over a very short time interval, for example the impact of a club on a golf ball. More generally, one can think of an *impulsive input* to any linear time-invariant system, for example, an electrical circuit. Our goal is to be able to determine the response of the system to an impulsive input.

4.5.1 Impulsive forces of finite duration

Suppose that an impulsive force acts, during the time interval $t_1 \leq t \leq t_2$ on a particle of mass m in linear motion. Newton's Second Law reads

$$\frac{d}{dt}(mv) = f(t),$$

which when integrated from t_1 to t_2 yields

$$mv(t_2) - mv(t_1) = \int_{t_1}^{t_2} f(t)dt.$$

Thus the principal effect of the impulsive force $f(t)$ is a sudden change in momentum, given by the integral

$$\int_{t_1}^{t_2} f(t)dt$$

called the *impulse of the force over the interval* $t_1 \leq t \leq t_2$.

As a mathematical model of an impulsive force of unit impulse acting at time $t = 0$ we take the function $\delta_\varepsilon(t)$, defined by

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & \text{if} \quad 0 \leq t < \varepsilon \\ 0, & \text{otherwise.} \end{cases} \tag{4.66}$$

The translate $\delta_\varepsilon(t-a)$ represents an impulsive force acting at time $t = a$. The graphs of these functions are shown in Figure 4.9.

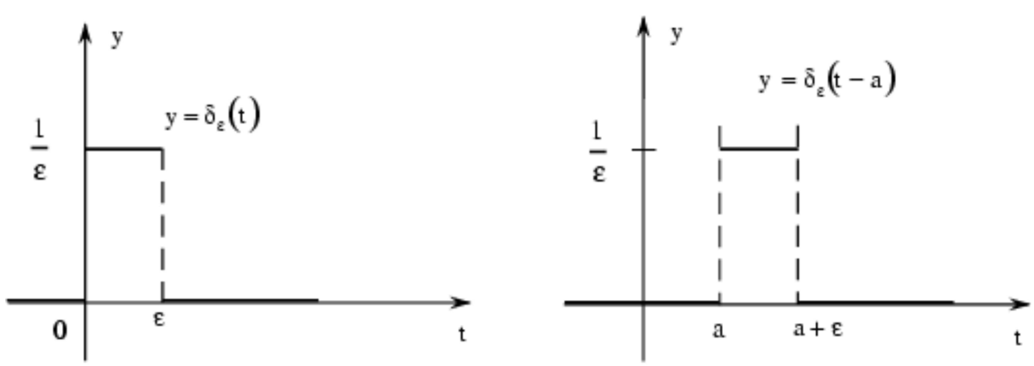


Figure 4.9: Graphs of the impulse functions $\delta_\varepsilon(t)$ and $\delta_\varepsilon(t-a)$.

Thinking of $\delta_\varepsilon(t-a)$ is a one-parameter family of functions labelled by $\varepsilon > 0$, we ask whether the limit $\lim\limits_{\varepsilon\to 0^+} \delta_\varepsilon(t-a)$ defines a function. Referring to Figure 4.9, we see that

$$\lim_{\varepsilon\to 0^+} \delta_\varepsilon(t-a) = \begin{cases} 0 & \text{if} \quad t \neq a \\ +\infty & \text{if} \quad t = a. \end{cases} \tag{4.67}$$

Thus the limit does not define a function. However the limit of the integral of the impulse function is well-defined, since[3]

$$\int_0^\infty \delta_\varepsilon(t-a)dt = 1$$

for all $\varepsilon > 0$ and $a \geq 0$.

More generally we have the following result, called the "sifting property".

Proposition

If g is continuous on some neighbourhood of a, then

$$\lim_{\varepsilon\to 0^+} \int_0^\infty g(t)\delta_\varepsilon(t-a)dt = g(a), \tag{4.68}$$

where $\delta_\epsilon(t)$ is the impulse function (4.66).

Proof: By (4.66)

$$\int_0^\infty g(t)\delta_\varepsilon(t-a)dt = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} g(t)dt.$$

For ε sufficiently close to zero, we can apply the mean value theorem for integrals, obtaining

$$\frac{1}{\varepsilon}\int_a^{a+\varepsilon} g(t)dt = g(\bar{t}),$$

where $a < \bar{t} < a+\varepsilon$. Since g is continuous at a,

$$\lim_{\varepsilon\to 0^+} g(\bar{t}) = g(a),$$

and the result follows. $\square$

We now solve a problem involving an impulsive force in order to motivate the future developments.

Example 3:

Find the zero state response of an undamped mass-spring system to a constant impulsive force of duration ε and total impulse p acting at time $t=0$. Show the limit of

[3] From Figure 4.9 the area under the graph of $y = \delta_\varepsilon(t-a)$ equals 1, for any $\varepsilon > 0$.

the response exists as $\varepsilon \to 0^+$.

Solution:
The equation of motion is

$$my'' + ky = p\delta_\varepsilon(t),$$

with

$$y(t) = 0 \quad \text{and} \quad y'(t) = 0 \quad \text{for} \quad t \le 0.$$

Dividing by m gives

$$y'' + \omega^2 y = \frac{p}{m}\delta_\varepsilon(t). \tag{4.69}$$

Using the result of Example 2, with input

$$u_\varepsilon(t) = \frac{p}{m}\delta_\varepsilon(t),$$

we get[a]

$$y_\varepsilon(t) = \frac{p}{m\omega}\int_0^t \sin\,\omega(t-\tau)\delta_\varepsilon(\tau)d\tau. \tag{4.70}$$

By (4.66)

$$y_\varepsilon(t) = \frac{p}{m\omega\varepsilon}\begin{cases} \displaystyle\int_0^\varepsilon \sin\omega(t-\tau)d\tau, & \text{if} \quad t \ge \varepsilon \\ \displaystyle\int_0^t \sin\omega(t-\tau)d\tau, & \text{if} \quad 0 \le t < \varepsilon. \end{cases}$$

Evaluating the integrals gives

$$y_\varepsilon(t) = \frac{p}{m\omega^2\varepsilon}\begin{cases} \cos\omega(t-\varepsilon) - \cos\omega t, & \text{if} \quad t \ge \varepsilon \\ 1 - \cos\omega t, & \text{if} \quad 0 \le t < \varepsilon. \end{cases} \tag{4.71}$$

The first and second derivatives are

$$y_\varepsilon'(t) = \frac{p}{m\omega\varepsilon}\begin{cases} \sin\omega t - \sin\omega(t-\varepsilon), & \text{if} \quad t \ge \varepsilon \\ \sin\omega t, & \text{if} \quad 0 \le t < \varepsilon \end{cases} \tag{4.72}$$

and

$$y_\varepsilon''(t) = \frac{p}{m\varepsilon}\begin{cases} \cos\omega t - \cos\omega(t-\varepsilon), & \text{if} \quad t > \varepsilon \\ \cos\omega t, & \text{if} \quad 0 < t < \varepsilon. \end{cases} \tag{4.73}$$

We now show that $\lim_{\varepsilon\to 0^+} y_\varepsilon(t)$ exists for any $t > 0$. Given $t > 0$ we choose ε close enough to zero to ensure that $0 < \varepsilon < t$. Then the first line in (4.71) applies. By l'Hôpital's rule,

$$\lim_{\varepsilon\to 0^+}\frac{\cos\omega(t-\varepsilon) - \cos\omega t}{\omega\varepsilon} = \lim_{\varepsilon\to 0^+}\frac{\omega\sin\omega(t-\varepsilon)}{\omega} = \sin\omega t.$$

It follows that[b]

$$\lim_{\varepsilon \to 0^+} y_\varepsilon(t) = \frac{p}{m\omega} \sin \omega t,$$

for any $t > 0$. We thus regard the function

$$y(t) = \frac{p}{m\omega} \sin \omega t, \quad t > 0, \tag{4.74}$$

with

$$y(t) = 0, \qquad t < 0, \tag{4.75}$$

as the response of the mass-spring system to an (instantaneous) impulse of magnitude p/m per unit mass.

The graphs of $y_\varepsilon(t)$ and $y(t)$ and their first two derivatives are shown in Figure 4.10. We note that $y_\varepsilon(t)$ is of class C^1, i.e. $y'_\varepsilon(t)$ is continuous at $t = 0$, while $y(t)$ is only continuous: observe that $y'(t)$ has a jump discontinuity at $t = 0$. The idealized impulse (i.e. $\varepsilon \to 0^+$) transfers momentum instantaneously to the mass at time $t = 0$. The function $y(t)$ does not describe the physical behaviour of the acceleration at $t = 0$, which is given by the limit of $y''_\varepsilon(t)$ as $\varepsilon \to 0^+$. Evaluating $y''_\varepsilon(t)$ at the centre of the ε-interval and letting $\varepsilon \to 0^+$ gives

$$\lim_{\varepsilon \to 0^+} y''_\varepsilon\left(\frac{\varepsilon}{2}\right) = \lim_{\varepsilon \to 0^+} \left(\frac{p}{m}\right) \frac{\cos \frac{1}{2}\omega\varepsilon}{\varepsilon} = +\infty.$$

This singular behaviour is indicated by the vertical arrow at $t = 0$ in the graph of $y''(t)$ in Figure 4.10.

[a]We denote the solution by $y_\varepsilon(t)$ to emphasize the dependence on ε.

[b]This result can be obtained more quickly by applying the proposition to equation (4.70).

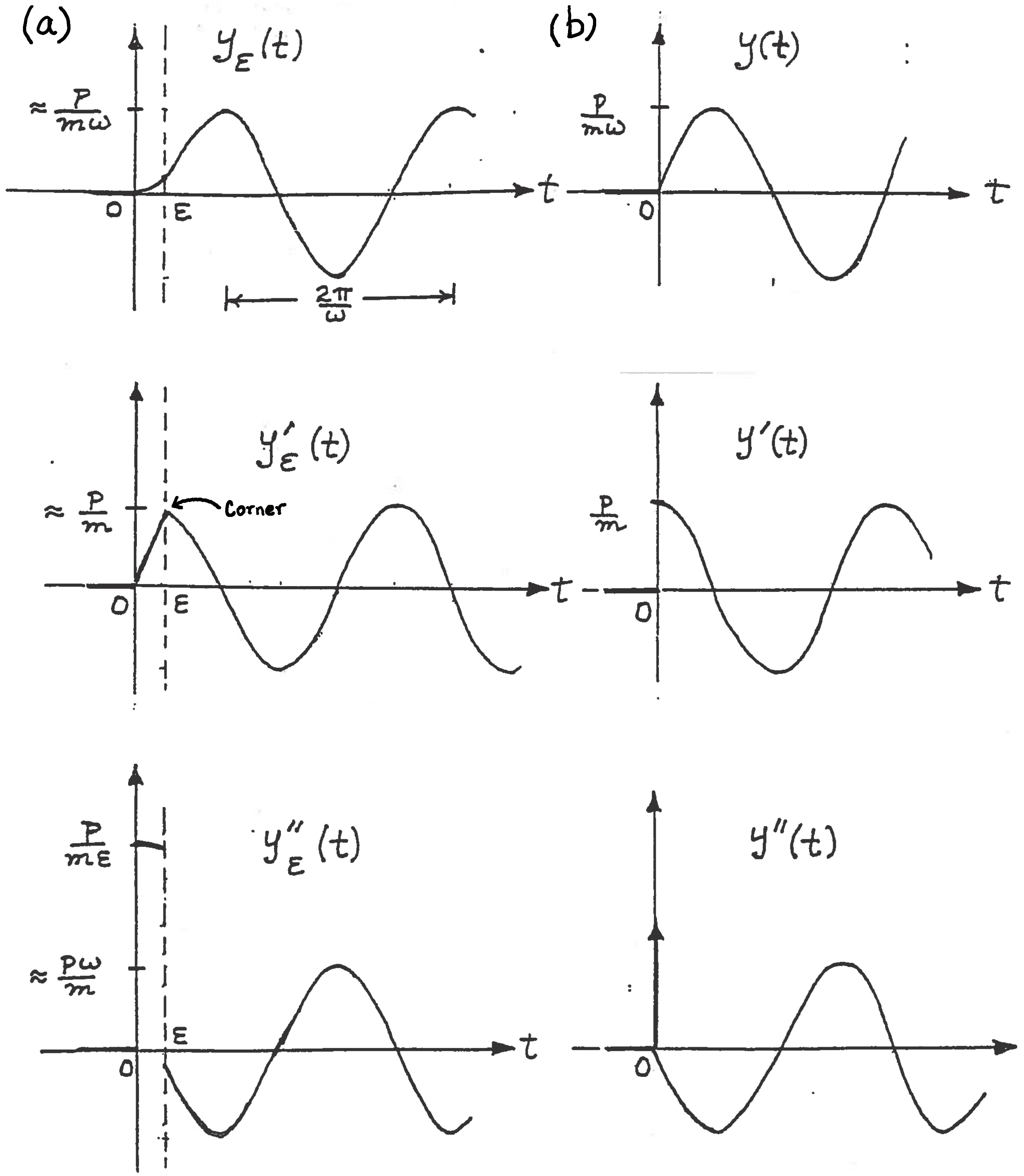


Figure 4.10: Response of an undamped mass-spring system to an impulsive force of impulse p/m per unit mass, a) for finite duration $\varepsilon > 0$ and b) in the limit as $\varepsilon \to 0$.

4.5.2 The Dirac delta symbol

In this section we develop the formal procedure that is used for calculating the response of a linear time-invariant system to an impulsive input, in the limit as $\varepsilon \to 0^+$, where ε is the duration of the impulse.

We have seen that the family of impulse functions $\delta_\varepsilon(t-a)$, defined by (4.66), satisfy (4.67), i.e.

$$\lim_{\varepsilon \to 0^+} \delta_\varepsilon(t-a) = \begin{cases} 0 & \text{if} \quad t \neq a \\ +\infty & \text{if} \quad t = a. \end{cases}$$

Although this limit does not exist it is convenient to represent it by the symbol

$$\delta(t-a),$$

called the *Dirac delta symbol*. Using this notation we write (4.68) symbolically as

$$\int_0^\infty g(t)\delta(t-a)dt = g(a), \tag{4.76}$$

where g is any continuous function, and $a \geq 0$. In particular, for $g(t) = 1$,

$$\int_0^\infty \delta(t-a)dt = 1,$$

for $a \geq 0$.

Physically we think of $\delta(t)$ as representing an idealized impulsive force with unit impulse and zero duration, acting at time $t = 0$. More generally, one can think of $\delta(t-a)$ as an impulsive input of unit magnitude, acting at time $t = a$. Thinking in these terms, we write the DE (4.69) in the limit $\varepsilon \to 0^+$ symbolically as

$$y'' + \omega^2 y = \frac{p}{m}\delta(t). \tag{4.77}$$

It is natural to ask whether one can solve this equation to obtain (4.74) by using the Laplace transform. Equation (4.76) with $g(t) = e^{-st}$ reads

$$\int_0^\infty e^{-st}\delta(t-a)dt = e^{-sa},$$

which motivates the following *definition of the Laplace transform of the Dirac delta symbol*:

$$\mathcal{L}[\delta(t-a)] = e^{-as}, \tag{4.78}$$

for $a \geq 0$. In particular, for $a = 0$,

$$\mathcal{L}[\delta(t)] = 1. \tag{4.79}$$

Alternative derivation of equation (4.78):

The idea is to think of $\mathcal{L}[\delta(t-a)]$ as $\lim_{\varepsilon\to 0^+} \mathcal{L}[\delta_\varepsilon(t-a)]$.

Since

$$\mathcal{L}[\delta_\varepsilon(t-a)] = \begin{cases} 0 & \text{for } t < a \\ \frac{1}{\varepsilon} & \text{for } a \le t < a+\varepsilon \\ 0 & \text{for } t \ge a+\varepsilon \end{cases},$$

we have

$$\begin{aligned} \mathcal{L}[\delta_\varepsilon(t-a)] &= 0 + \left(\frac{1}{\varepsilon} - 0\right) H(t-a) + \left(0 - \frac{1}{\varepsilon}\right) H(t-(a-\varepsilon)) \\ &= \frac{1}{\varepsilon}(H(t-a) - H(t-(a+\varepsilon)) \end{aligned}$$

where H is the Heaviside step function. By the second shift theorem,

$$\begin{aligned} \mathcal{L}[\delta_\varepsilon(t-a)] &= \frac{1}{\varepsilon}\left[\frac{1}{s}e^{-as} - \frac{1}{s}e^{-(a+\varepsilon)s}\right] \\ &= e^{-as} \cdot \frac{1-e^{-\varepsilon s}}{\varepsilon s} \\ &\to e^{-as} \text{ as } \varepsilon \to 0^+ \end{aligned}$$

by L'Hôpital's rule or Taylor expansion.

We now rework Example 3 using the Dirac delta symbol.

Example 4:

Find the zero state response of an undamped mass-spring system to an impulse of magnitude p/m per unit mass, acting at time $t = 0$.

Solution:
The equation of motion is (4.77), i.e.

$$y'' + \omega^2 y = \frac{p}{m}\delta(t-0) = \frac{p}{m}\delta(t),$$

with

$$y(0) = 0 = y'(0). \tag{4.80}$$

Take the Laplace transform of the DE using (4.16), (4.79) and (4.80) to obtain

$$s^2Y(s) + \omega^2 Y(s) = \frac{p}{m}e^{0s} = \frac{p}{m},$$

where $Y(s) = \mathcal{L}[y(t)]$. Solving for $Y(s)$ gives

$$Y(s) = \left(\frac{p}{m}\right)\frac{1}{s^2+\omega^2}.$$

Taking the inverse Laplace transform and recalling

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} \tag{4.81}$$

gives

$$y(t) = \frac{p}{m\omega} \sin \omega t,$$

for $t > 0$, the same expression as found by solving the problem with $\varepsilon > 0$ and then taking the limit as $\varepsilon \to 0^+$.

Example 5:

An undamped mass-spring system of natural frequency ω is initially at rest. At each time $t = 0, \frac{\pi}{\omega}, \frac{2\pi}{\omega}, \ldots$ the mass is struck with a hammer, which imparts an impulse of magnitude p per unit mass in the positive direction. Determine the resulting motion.

Solution: The DE is

$$y'' + \omega^2 y = p \sum_{n=0}^{\infty} \delta\left(t - \frac{n\pi}{\omega}\right), \tag{4.82}$$

with initial conditions

$$y(0) = 0, \quad y'(0) = 0.$$

Applying the Laplace transform as in Example 4 and using (4.78) gives

$$s^2 Y(s) + \omega^2 Y(s) = p \sum_{n=0}^{\infty} e^{-\frac{n\pi s}{\omega}}$$

and hence

$$Y(s) = p \sum_{n=0}^{\infty} \frac{e^{-\frac{n\pi s}{\omega}}}{s^2 + \omega^2}. \tag{4.83}$$

By the Second Shift Theorem and equation (4.81),

$$\mathcal{L}^{-1}\left[\frac{e^{-\frac{n\pi s}{\omega}}}{s^2 + \omega^2}\right] = \frac{1}{\omega} H\left(t - \frac{n\pi}{\omega}\right) \sin(\omega t - n\pi).$$

Thus applying $\mathcal{L}^{-1}$ to (4.83) gives the response in the form

$$y(t) = \left(\frac{p}{\omega}\right) \sum_{n=0}^{\infty} H\left(t - \frac{n\pi}{\omega}\right) \sin(\omega t - n\pi).$$

Since $\sin(\omega t - n\pi) = (-1)^n \sin \omega t$ and $H\left(t - \frac{n\pi}{\omega}\right) = 0$ for $t < \frac{n\pi}{\omega}$, it follows that if $\frac{n\pi}{\omega} < t < \frac{(n+1)\pi}{\omega}$, then

$$y(t) = \frac{p}{\omega} \left[\sin \omega t - \sin \omega t + \sin \omega t + \cdots + (-1)^n \sin \omega t\right],$$

i.e.

$$y(t) = \begin{cases} \frac{p}{\omega}\sin\omega t, & \text{if} \quad n \quad \text{is even} \\ 0, & \text{if} \quad n \quad \text{is odd.} \end{cases}$$

The graph of the response $y(t)$ is shown in Figure 4.11. The physical interpretation is as follows: the first blow starts the mass moving in the positive direction; just as it returns to the origin the second blow stops it dead; it remains at rest until the third blow sets it in motion again, and so on.

□

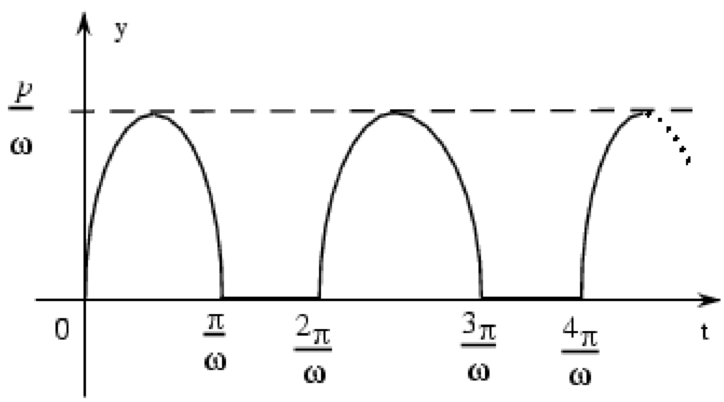


Figure 4.11: Response $y(t)$ from example 5.

Warning:

It should be kept in mind that the calculations in Section 4.5.2 are of a formal nature, since the Dirac delta symbol $\delta(t-a)$ is not a well-defined mathematical quantity: the limit (4.67) does not exist and hence $\delta(t-a)$ *is not a function*. Thus, statements involving $\delta(t-a)$, have to be interpreted as limits. In particular

$$\text{“}\mathcal{L}[\delta(t)] = 1\text{”} \quad \text{means} \quad \lim_{\varepsilon\to0^+}\mathcal{L}[\delta_\varepsilon(t)] = 1,$$

$$\text{“}\int_0^\infty g(t)\delta(t-a)dt = g(a)\text{”} \quad \text{means} \quad \lim_{\varepsilon\to0^+}\int_0^\infty g(t)\delta_\varepsilon(t-a)dt = g(a),$$

and

$$\text{“}y(t) \quad \text{is a solution of the initial value problem}$$

$$y'' + \omega^2 y = \delta(t), \qquad y(0) = 0, \qquad y'(0) = 0\text{”},$$

means

$$y(t) = \lim_{\varepsilon\to0^+} y_\varepsilon(t), \quad \text{where} \quad y_\varepsilon(t) \quad \text{is a solution of}$$

$$y'' + \omega^2 y = \delta_\varepsilon(t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

The final results of the calculations are, however, mathematically valid, since they can be derived rigorously by using the pulse function $\delta_\varepsilon(t-a)$ and then taking the limit as $\varepsilon \to 0^+$ once the solution of the DE has been found. We illustrated this process in Examples 3 and 4, first giving the rigorous method with $\varepsilon > 0$, and then giving the quick formal solution.

We finally note that the Dirac delta symbol has been placed on a sound mathematical footing[4] by introducing the class of "generalized functions" or "distributions". In this setting one can write statements such as

$$H'(t-a) = \delta(t-a),$$

formally capturing the intuitive idea that the unit step function has an "infinite rate of change" at the step.

4.5.3 The unit impulse response

We have seen that in the frequency domain the response $Y(s)$ is related to the input $U(s)$ in a particularly simple way, namely, multiplication by the transfer function $G(s)$:

$$Y(s) = G(s)U(s). \tag{4.84}$$

We have seen in Section 4.5.2 that it is useful to consider an *idealized unit impulse*, denoted by $\delta(t)$, as the input, since the response is mathematically simpler than the response to an impulse of duration ε. Using the definition

$$\mathcal{L}[\delta(t)] = 1,$$

we get $U(s) = 1$, and so (4.84) reduces to

$$Y(s) = G(s).$$

Thus the response in the time domain to the idealized unit impulse $\delta(t)$ is

$$y(t) = \mathcal{L}^{-1}[G(s)] = g(t),$$

for $t > 0$. The function $g(t)$ is thus called the *unit impulse response* of the linear time-invariant system. In the time domain, the response $y(t)$ to an input $u(t)$ is given by

$$y(t) = (g * u)(t) = \int_0^t g(t-\tau)u(\tau)d\tau. \tag{4.85}$$

Thus (4.85) shows that *the unit impulse response* $g(t)$ *determines the response to an arbitrary input* $u(t)$ via the convolution operation.

Example 6:

Consider the undamped mass-spring system with equation of motion

$$y'' + \omega^2 y = u(t)$$

[4] By Laurent Schwarz, in the 1950's.

with $y(0) = 0 = y'(0)$. Following example 4, we see that the response to a unit impulse $u(t) = \delta(t)$ is

$$y_{\text{impulse}}(t) = g(t) = \frac{1}{\omega}\sin\omega t$$

(just set $\frac{p}{m} = 1$). Find the **step response**, i.e. the response to a unit step (Heaviside) input $u(t) = H(t)$.

Solution: By (4.85), the response is

$$\begin{aligned} y_{\text{step}}(t) &= \int_0^t g(t-\tau)u(\tau)d\tau \\ &= \int_0^t \tfrac{1}{\omega}\sin\omega(t-\tau)H(\tau)d\tau \\ &= \tfrac{1}{\omega}\int_0^t \sin\omega(t-\tau)d\tau \\ &= \tfrac{1}{\omega^2}\cos\omega(t-\tau)\big|_0^t \\ &= \tfrac{1}{\omega^2}(1-\cos\omega t). \end{aligned}$$

Chapter 5

Linear Vector Differential Equations (Systems of DEs)

In this Chapter we generalize the concept of differential equations to the case where the unknown is a vector function in $\mathbb{R}^2$. We begin by discussing two familiar physical systems from a different point of view, in order to motivate the idea.

5.1 Introduction

5.1.1 Coupled Mixing Tanks

Consider a system of two coupled mixing tanks each of volume V, with flow rates shown in figure 5.1, and constant inflow concentration c. Let $m_1(t)$ and $m_2(t)$ denote the mass of chemical in the two tanks at time t, respectively. Introduce a characteristic time $t_c = V/f$ and a dimensionless time $\tau = t/t_c$ (see equation (2.11)). Then the mass balance equation (1.49) applied to each tank separately leads to the two DEs

$$\begin{aligned} m_1' &= -2m_1 + m_2 + cV \\ m_2' &= 2m_1 - 2m_2, \end{aligned}$$

where $'$ denotes differentiation with respect to τ. Fill in the details as an exercise, referring to Sections 1.3.1 and 2.1.3, if necessary. We describe the *state of this system* at time τ by the vector

$$\mathbf{x}(\tau) = \begin{pmatrix} m_1(\tau) \\ m_2(\tau) \end{pmatrix} \in \mathbb{R}^2.$$

The two scalar DEs can be written as one vector DE

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}' = \begin{pmatrix} -2 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} cV \\ 0 \end{pmatrix}, \tag{5.1}$$

using the 2×2 coefficient matrix. Using the state vector $\mathbf{x}$ we write (5.1) in vector notation as

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}, \tag{5.2}$$

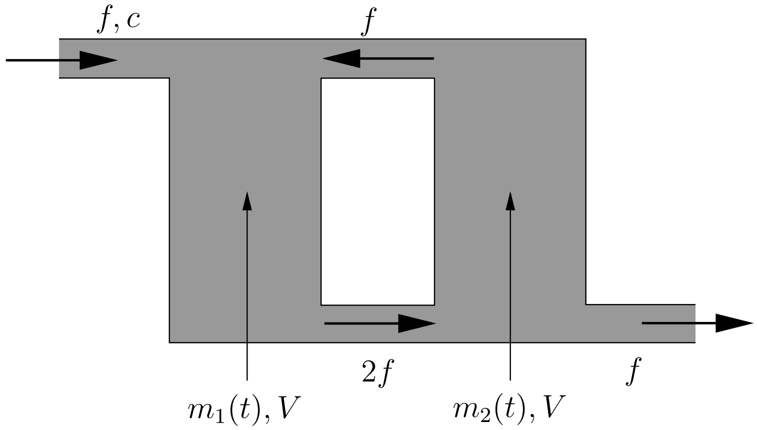


Figure 5.1: A system of two coupled mixing tanks.

where

$$A = \begin{pmatrix} -2 & 1 \\ 2 & -2 \end{pmatrix}, \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} cV \\ 0 \end{pmatrix}.$$

We refer to (5.2) as a *first order vector DE in* $\mathbb{R}^2$. □

Remark

The notation used in equation (5.2) makes sense in $\mathbb{R}^n$. For example, one can imagine a system of 3 coupled mixing tanks leading to a vector DE in $\mathbb{R}^3$.

5.1.2 The mechanical oscillator

The motion of a damped mass-spring system with applied force is described by the second order DE

$$my'' + cy' + ky = F(t),$$

(see equation (3.4)). The dimensionless version of this DE is

$$y'' + 2\zeta y' + y = f(\tau), \tag{5.3}$$

where ζ is the dimensionless damping constant (see equation (3.88) in Section 3.3.3).

To describe the state of the system at an instant of time it is not enough to give the *displacement* y: one has also to give the *velocity* $\frac{dy}{dt}$, or $\frac{dy}{d\tau}$, in terms of the dimensionless time τ.

So we introduce the state vector

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \in \mathbb{R}^2,$$

with components

$$x_1 = y \quad \text{and} \quad x_2 = y', \tag{5.4}$$

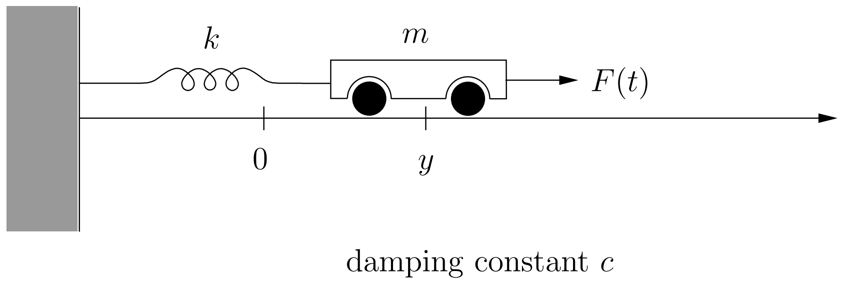


Figure 5.2: A damped mass-spring system.

where $'$ denotes differentiation with respect to τ. But what vector DE does the state vector satisfy? Well, from (5.4),

$$x_1' = y' = x_2,$$

and from (5.3) and (5.4)

$$\begin{aligned} x_2' = y'' &= -2\zeta y' - y + f(\tau) \\ &= -2\zeta x_2 - x_1 + f(\tau), \end{aligned}$$

Collecting the results, we have

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - 2\zeta x_2 + f(\tau). \end{aligned}$$

In vector form this reads

$$\mathbf{x}' = Ax + \mathbf{f}, \tag{5.5}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2\zeta \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} 0 \\ f \end{pmatrix}. \qquad \square$$

One can obtain vector DEs in higher dimensions in this context. The system shown in figure 5.3 will have a state vector

$$\mathbf{x} = \begin{pmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{pmatrix} \in \mathbb{R}^4.$$

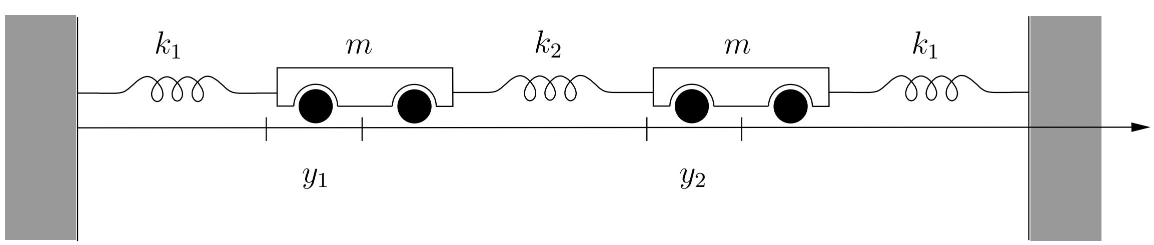


Figure 5.3: A two-mass oscillator.

Exercise

The procedure leading from the second order linear DE (5.3) to the linear vector DE (5.5) can be used to write *any* second order linear DE as a vector DE. Show that

$$y'' + py' + qy = 0 \tag{5.6}$$

is equivalent to

$$\mathbf{x}' = A\mathbf{x}, \quad \text{with} \quad \mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}. \tag{5.7}$$

5.1.3 Overview

From a mathematical point of view the object of study in this chapter is a *linear vector DE* of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}(t),$$

or more concisely,

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t). \tag{5.8}$$

Here

$$\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{R}^2 \tag{5.9}$$

is the *unknown vector-valued function*,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{5.10}$$

is the 2×2 *coefficient matrix*, whose entries are constants, and

$$\mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \tag{5.11}$$

is a given vector-valued function. The *initial condition* is of the form

$$\mathbf{x}(0) = \mathbf{a}, \tag{5.12}$$

where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is a given vector.

The DE (5.8) should be thought of as describing the *evolution in time* of a physical system (e.g. coupled mixing tanks or mechanical oscillator). The unknown vector-valued function $\mathbf{x}$ is the *state vector* of the system. The constant coefficient matrix A describes the *internal characteristics* of the system (e.g. flow rate, damping), and the vector-valued function $\mathbf{f}(t)$ describes the *external input* to the system.

The system can be represented symbolically in a so-called *block diagram* (see figure 5.4).

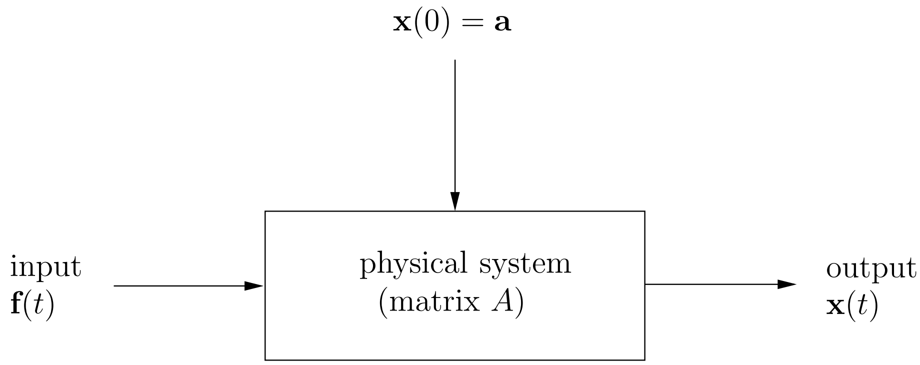


Figure 5.4: Block diagram for the physical system described by the linear vector DE $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$.

The goal is to determine the state $\mathbf{x}(t)$ of the system at time t (the "output"), given the input $\mathbf{f}(t)$ and initial state $\mathbf{x}(0) = \mathbf{a}$. In this chapter we shall develop algorithms to solve this problem.

Finally we note that it is often helpful to write a vector DE (5.8) in *component form.* Using (5.9)-(5.11), (5.7) can be written as

$$\begin{cases} x_1' &= a_{11}x_1 + a_{12}x_2 + f_1(t) \\ x_2' &= a_{21}x_1 + a_{22}x_2 + f_2(t), \end{cases} \tag{5.13}$$

which is referred to as a *system of linear DEs.* Indeed the terms "linear vector DE" and "system of linear DEs" have the same meaning.

5.1.4 Linearity and Superposition

We shall begin by considering the case of zero external input, in which case the DE (5.8) reduces to

$$\mathbf{x}' = A\mathbf{x}, \tag{5.14}$$

referred to as a *homogeneous* linear vector DE (the zero function $\mathbf{x}(t) = \mathbf{0}$ is a solution).

As in the case of a homogeneous linear DE for a scalar (see Section 3.2.1), the Principle of Superposition holds for a homogeneous linear vector DE.

Proposition

If $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are solutions of

$$\mathbf{x}' = A\mathbf{x},$$

then

$$c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$$

is also a solution for any constant scalars c_1 and c_2.

Proof: We are given that

$$\mathbf{x}^{(1)\prime} = A\mathbf{x}^{(1)} \quad \text{and} \quad \mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)}.$$

Multiply the first by c_1, the second by c_2, and add, using the matrix property

$$A(c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}) = c_1A\mathbf{x}^{(1)} + c_2A\mathbf{x}^{(2)},$$

to get

$$(c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)})' = A(c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}),$$

as required. □

For a linear vector DE (5.14) in $\mathbb{R}^2$, the general solution must contain *two* arbitrary constants because the initial condition (5.12) contains two arbitrary constants, i.e. the components of the vector $\mathbf{a}$. We can thus state:

> the *general solution of the homogeneous linear vector DE in* $\mathbb{R}^2$,
>
> $$\mathbf{x}' = A\mathbf{x},$$
>
> *is of the form*
>
> $$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t), \tag{5.15}$$
>
> *where* $\mathbf{x}^{(1)}$ *and* $\mathbf{x}^{(2)}$ *are two linearly independent solutions of the DE, and* c_1, c_2 *are arbitrary constants.*

5.2 Solving vector DEs using eigenvectors

5.2.1 The method

Consider a *homogeneous* linear vector DE in $\mathbb{R}^2$:

$$\mathbf{x}' = A\mathbf{x}. \tag{5.16}$$

As discussed in Section 5.1.4 we need to obtain two linearly independent solutions. We consider a trial function containing an *exponential* (surprise!)

$$\mathbf{x} = e^{\lambda t}\mathbf{v}, \tag{5.17}$$

where $\mathbf{v} \in \mathbb{R}^2$ is a constant vector, and λ is a constant scalar (which may be complex).

The derivative of (5.17) is

$$\mathbf{x}' = \lambda e^{\lambda t}\mathbf{v}.$$

Substituting in (5.16) gives

$$\lambda e^{\lambda t}\mathbf{v} = A(e^{\lambda t}\mathbf{v}) = e^{\lambda t}A\mathbf{v}.$$

Multiply by the scalar $e^{-\lambda t}$ obtaining

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{5.18}$$

This equation states that the scalar λ in the trial function (5.17) is an *eigenvalue* of the coefficient matrix A, and $\mathbf{v}$ is an associated *eigenvector*.

The eigenvalues can be found by rewriting (5.18) in the form

$$(A - \lambda I)\mathbf{v} = \mathbf{0},$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the 2×2 identity matrix. This equation will have a non-zero solution for $\mathbf{v}$ iff the matrix $A - \lambda I$ is non-invertible i.e., the inverse matrix $(A - \lambda I)^{-1}$ does not exist. This is the case iff the determinant of $A - \lambda I$ is zero. We let

$$h(\lambda) = \det(A - \lambda I). \tag{5.19}$$

Then the eigenvalues of A are the roots of the equation

$$h(\lambda) = 0, \tag{5.20}$$

which is called the *characteristic equation* of A. The function $h(\lambda)$ defined by (5.19) is in fact a polynomial of degree two:

$$h(\lambda) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}.$$

Here we used the usual formula for the determinant of a 2×2 matrix. The function $h(\lambda)$ is called the *characteristic polynomial* of the matrix A.

If λ is an eigenvalue of A (i.e. a solution of (5.20)) then (5.18) will have a non-zero solution for $\mathbf{v}$, and (5.17) will be a solution of the DE (5.16). There are three cases:

A) Unequal real eigenvalues

B) Complex eigenvalues

C) Equal real eigenvalues,

which we illustrate with examples in the rest of this section. In case A) we immediately get two linearly independent solutions of the DE (5.16) (since eigenvectors associated with distinct eigenvalues are linearly independent). In case B) we have to take real and imaginary parts of the solutions, while case C) requires special treatment.

5.2.2 Unequal real eigenvalues

Example 1:

Find the general solution of the vector DE

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{pmatrix} -4 & 1 \\ -2 & -1 \end{pmatrix}. \tag{5.21}$$

Solution: We consider a trial function

$$\mathbf{x} = e^{\lambda t}\mathbf{v}. \tag{5.22}$$

Substituting (5.22) in (5.21) leads to

$$(A - \lambda I)\mathbf{v} = 0. \tag{5.23}$$

Thus λ must satisfy

$$h(\lambda) = 0,$$

where the characteristic polynomial is

$$\begin{aligned} h(\lambda) = \det(A - \lambda I) &= \det \begin{pmatrix} -4-\lambda & 1 \\ -2 & -1-\lambda \end{pmatrix} \\ &= (-4-\lambda)(-1-\lambda) - (-2)(1) \\ &= \lambda^2 + 5\lambda + 6 \\ &= (\lambda+2)(\lambda+3). \end{aligned}$$

Thus the eigenvalues of A are

$$\lambda = -2 \quad \text{and} \quad -3.$$

First, considering $\lambda = -2$, equation (5.23) becomes

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

leading to

$$\begin{aligned} -2v_1 + v_2 &= 0 \\ -2v_1 + v_2 &= 0, \end{aligned}$$

which requires that $v_2 = 2v_1$ (the two equations are identical). In other words, any vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix}$$

with $v_1 \neq 0$, is an eigenvector of A associated with the eigenvalue $\lambda = -2$. Choosing $v_1 = 1$, it follows from (5.22) that

$$\mathbf{x} = e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{5.24}$$

is a solution of the DE (5.21).
Second, considering $\lambda = -3$, equation (5.23) becomes

$$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which implies $v_1 = v_2$. In other words, any vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix},$$

with $v_1 \neq 0$, is an eigenvector of A with eigenvalue $\lambda = -3$. Choosing $v_1 = 1$ it follows from (5.22) that

$$\mathbf{x} = e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.25}$$

is a solution of the DE (5.21).

Finally, since the vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are linearly independent, the solutions (5.24) and (5.25) are linearly independent. Thus by Section 5.1.4 (see equation (5.15)) the general solution of the DE (5.21) is

$$\mathbf{x} = c_1 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{5.26}$$

where c_1 and c_2 are arbitrary constants. □

Initial conditions:

The constants c_1 and c_2 are determined by the initial condition

$$\mathbf{x}(0) = \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Setting $t = 0$ in (5.26) gives

$$\mathbf{a} = \begin{pmatrix} c_1 + c_2 \\ 2c_1 + c_2 \end{pmatrix},$$

which can be solved to give

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_2 - a_1 \\ 2a_1 - a_2 \end{pmatrix}. \tag{5.27}$$

Two special cases will be important later, namely $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. From (5.26) and (5.27) we obtain:

$$\mathbf{x} = \begin{pmatrix} -e^{-2t} + 2e^{-3t} \\ -2e^{-2t} + 2e^{-3t} \end{pmatrix}, \qquad \text{with} \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{5.28}$$

and

$$\mathbf{x} = \begin{pmatrix} e^{-2t} - e^{-3t} \\ 2e^{-2t} - e^{-3t} \end{pmatrix}, \qquad \text{with} \qquad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.29}$$

□

5.2.3 Complex eigenvalues

Example 2:

Find the general solution of the vector DE

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix}. \tag{5.30}$$

Solution: We consider a trial function

$$\mathbf{x} = e^{\lambda t}\mathbf{v}. \tag{5.31}$$

As in Example 1, λ must be an eigenvalue of A i.e. λ must be a solution of

$$h(\lambda) = \det(A - \lambda I) = 0,$$

and $\mathbf{v}$ must be an eigenvector, i.e. a non-zero solution of

$$(A - \lambda I)\mathbf{v} = \mathbf{0}. \tag{5.32}$$

The characteristic equation is

$$h(\lambda) = \det \begin{pmatrix} 1-\lambda & 5 \\ -1 & -3-\lambda \end{pmatrix} = (\lambda+1)^2 + 1,$$

after simplifying and completing the square. Setting $h(\lambda) = 0$, we obtain the eigenvalues

$$\lambda = -1 + i, \quad -1 - i.$$

Choosing $\lambda = -1 + i$, equation (5.32) becomes

$$\begin{pmatrix} 2-i & 5 \\ -1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

leading to

$$(2-i)v_1 + 5v_2 = 0,$$
$$-v_1 - (2+i)v_2 = 0.$$

These two equations are essentially the same, since multiplying the second by $-(2-i)$ yields the first. Thus

$$v_2 = -\tfrac{1}{5}(2-i)v_1,$$

and choosing $v_1 = 5$ gives

$$\mathbf{v} = \begin{pmatrix} 5 \\ -2+i \end{pmatrix}$$

as a solution of equation (5.32) in the case $\lambda = -1 + i$. Then, using (5.31), we obtain

$$\mathbf{x} = e^{(-1+i)t}\begin{pmatrix} 5 \\ -2+i \end{pmatrix} \tag{5.33}$$

as a *complex* solution of the DE (5.30). The real and imaginary parts of this solution are themselves solutions of (5.30). (Why? In general, if you write the solution as $\mathbf{x} = \mathbf{x}^{(1)} + i\mathbf{x}^{(2)}$ and substitute into the DE, equating real and imaginary parts leads to $\mathbf{x}^{(1)\prime} = A\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)\prime} = A\mathbf{x}^{(2)}$ — try it!) Using Euler's formula we decompose (5.33) into its real and imaginary parts:

$$\begin{aligned}\mathbf{x} &= e^{-t}(\cos t + i\sin t)\left[\begin{pmatrix}5\\-2\end{pmatrix} + i\begin{pmatrix}0\\1\end{pmatrix}\right]\\ &= e^{-t}\left[\left\{\cos t\begin{pmatrix}5\\-2\end{pmatrix} - \sin t\begin{pmatrix}0\\1\end{pmatrix}\right\} + i\left\{\sin t\begin{pmatrix}5\\-2\end{pmatrix} + \cos t\begin{pmatrix}0\\1\end{pmatrix}\right\}\right]\\ &= e^{-t}\begin{pmatrix}5\cos t\\-2\cos t - \sin t\end{pmatrix} + ie^{-t}\begin{pmatrix}5\sin t\\\cos t - 2\sin t\end{pmatrix}.\end{aligned}$$

These two solutions are linearly independent since one is not a multiple of the other. Thus, by equation (5.15), the general solution of the DE (5.30) is

$$\mathbf{x} = e^{-t}\left[c_1\begin{pmatrix}5\cos t\\-2\cos t - \sin t\end{pmatrix} + c_2\begin{pmatrix}5\sin t\\\cos t - 2\sin t\end{pmatrix}\right], \tag{5.34}$$

where c_1 and c_2 are arbitrary constants. □

5.2.4 Equal real eigenvalues

Example 3:

Find the general solution of the vector DE

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix}1 & -1\\1 & 3\end{pmatrix}. \tag{5.35}$$

Solution: We consider a trial function

$$\mathbf{x} = e^{\lambda t}\mathbf{v}. \tag{5.36}$$

As in Example 1, λ must be a solution of

$$h(\lambda) = \det(A - \lambda I) = 0,$$

and $\mathbf{v}$ must be a non-zero solution of

$$(A - \lambda I)\mathbf{v} = \mathbf{0}. \tag{5.37}$$

The characteristic equation is

$$h(\lambda) = \det\begin{pmatrix}1-\lambda & -1\\1 & 3-\lambda\end{pmatrix} = (\lambda - 2)^2,$$

after simplifying, and so we have equal eigenvalues

$$\lambda_1 = \lambda_2 = 2.$$

Equation (5.37) becomes

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

giving $v_1 + v_2 = 0$. Choosing $v_1 = 1$ (for convenience) gives

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{5.38}$$

and so by (5.36),

$$\mathbf{x} = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{5.39}$$

is a solution of the DE (5.35).
We are now faced with the problem of finding a second linearly independent solution.[a] In the past we have found that in such situations "multiplying by t" was a good thing to do. So we consider

$$\mathbf{x} = te^{2t}\mathbf{v}$$

as a new trial function ("plan A"). When substituted in the DE (5.35), this choice leads to a contradiction. We presumably need "more constants" in the trial function. So we try "plan B", a trial function of the form

$$\mathbf{x} = te^{2t}\mathbf{v} + e^{2t}\mathbf{w}. \tag{5.40}$$

Differentiating and simplifying gives

$$\mathbf{x}' = e^{2t}[2t\mathbf{v} + (\mathbf{v} + 2\mathbf{w})].$$

Substitute in the DE (5.35) and divide by e^{2t} to obtain

$$2t\mathbf{v} + (\mathbf{v} + 2\mathbf{w}) = tA\mathbf{v} + A\mathbf{w},$$

which must hold for all t. Equating coefficients gives

$$A\mathbf{v} = 2\mathbf{v} \quad \text{and} \quad A\mathbf{w} = \mathbf{v} + \mathbf{2w},$$

i.e.

$$(A - 2I)\mathbf{v} = 0 \quad \text{and} \quad (A - 2I)\mathbf{w} = \mathbf{v}.$$

We can use (5.38) as a solution of the first equation, i.e. $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then the second equation becomes

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solving this linear system gives $w_1 = 0$, $w_2 = -1$. Thus, the trial function (5.40) gives the solution

$$\mathbf{x} = e^{2t}\left[\begin{pmatrix}0\\-1\end{pmatrix} + t\begin{pmatrix}1\\-1\end{pmatrix}\right]. \tag{5.41}$$

of the DE (5.35).

The solutions (5.39) and (5.41) are linearly independent by inspection. Thus by equation (5.15) the general solution of the DE (5.35) is

$$\mathbf{x} = c_1 e^{2t}\begin{pmatrix}1\\-1\end{pmatrix} + c_2 e^{2t}\left\{\begin{pmatrix}0\\-1\end{pmatrix} + t\begin{pmatrix}1\\-1\end{pmatrix}\right\},$$

which can be rearranged as

$$\mathbf{x} = e^{2t}\left[(c_1 + tc_2)\begin{pmatrix}1\\-1\end{pmatrix} + c_2\begin{pmatrix}0\\-1\end{pmatrix}\right]. \qquad \square$$

[a] In some cases (e.g. diagonal 2×2 matrix), one can find two linearly independent eigenvectors when solving (5.37); in these cases there is no difficulty in finding the second linearly independent solution.

Exercises

1) Solve $\mathbf{x}' = \begin{pmatrix}3 & -2\\2 & -2\end{pmatrix}\mathbf{x}$.

Answer: $\mathbf{x} = c_1 e^{-t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}2\\1\end{pmatrix}$.

2) Solve $\mathbf{x}' = \begin{pmatrix}-1 & -4\\1 & -1\end{pmatrix}\mathbf{x}$.

Answer: $\mathbf{x} = e^{-t}\left[c_1\begin{pmatrix}-2\sin 2t\\\cos 2t\end{pmatrix} + c_2\begin{pmatrix}2\cos 2t\\\sin 2t\end{pmatrix}\right]$

3) Solve $\mathbf{x}' = \begin{pmatrix}2 & 1\\-1 & 4\end{pmatrix}\mathbf{x}$.

Answer: $\mathbf{x} = e^{3t}\left[(c_1 + tc_2)\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix}\right]. \qquad \square$

5.3 Solving vector DEs using the Laplace transform

5.3.1 The method

Consider a *homogeneous* linear vector DE

$$\mathbf{x}' = A\mathbf{x}, \tag{5.42}$$

with initial condition

$$\mathbf{x}(0) = \mathbf{a}. \tag{5.43}$$

We wish to apply the Laplace transform operator $\mathcal{L}$ to this DE. In order to do this we first have to define the Laplace transform of a vector-valued function $\mathbf{x}(t)$.

Definition

The *Laplace transform of a vector-valued function* $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is defined by

$$\mathcal{L}[\mathbf{x}(t)] = \begin{pmatrix} \mathcal{L}[x_1(t)] \\ \mathcal{L}[x_2(t)] \end{pmatrix},$$

provided that the Laplace transform of each component function exists. □

NOTE: The Laplace transform of $\mathbf{x}(t)$ is itself a vector-valued function, which we denote by $\mathbf{X}(s)$, i.e.

$$\mathbf{X}(s) = \mathcal{L}[\mathbf{x}(t)].$$

We now need to relate $\mathcal{L}[\mathbf{x}']$ and $\mathcal{L}[A\mathbf{x}]$ to $\mathcal{L}[\mathbf{x}]$. These matters are taken care of in the following propositions.

Proposition 1:

Given a vector-valued function with derivative $\mathbf{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$. If $\mathcal{L}[x_1'(t)]$ and $\mathcal{L}[x_2'(t)]$ exist, then

$$\mathcal{L}[\mathbf{x}'(t)] = s\mathcal{L}[\mathbf{x}(t)] - \mathbf{x}(0).$$

Proof: By the definition,

$$\begin{aligned}
\mathcal{L}[\mathbf{x}'(t)] &= \begin{pmatrix} \mathcal{L}[x_1'(t)] \\ \mathcal{L}[x_2'(t)] \end{pmatrix} \\
&= \begin{pmatrix} s\mathcal{L}[x_1(t)] - x_1(0) \\ s\mathcal{L}[x_2(t)] - x_2(0) \end{pmatrix} \quad \text{(Laplace transform of the derivative of a scalar function)} \\
&= s\begin{pmatrix} \mathcal{L}[x_1(t)] \\ \mathcal{L}[x_2(t)] \end{pmatrix} - \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \quad \text{(using standard operations on vectors)} \\
&= s\mathcal{L}[\mathbf{x}(t)] - \mathbf{x}(0),
\end{aligned}$$

again using the definition. □

Proposition 2:

If A is a constant matrix, and $\mathcal{L}[\mathbf{x}(t)]$ exists, then

$$\mathcal{L}[A\mathbf{x}(t)] = A\mathcal{L}[\mathbf{x}(t)].$$

Proof: Consider

$$\begin{aligned}
\mathcal{L}[A\mathbf{x}(t)] &= \mathcal{L}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}\right] \\
&= \mathcal{L}\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} \quad \text{(a matrix acting on a vector)} \\
&= \begin{pmatrix} \mathcal{L}[a_{11}x_1 + a_{12}x_2] \\ \mathcal{L}[a_{21}x_1 + a_{22}x_2] \end{pmatrix} \quad \text{(by the definition)} \\
&= \begin{pmatrix} a_{11}\mathcal{L}[x_1] + a_{12}\mathcal{L}[x_2] \\ a_{21}\mathcal{L}[x_1] + a_{22}\mathcal{L}[x_2] \end{pmatrix} \quad \text{(since } \mathcal{L} \text{ is a linear operator)} \\
&= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \mathcal{L}[x_1] \\ \mathcal{L}[x_2] \end{pmatrix} \quad \text{(a matrix acting on a vector)} \\
&= A\mathcal{L}[\mathbf{x}(t)], \quad \text{by the definition.} \qquad \square
\end{aligned}$$

We can now apply $\mathcal{L}$ to the DE (5.42):

$$\mathcal{L}[\mathbf{x}'] = \mathcal{L}[A\mathbf{x}].$$

By Propositions 1 and 2 this becomes

$$s\mathcal{L}[\mathbf{x}(t)] - \mathbf{x}(0) = A\mathcal{L}[\mathbf{x}(t)].$$

Writing $\mathcal{L}[\mathbf{x}(t)] = \mathbf{X}(s)$, and using equation (5.43), we get

$$s\mathbf{X}(s) - \mathbf{a} = A\mathbf{X}(s),$$

which can be rearranged to read

$$(sI - A)\mathbf{X}(s) = \mathbf{a}, \tag{5.44}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix. Equation (5.44) is a system of linear algebraic equations with coefficient matrix $(sI - A)$. Provided that $sI - A$ is invertible, the solution of (5.44) is

$$\mathbf{X}(s) = (sI - A)^{-1}\mathbf{a}. \tag{5.45}$$

Having obtained $\mathbf{X}(s)$, the solution $\mathbf{x}(t)$ of the DE is obtained by taking the inverse Laplace transform:

$$\mathbf{x}(t) = \mathcal{L}^{-1}[\mathbf{X}(s)],$$

i.e. apply $\mathcal{L}^{-1}$ to the components of $\mathbf{X}(s)$. □

Comment: Finding the inverse of a 2×2 matrix to obtain the solution (5.45) is easy:

$$\text{if} \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{then} \quad B^{-1} = \frac{1}{\det(B)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{5.46}$$

where $\det(B) = ad - bc$.

This can be verified by showing that $BB^{-1} = I$. □

We now illustrate the method with an example.

5.3.2 An example

Example

Solve the vector DE

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} -4 & 1 \\ -2 & -1 \end{pmatrix}, \tag{5.47}$$

with initial condition

$$\mathbf{x}(0) = \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Solution: Apply $\mathcal{L}$ to the DE (5.47) and use Propositions 1 and 2 to obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s)$$

where we have written $\mathcal{L}[\mathbf{x}(t)] = \mathbf{X}(s)$, as usual. Rearrange and use the initial condition to get

$$(sI - A)\mathbf{X}(s) = \mathbf{a},$$

i.e.

$$\begin{pmatrix} s+4 & -1 \\ 2 & s+1 \end{pmatrix} \mathbf{X}(s) = \mathbf{a}. \tag{5.48}$$

To find the inverse matrix we note

$$\det \begin{pmatrix} s+4 & -1 \\ 2 & s+1 \end{pmatrix} = (s+2)(s+3),$$

after simplifying. Thus by equation (5.46),

$$\begin{pmatrix} s+4 & -1 \\ 2 & s+1 \end{pmatrix}^{-1} = \frac{1}{(s+2)(s+3)} \begin{pmatrix} s+1 & 1 \\ -2 & s+4 \end{pmatrix},$$

and so the solution of (5.48) is

$$\mathbf{X}(s) = \frac{1}{(s+2)(s+3)} \begin{pmatrix} s+1 & 1 \\ -2 & s+4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{5.49}$$

for $s \neq -2, -3$. In component form

$$X_1(s) = \frac{s+1}{(s+2)(s+3)}a_1 + \frac{1}{(s+2)(s+3)}a_2,$$
$$X_2(s) = \frac{-2}{(s+2)(s+3)}a_1 + \frac{s+4}{(s+2)(s+3)}a_2.$$

Performing the partial fraction expansions yields

$$X_1(s) = \left(\frac{2}{s+3} - \frac{1}{s+2}\right)a_1 + \left(\frac{1}{s+2} - \frac{1}{s+3}\right)a_2,$$
$$X_2(s) = 2\left(\frac{1}{s+3} - \frac{1}{s+2}\right)a_1 + \left(\frac{2}{s+2} - \frac{1}{s+3}\right)a_2.$$

We can calculate the inverse Laplace transforms using $\mathcal{L}^{-1}\left[\frac{1}{s-\alpha}\right] = e^{\alpha t}$. This gives

$$\begin{aligned} x_1(t) &= \mathcal{L}^{-1}[X_1(s)] = (2e^{-3t} - e^{-2t})a_1 + (e^{-2t} - e^{-3t})a_2, \\ x_2(t) &= \mathcal{L}^{-1}[X_2(s)] = 2(e^{-3t} - e^{-2t})a_1 + (2e^{-2t} - e^{-3t})a_2. \end{aligned} \tag{5.50}$$

These equations can be written in vector form as

$$\mathbf{x}(t) = \begin{pmatrix} 2e^{-3t} - e^{-2t} & e^{-2t} - e^{-3t} \\ 2(e^{-3t} - e^{-2t}) & 2e^{-2t} - e^{-3t} \end{pmatrix}\mathbf{a}, \tag{5.51}$$

giving the solution of the vector DE (5.47) which satisfies the initial condition $\mathbf{x}(0) = \mathbf{a}$.
□

Exercise

Use the Laplace transform method to solve the vector DE

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix},$$

with initial condition $\mathbf{x}(0) = \mathbf{a}$. This is example 2 in Section 5.2.3.

Answer: $\mathbf{x}(t) = e^{-t}\begin{pmatrix} \cos t + 2\sin t & 5\sin t \\ -\sin t & \cos t - 2\sin t \end{pmatrix}\mathbf{a}.$

HINT: $\dfrac{s+3}{s^2+2s+2} = \dfrac{(s+1)+2}{(s+1)^2+1}.$ □

5.3.3 The Fundamental Matrix

The solution (5.51) has the following form

$$\mathbf{x}(t) = \Phi(t)\mathbf{a}, \tag{5.52}$$

where $\Phi(t)$ is the 2×2 matrix in (5.51). This form of the solution shows directly how the state $\mathbf{x}(t)$ at time t depends on the initial state $\mathbf{x}(0) = \mathbf{a}$. The matrix $\Phi(t)$ is called *the fundamental matrix of the DE* (5.47).

A different form of the solution was obtained using the eigenvalue method for the same example (see Example 1 in Section 5.2.2). The solution was obtained in the form (see equation (5.26))

$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.53}$$

This form of the solution is simpler algebraically and shows clearly that there are two distinct rates of decay i.e. e^{-2t} and e^{-3t}.

How are the two forms of the solution related? By equation (5.51) the columns of the fundamental matrix are

$$\mathbf{\Phi}^1(t) = \begin{pmatrix} 2e^{-3t} - e^{-2t} \\ 2(e^{-3t} - e^{-2t}) \end{pmatrix}, \quad \mathbf{\Phi}^2(t) = \begin{pmatrix} e^{-2t} - e^{-3t} \\ 2e^{-2t} - e^{-3t} \end{pmatrix}. \tag{5.54}$$

Each column vector is a solution of the DE. The column $\mathbf{\Phi}^1(t)$ is the solution corresponding to the initial condition $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and the column $\mathbf{\Phi}^2(t)$ is the solution corresponding to the initial condition $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. [Choose $a_1 = 1$, $a_2 = 0$ and $a_1 = 0, a_2 = 1$ in (5.50) to get these solutions.] If we have the solution in eigenvector form (5.53), we can construct the fundamental matrix $\Phi(t)$ by finding the two special solutions (5.54): impose the initial conditions $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ successively and determine the constants c_1 and c_2 in (5.53) (this was done in equations (5.28) and (5.29) in Section 5.2.2). □

In the Laplace transform method, the Laplace transform of the solution $\mathbf{x}(t)$, denoted $\mathbf{X}(s)$, is given by

$$\mathbf{X}(s) = (sI - A)^{-1}\mathbf{a} \tag{5.55}$$

The solution $\mathbf{x}(t) = \mathcal{L}^{-1}[\mathbf{X}(s)]$ is obtained by applying $\mathcal{L}^{-1}$ to (5.55). In the example in this section we wrote (5.55) in component form and applied $\mathcal{L}^{-1}$ to each component. One can apply $\mathcal{L}^{-1}$ directly to (5.55) obtaining

$$\mathbf{x}(t) = \mathcal{L}^{-1}[(sI - A)^{-1}]\mathbf{a},$$

where the Laplace transform of the matrix is obtained by applying $\mathcal{L}^{-1}$ to each entry. In the example,

$$\mathbf{X}(s) = \begin{pmatrix} \frac{2}{s+3} - \frac{1}{s+2} & \frac{1}{s+2} - \frac{1}{s+3} \\ \frac{2}{s+3} - \frac{2}{s+2} & \frac{2}{s+2} - \frac{1}{s+3} \end{pmatrix} \mathbf{a}.$$

Thus

$$\begin{aligned} \mathbf{x}(t) = \mathcal{L}^{-1}\mathbf{X}(s) &= \begin{pmatrix} \mathcal{L}^{-1}[\quad] & \mathcal{L}^{-1}[\quad] \\ \mathcal{L}^{-1}[\quad] & \mathcal{L}^{-1}[\quad] \end{pmatrix} \mathbf{a} \\ &= \begin{pmatrix} 2e^{-3t} - e^{-2t} & e^{-2t} - e^{-3t} \\ 2e^{-3t} - 2e^{-2t} & 2e^{-2t} - e^{-3t} \end{pmatrix} \mathbf{a}, \end{aligned}$$

in agreement with equation (5.51). $\square$

We now summarize the Laplace transform method schematically, writing $\mathcal{L}[\mathbf{x}(t)] = \mathbf{X}(s)$, as usual.

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{a} \end{cases} \xrightarrow[\text{and rearrange}]{\text{apply } \mathcal{L}} \mathbf{X}(s) = (sI - A)^{-1}\mathbf{a} \xrightarrow{\text{apply } \mathcal{L}^{-1}} \mathbf{x}(t) = \Phi(t)\mathbf{a}$$

Thus, the fundamental matrix $\Phi(t)$ is related to the coefficient matrix A by

$$\mathcal{L}[\Phi(t)] = (sI - A)^{-1},$$

or equivalently

$$\mathcal{L}^{-1}[(sI - A)^{-1}] = \Phi(t). \qquad \square$$

5.4 Orbits of a vector DE in state space

In this section we show how to sketch the solutions of the homogeneous, linear vector DE

$$\mathbf{x}' = A\mathbf{x}$$

as a family of curves in the state space $\mathbb{R}^2$. These curves are called the *orbits of the DE*. The goal is to use this picture of the solutions (sometimes called the "phase portrait") to understand the behaviour of the underlying physical system (e.g. oscillator or coupled mixing tanks):

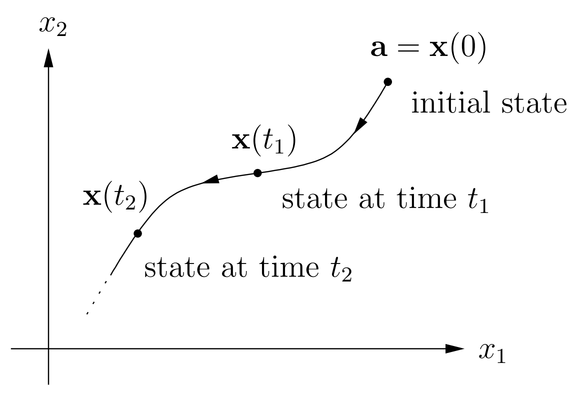


Figure 5.5: An orbit of the DE $\mathbf{x}' = A\mathbf{x}$.

Oscillator:

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

Coupled mixing tanks:

$$\mathbf{x} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

The evolution of a physical system described by the DE $\mathbf{x}' = A\mathbf{x}$ is thus represented by a *moving point* $\mathbf{x}(t)$ *in the state space* $\mathbb{R}^2$, i.e. as time passes, the point $\mathbf{x}(t)$ moves along the orbit, as shown in Figure 5.5. From the shape of the orbit one can draw various conclusions e.g. are the individual components x_1 and x_2 increasing or decreasing, do they attain a maximum or minimum etc.?

We have seen that the second order oscillator DE with zero driving force, i.e.

$$y'' + 2\zeta y' + y = 0$$

can be written as a first order vector DE

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \in \mathbb{R}^2, \tag{5.56}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2\zeta \end{pmatrix}, \tag{5.57}$$

and ζ is the dimensionless damping constant (see Section 5.1.2). We are going to sketch the orbits for two values of ζ, one for which the system is overdamped $\left(\zeta = \frac{5}{4} > 1\right)$ and one for which the system is underdamped $\left(\zeta = \frac{3}{5} < 1\right)$.

5.4.1 Phase portraits: unequal real eigenvalues

Example 1:

Give a qualitative sketch of the orbits of the vector DE

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{pmatrix} \tag{5.58}$$

in the state space $\mathbb{R}^2$.

Solution: It is best to use the form of the solution obtained using the eigenvector method. One obtains (exercise):

$$\mathbf{x} = c_1 e^{-\frac{1}{2}t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \tag{5.59}$$

where c_1 and c_2 are arbitrary constants. This equation determines a solution for any initial state $\mathbf{x}(0) = \mathbf{a}$, and thus determines an orbit through each point of $\mathbb{R}^2$. To sketch the orbits we rely on three properties of the family of solutions.

1) Exceptional solutions:

There are *three* exceptional solutions in the family (5.59). First, the equilibrium solution

$$\mathbf{x} = \mathbf{0},$$

given by $c_1 = 0 = c_2$. Second, the solution given by $c_2 = 0$, $c_1 \neq 0$, i.e.

$$\mathbf{x} = c_1 e^{-\frac{1}{2}t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{5.60}$$

Third, the solution given by $c_1 = 0$, $c_2 \neq 0$, i.e.

$$\mathbf{x} = c_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{5.61}$$

The solutions (5.60) and (5.61) are represented by straight lines in the state space: eliminating t gives $x_2 = -\frac{1}{2}x_1$ for (5.60) and $x_2 = -2x_1$ for (5.61). *2) Asymptotic behaviour as $t \to +\infty$:*

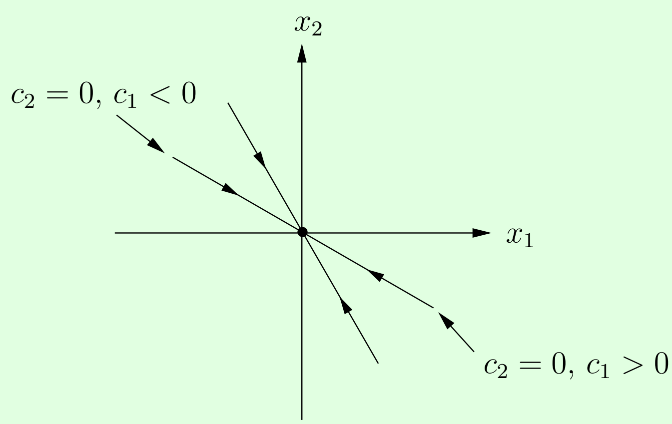
It follows immediately from (5.59) that

$$\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{0},$$

for all values of the constants c_1 and c_2. This result means that *every orbit approaches the equilibrium orbit* $\mathbf{x} = \mathbf{0}$ *as* $t \to +\infty$. Moreover, since e^{-2t} is small compared to $e^{-\frac{1}{2}t}$ as $t \to +\infty$, we can write

$$\mathbf{x} \approx c_1 e^{-\frac{1}{2}t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{as} \quad t \to +\infty,$$

i.e. any solution with $c_1 \neq 0$ is approximated by the exceptional solution (5.60) as $t \to \infty$. This result means that all orbits with $c_1 \neq 0$ approach the origin along the line $x_2 = -\frac{1}{2}x_1$ i.e. this line "attracts" other orbits as they approach the origin.



3) Sign of the slope of the orbits:

In component form the DE (5.58) reads

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - \tfrac{5}{2}x_2. \end{aligned}$$

Keeping in mind that the vector $\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$ is tangent to the orbit $\mathbf{x} = \mathbf{x}(t)$, and that their slope is $\frac{dx_2}{dx_1} = \frac{x_2'}{x_1'}$, we conclude that

(i) the tangent line to an orbit is *vertical* whenever an orbit crosses the line $x_2 = 0$ (i.e. $x_1' = 0$), called the *vertical isocline*,

(ii) the tangent line to an orbit is *horizontal* whenever an orbit crosses the line $x_2 = -\frac{2}{5}x_1$ (i.e. $x_2' = 0$), called the *horizontal isocline*, and

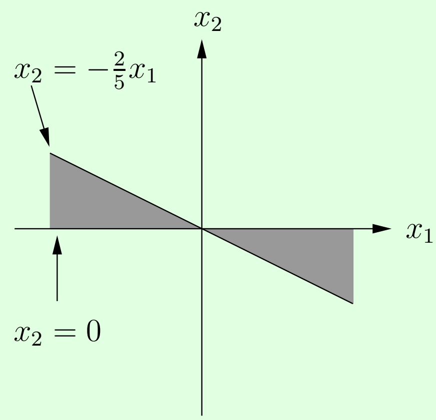
(iii) the slope of an orbit is *positive* when it lies in the region defined by

$$-\tfrac{2}{5}x_1 < x_2 < 0,$$

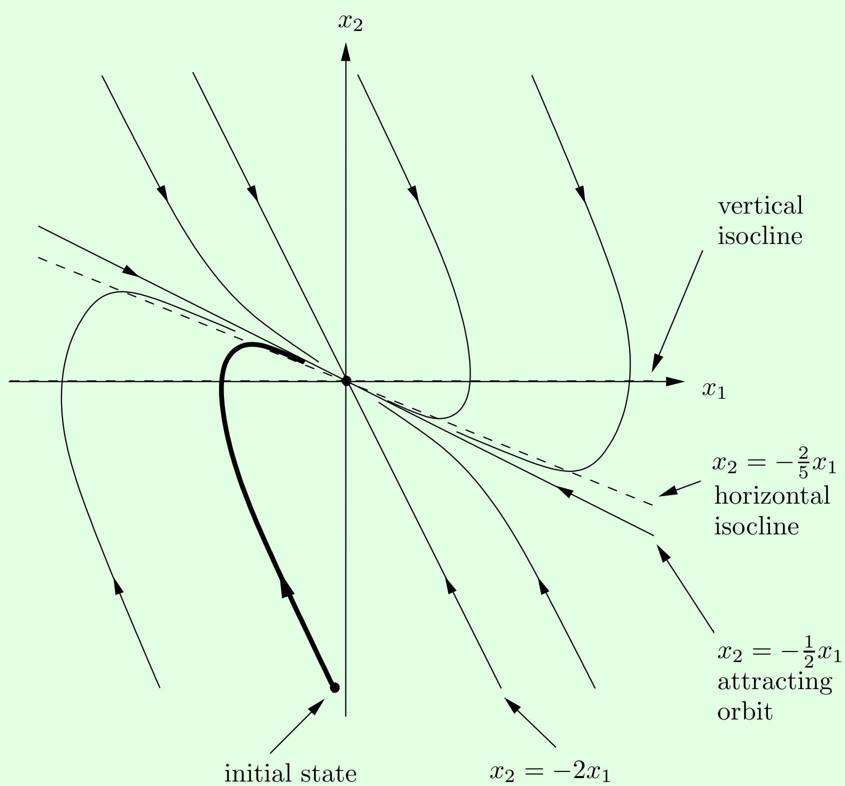
or

$$0 < x_2 < -\tfrac{2}{5}x_1,$$

i.e. the shaded region shown to the right.

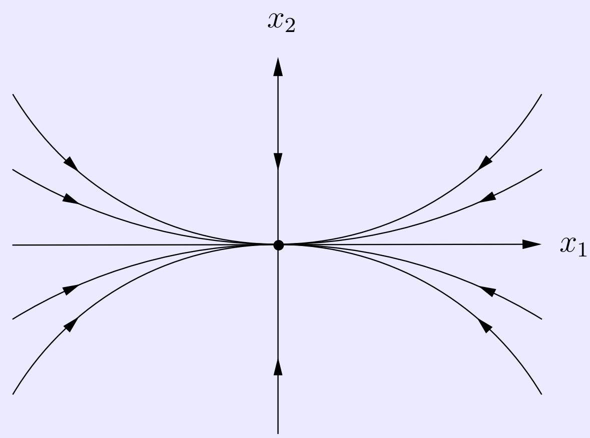


The information obtained under 1)–3) leads to the following:



Remark

The phase portrait for any DE $x' = Ax$ with unequal negative real eigenvalues will have the general form shown in Figure 5.8. The specific shape will depend on the location of the attracting orbit and of the horizontal and vertical isoclines. If the matrix A is diagonal, the horizontal and vertical isoclines coincide with the exceptional solutions giving the simple picture shown here:



Physical interpretation of the phase portrait for an oscillator:

In example 1, if the state vector $\mathbf{x}$ is given by $\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$, where y is the displacement of the trolley and y' is its velocity, then we can use the picture to describe the motion of the oscillator. Thus, when an orbit cuts the horizontal axis ($x_2 = 0$) the velocity is zero and when an orbit cuts the vertical axis ($x_1 = 0$) the displacement is zero. In the final picture, the dark orbit represents the situation in which the trolley is given a negative velocity ($x_2 < 0$) initially, after which it comes momentarily to rest ($x_2 = 0$) as it reaches maximum (negative) displacement; then, returning to the equilibrium position, it reaches maximum velocity (locally) then slows down to a state of rest.

5.4.2 Phase portraits: complex eigenvalues

We begin by considering an example with complex eigenvalues that is simple algebraically.

Example 2:

Give a qualitative analysis of the orbits of the DE

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}. \tag{5.62}$$

Solution: We use the eigenvalue method to get the general solution

$$\begin{aligned} \mathbf{x} &= e^{-t}\left[c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right] \\ &= e^{-t}\begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}, \end{aligned}$$

(exercise). We can write each component as a single cosine or sine respectively:

$$\mathbf{x} = be^{-t}\begin{pmatrix} \cos(t-\delta) \\ -\sin(t-\delta) \end{pmatrix}, \tag{5.63}$$

where

$$b = \sqrt{c_1^2 + c_2^2}, \quad \cos\delta = \frac{c_1}{b}, \quad \sin\delta = \frac{c_2}{b}.$$

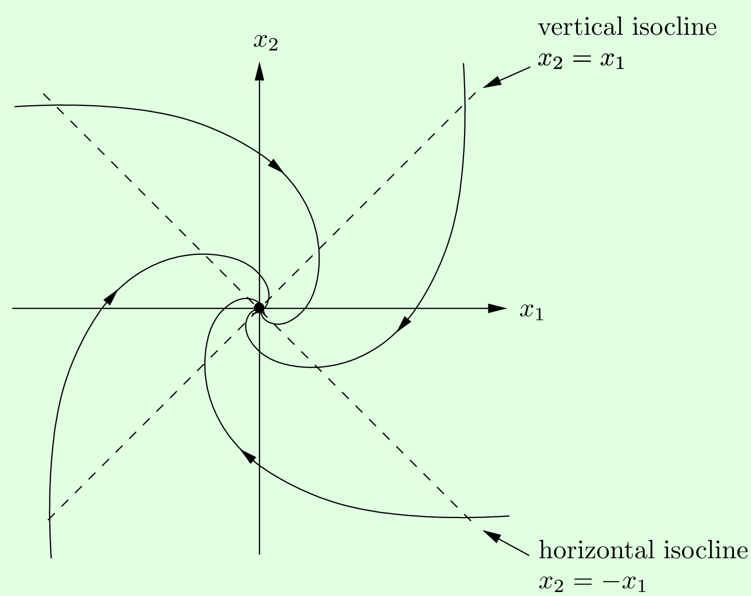
In this case, the only exceptional solution is the equilibrium solution $\mathbf{x} = \mathbf{0}$ (i.e. $b = 0$), and all other solutions are asymptotic to it as $t \to +\infty$:

$$\lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{0}.$$

Because of the cos and sin in (5.63), the orbits spiral around in a clockwise manner as they approach the origin. [Recall that $\mathbf{x} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ describes a circle traversed clockwise.] The shape of the spirals is determined by the horizontal and vertical isoclines, which are obtained by writing the DE (5.62) in component form

$$x_1' = -x_1 + x_2$$
$$x_2' = -x_1 - x_2.$$

The horizontal isocline ($x_2' = 0$) is $x_2 = -x_1$ and the vertical isocline ($x_1' = 0$) is $x_2 = x_1$. The orbits are shown in the figure below.



The phase portrait for any DE $\mathbf{x}' = A\mathbf{x}$ having complex eigenvalues with negative real parts, is qualitatively the same as the figure above – a family of spirals focusing on the origin. The specific shape will depend on the horizontal and vertical isoclines, which in general will not be orthogonal, unlike the above special case.

We now consider the example that is a special case of the oscillator DE (5.56)-(5.57).

Example 3:

Give a qualitative sketch of orbits of the DE

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{6}{5} \end{pmatrix}.$$

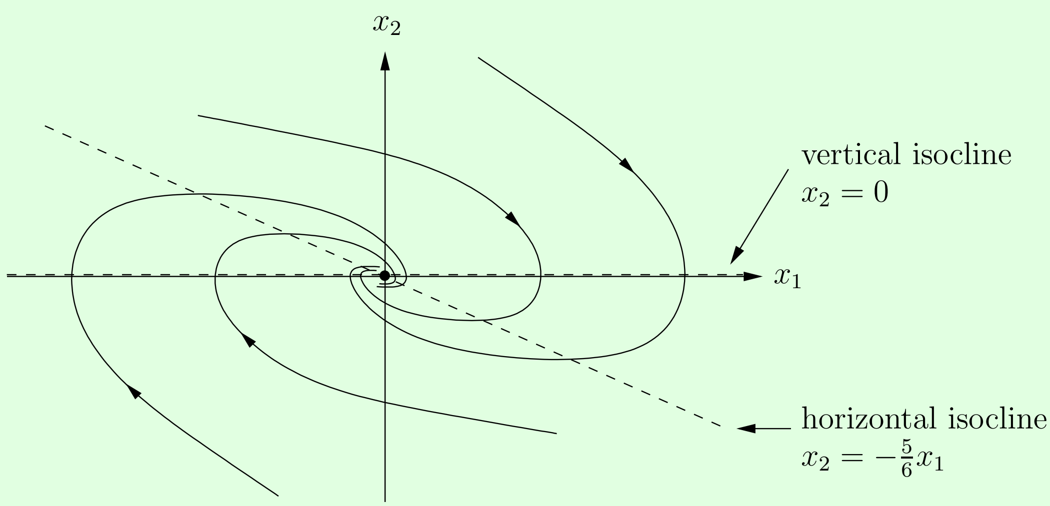
Solution: We use the eigenvalue method to find the general solution:

$$\mathbf{x} = e^{-\frac{3}{5}t}\left[\left(c_1 \cos \tfrac{4}{5}t + c_2 \sin \tfrac{4}{5}t\right)\begin{pmatrix} 1 \\ -\frac{3}{5} \end{pmatrix} + \left(-c_1 \sin \tfrac{4}{5}t + c_2 \cos \tfrac{4}{5}t\right)\begin{pmatrix} 0 \\ \frac{4}{5} \end{pmatrix}\right],$$

(exercise).
Since the real part of the eigenvalues is negative $\left(Re(\lambda) = -\frac{3}{5}\right)$, the solution contains a decaying exponential, and so the orbits spiral into the origin. The shape of the spiral is determined by the isoclines. In component form the DE is

$$x_1' = x_2$$
$$x_2' = -x_1 - \tfrac{6}{5}x_2.$$

Thus the horizontal isocline ($x_2' = 0$) is $x_2 = -\frac{5}{6}x_1$, and the vertical isocline ($x_1' = 0$) is $x_2 = 0$. The orbits are shown here:



5.4.3 Long term evolution and stability

The examples of homogeneous vector DEs $\mathbf{x}' = A\mathbf{x}$ in this section have the property that all solutions satisfy

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}. \tag{5.64}$$

Thus, for all initial states the system approaches the equilibrium state $\mathbf{x} = \mathbf{0}$ in the long term. In this case we say that the equilibrium state is *asymptotically stable*.

One can predict whether or not the system is asymptotically stable without finding the solutions explicitly, simply by considering the eigenvalues of the coefficient matrix A. The solutions are linear combinations of the trial functions

$$\mathbf{x} = e^{\lambda t}\mathbf{v}. \tag{5.65}$$

We have seen that λ has to be an eigenvalue of the matrix A. If all eigenvalues of A satisfy

$$Re(\lambda) < 0,$$

then every solution (5.65) will contain a decaying exponential and hence all solutions will satisfy (5.64). We summarize this result in the following Proposition.

Proposition

If all eigenvalues of the matrix A satisfy

$$Re(\lambda) < 0,$$

then all solutions of the DE $\mathbf{x}' = A\mathbf{x}$ satisfy

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0},$$

i.e. the equilibrium state $\mathbf{x} = \mathbf{0}$ is asymptotically stable. $\square$

5.5 Solving inhomogeneous linear vector DEs

In this Section we show how to solve the inhomogeneous vector DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t). \tag{5.66}$$

This DE should be thought of as describing the state $\mathbf{x}(t)$ of a linear physical system, with input function $\mathbf{f}(t)$ (see Section 5.1.3).

As with any linear DE, the general solution will be of the form

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t),$$

where $\mathbf{x}_h(t)$ is the general solution of the homogeneous DE $\mathbf{x}' = A\mathbf{x}$, and $\mathbf{x}_p(t)$ is a particular solution of the inhomogeneous DE (5.66)

Three methods are available to find $\mathbf{x}_p(t)$:

(i) the method of undetermined coefficients (a generalization of the method used for second order scalar DEs),

(ii) the Laplace transform method (an extension of the method used in Section 5.3), and

(iii) the method of "variation of parameters", which makes use of the fundamental matrix $\Phi(t)$.

Methods (i) and (ii) are limited as regards the possible forms of the input function $\mathbf{f}(t)$, and also entail extensive algebraic manipulation.

Method (iii) is generally applicable, and has a very simple formulation in terms of the fundamental matrix $\Phi(t)$ (although the algebra can be tedious in some examples). Because of these advantages it is the method we use most often.

Before we can develop the method we need to discuss some properties of $\Phi(t)$.

5.5.1 Properties of the Fundamental Matrix

In Section 5.3.3 we showed that the general solution of the homogeneous DE $\mathbf{x}' = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \mathbf{a}$ has the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{a}, \tag{5.67}$$

where $\Phi(t)$ is a 2×2 time-dependent matrix called the fundamental matrix of the DE. It follows by setting $t = 0$ in (5.67) that $\Phi(0)\mathbf{a} = \mathbf{a}$. Since $\mathbf{a}$ is arbitrary, this equation implies that

$$\Phi(0) = I, \tag{5.68}$$

where I is the 2×2 identity matrix.

We need two additional properties of $\Phi(t)$, the first of which follows directly from the fact that (5.67) is a solution of the homogeneous DE

$$\mathbf{x}' = A\mathbf{x}. \tag{5.69}$$

Proposition 1:

The fundamental matrix $\Phi(t)$ of the DE $\mathbf{x}' = A\mathbf{x}$ satisfies

$$\Phi'(t) = A\Phi(t). \tag{5.70}$$

Proof: Since (5.67) is a solution of (5.69) we have

$$[\Phi(t)\mathbf{a}]' = A[\Phi(t)\mathbf{a}],$$

which yields

$$[\Phi'(t) - A\Phi(t)]\mathbf{a} = \mathbf{0},$$

after rearranging. Since this holds for all $\mathbf{a} \in \mathbb{R}^2$, it follows that

$$\Phi'(t) - A\Phi(t) = 0,$$

where "0" denotes the zero matrix, which gives the required result. $\square$

We shall refer to (5.70) as *the derivative property of* $\Phi(t)$.

The second property of $\Phi(t)$ follows from the geometric interpretation of equation (5.67). One can think of $\Phi(t)$ as an operator that transforms an initial state $\mathbf{x}(0) = \mathbf{a}$ into the state $\mathbf{x}(t) = \Phi(t)\mathbf{a}$ at time t.

Consider the situation shown in Figure 5.12. At time $t = 0$ the system is in state $\mathbf{a}$, and after a time t_1 it evolves into state $\mathbf{b}$, given by

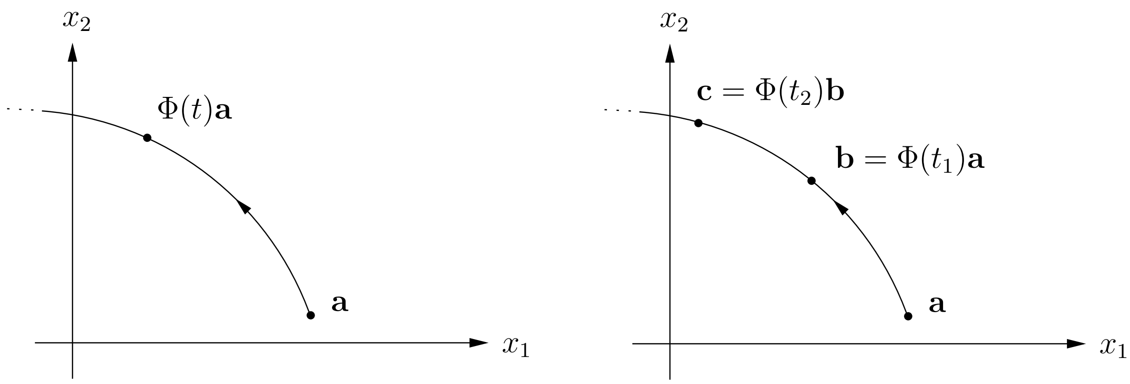


Figure 5.6: The action of the fundamental matrix.

$$\mathbf{b} = \Phi(t_1)\mathbf{a}. \tag{5.71}$$

After a further time t_2, it evolves into state $\mathbf{c}$, so that

$$\mathbf{c} = \Phi(t_2)\mathbf{b}, \tag{5.72}$$

thinking of $\mathbf{b}$ as the initial state. On the other hand, in a time $t_2 + t_1$ the system will evolve from $\mathbf{a}$ to $\mathbf{c}$ so that

$$\mathbf{c} = \Phi(t_2 + t_1)\mathbf{a}. \tag{5.73}$$

Substituting (5.71) in (5.72) and comparing with (5.73) gives

$$\Phi(t_2 + t_1)\mathbf{a} = \Phi(t_2)\Phi(t_1)\mathbf{a}.$$

Since $\mathbf{a}$ is an arbitrary vector, it follows that

$$\Phi(t_2 + t_1) = \Phi(t_2)\Phi(t_1),$$

a matrix equation.

We summarize this result and the result (5.68) in the following Proposition.

Proposition 2:

The fundamental matrix $\Phi(t)$ of the DE $\mathbf{x}' = A\mathbf{x}$ satisfies

$$\Phi(0) = I,$$

and

$$\Phi(t_2 + t_1) = \Phi(t_2)\Phi(t_1), \tag{5.74}$$

for all $t_1, t_2 \in \mathbb{R}$. □

The property we need is an immediate consequence of this Proposition. Choosing $t_2 = -t_1$ and writing $t_1 = t$, equation (5.74) with (5.68) gives

$$\Phi(-t)\Phi(t) = I.$$

In other words, $\Phi(t)$ is invertible for any t, and

$$[\Phi(t)]^{-1} = \Phi(-t). \tag{5.75}$$

We shall refer to (5.75) as *the inverse property of the fundamental matrix.* This result is quite remarkable. It states that to find the inverse matrix of a fundamental matrix $\Phi(t)$, one simply replaces t by $-t$.

5.5.2 The method of Variation of Parameters

Given the DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \tag{5.76}$$

let $\Phi(t)$ be the fundamental matrix of the related homogeneous DE $\mathbf{x}' = A\mathbf{x}$. We know that $\mathbf{x}(t) = \Phi(t)\mathbf{a}$ is a solution of $\mathbf{x}' = A\mathbf{x}$ for any $\mathbf{a} \in \mathbb{R}^2$. This suggests that in order to find a particular solution of (5.76) we consider a trial function of the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{v}(t), \tag{5.77}$$

where $\mathbf{v}(t)$ is an arbitrary vector-valued function. In other words, we replace the constant vector $\mathbf{a}$ (whose components are two arbitrary parameters) by a time-dependent vector $\mathbf{v}(t)$ (whose components are two arbitrary scalar functions), i.e. we "vary the parameters". This choice of trial function is thus called the *method of variation of parameters.*

Differentiate (5.77) with respect to t using the Product Rule and the derivative property (5.70) of $\Phi(t)$ to obtain

$$\begin{aligned}\mathbf{x}'(t) &= \Phi'(t)\mathbf{v}(t) + \Phi(t)\mathbf{v}'(t)\\ &= A\Phi(t)\mathbf{v}(t) + \Phi(t)\mathbf{v}'(t)\\ &= A\mathbf{x}(t) + \Phi(t)\mathbf{v}'(t),\end{aligned}$$

by (5.77). Equating this expression for $\mathbf{x}'$ with $\mathbf{x}'$ in (5.76) gives

$$\Phi(t)\mathbf{v}'(t) = \mathbf{f}(t).$$

We now multiply by the matrix $\Phi(-t)$ and use the inverse property (5.75) to get

$$\mathbf{v}'(t) = \Phi(-t)\mathbf{f}(t). \tag{5.78}$$

Equations (5.77) *and* (5.78) *constitute the method of variation of parameters.* Given $\mathbf{f}(t)$, and having calculated $\Phi(t)$, one obtains $\mathbf{v}(t)$ by taking the antiderivative of (5.78). Then (5.77) gives a particular solution of the DE (5.76). □

Remark

When taking the antiderivative of (5.78), a constant of integration, which is a constant vector, arises. If one simply wants a particular solution one can choose this vector to be zero. Alternately, one can choose this vector so that

$$\mathbf{v}(0) = \mathbf{0}.$$

Then the particular solution

$$\mathbf{x}_p(t) = \Phi(t)\mathbf{v}(t)$$

satisfies

$$\mathbf{x}_p(0) = \mathbf{0}. \tag{5.79}$$

With this choice of particular solution, the general solution $\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$ can be written

$$\mathbf{x}(t) = \Phi(t)\mathbf{a} + \mathbf{x}_p(t). \tag{5.80}$$

Equation (5.79) ensures that

$$\begin{aligned}\mathbf{x}(0) &= \Phi(0)\mathbf{a} + \mathbf{x}_p(0)\\ &= I\mathbf{a} + \mathbf{0} = \mathbf{a},\end{aligned}$$

as required. □

Example

Solve the DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(0) = \mathbf{a}, \tag{5.81}$$

with

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

The fundamental matrix of $x' = Ax$ is

$$\Phi(t) = e^t \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \tag{5.82}$$

Solution: The solution is of the form (5.80), i.e.

$$\mathbf{x}(t) = \Phi(t)\mathbf{a} + \mathbf{x}_p(t). \tag{5.83}$$

Substituting a trial function

$$\mathbf{x}_p(t) = \Phi(t)\mathbf{v}(t) \tag{5.84}$$

in the DE leads to (as in Section 5.5.2)

$$\begin{aligned}\mathbf{v}'(t) &= \Phi(-t)\mathbf{f}(t)\\ &= e^{-t}\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}\\ &= e^{-t}\begin{pmatrix}1\\ 0\end{pmatrix}\end{aligned}$$

Taking the antiderivative yields

$$\mathbf{v}(t) = -e^{-t}\begin{pmatrix}1\\ 0\end{pmatrix} + \mathbf{c}. \tag{5.85}$$

Setting $t = 0$ gives $\mathbf{v}(0) = -\begin{pmatrix}1\\ 0\end{pmatrix} + \mathbf{c}$. We want $\mathbf{v}(0) = 0$, and so we choose $\mathbf{c} = \begin{pmatrix}1\\ 0\end{pmatrix}$. Thus (5.85) becomes

$$\mathbf{v}(t) = (1 - e^{-t})\begin{pmatrix}1\\ 0\end{pmatrix}. \tag{5.86}$$

By (5.82), (5.84) and (5.86) the particular solution is

$$\mathbf{x}_p(t) = (e^t - 1)\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = (e^t - 1)\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix},$$

and by (5.83) the full solution is

$$\mathbf{x}(t) = e^t\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\mathbf{a} + (e^t - 1)\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}. \qquad \square$$

Exercise 1

Solve the DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(0) = \mathbf{a},$$

with

$$A = \begin{pmatrix}-3 & 2\\ 2 & -3\end{pmatrix} \qquad \mathbf{f} = e^{-3t}\begin{pmatrix}1\\ 1\end{pmatrix}.$$

The fundamental matrix is

$$\Phi(t) = \tfrac{1}{2}\begin{pmatrix}e^{-t} + e^{-5t} & e^{-t} - e^{-5t}\\ e^{-t} - e^{-5t} & e^{-t} + e^{-5t}\end{pmatrix}.$$

Answer: $\mathbf{x}(t) = \Phi(t)\mathbf{a} + \tfrac{1}{2}(e^{-t} - e^{-3t})\begin{pmatrix}1\\ 1\end{pmatrix}.$ $\square$

Exercise 2

Think of a *scalar* DE $x' = ax + f(t)$ as a vector DE where a is a 1×1 matrix, and find $\Phi(t)$. Does $\Phi(-t)$ remind you of an important function that was used in Chapter 1 when solving such a DE?

Answer: $\Phi(t) = e^{at}$, so $\Phi(-t) = e^{-at}$ which is the integrating factor!

Remark

Exercise 2 above shows that the Fundamental Matrix generalizes the notion of an integrating factor.

5.5.3 Another version of Variation of Parameters

(Or, if you will allow a brief pun, a *variation* on the method from the previous subsection!)

Consider the vector DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$$

where A is a constant 2×2 matrix.

If the solution to the homogeneous DE $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}_h(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$$

then we try a solution of the form

$$\mathbf{x}(t) = u_1\mathbf{x}^{(1)}(t) + u_2\mathbf{x}^{(2)}(t)$$

where u_1 and u_2 are *functions* of t. Differentiation yields

$$\mathbf{x}' = u_1'\mathbf{x}^{(1)} + u_1\mathbf{x}^{(1)'} + u_2'\mathbf{x}^{(2)} + u_2\mathbf{x}^{(2)'}$$

so substitution into the full DE yields

$$\begin{aligned} u_1'\mathbf{x}^{(1)} + u_1\mathbf{x}^{(1)'} + u_2'\mathbf{x}^{(2)} + u_2\mathbf{x}^{(2)'} &= Au_1\mathbf{x}^{(1)} + Au_2\mathbf{x}^{(2)} + \mathbf{f}(t) \\ &= u_1A\mathbf{x}^{(1)} + u_2A\mathbf{x}^{(2)} + \mathbf{f}(t) \end{aligned}$$

since u_1 and u_2 are scalar functions. Now, by assumption, $\mathbf{x}^{(1)'} = A\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)'} = A\mathbf{x}^{(2)}$, so this leaves

$$u_1'\mathbf{x}^{(1)} + u_2'\mathbf{x}^{(2)} = \mathbf{f}(t), \tag{5.87}$$

We'll need to remember this! In components, it is

$$u_1'x_1^{(1)} + u_2'x_1^{(2)} = f_1(t) \tag{5.88}$$

$$u_1'x_2^{(1)} + u_2'x_2^{(2)} = f_2(t) \tag{5.89}$$

We will always be able to solve this for u_1' and u_2' !
(Why? Because we have $\begin{pmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, and the columns of the matrix are the *linearly independent* solutions to $\mathbf{x}' = A\mathbf{x}$.)

We can then integrate to find u_1 and u_2, and hence $\mathbf{x}(t)$.

If we retain the constants of integration we generate the full general solution; if we omit them we obtain just a particular solution.

Example

Solve

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$$

given that

$$\mathbf{x}_h(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

and the IC is $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Solution: We set

$$\mathbf{x} = u_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + u_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{5.90}$$

To find u_1 and u_2, we use equation (5.87) which becomes

$$u_1' \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} + u_2' \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$$

which leads to $4e^{3t}u_1' = 2e^{2t}$ (two times the first equation plus the second) and $4e^{-t}u_2' = 2e^{2t}$ (two times the first plus the second), leading to $u_1 = \frac{-1}{2}e^{-t} + c_1$ and $u_2 = \frac{1}{6}e^{3t} + c_2$.

Thus, after substituting u_1 and u_2 into eq. (5.90), we get

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} - \frac{1}{2} e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{6} e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{2t} \begin{pmatrix} -\frac{1}{3} \\ -\frac{4}{3} \end{pmatrix} \end{aligned}$$

Applying the IC $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ yields $c_1 = 2$, $c_2 = -\frac{2}{3}$ (try it), and therefore

$$\mathbf{x}(t) = e^{3t} \begin{pmatrix} 2 \\ 4 \end{pmatrix} + e^{-t} \begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix} + e^{2t} \begin{pmatrix} -\frac{1}{3} \\ -\frac{4}{3} \end{pmatrix}.$$

Remark 1

With the DE in the above example, this method yields a shorter solution than the Fundamental Matrix method. However, the example on page 158 is more easily solved with the Fundamental Matrix.

Remark 2

The above method can also be applied in a variety of circumstances. For example, if a second-order linear scalar DE $y'' + P(x)y' + Q(x)y = f(x)$ has homogeneous solution $y_h = c_1y_1 + c_2y_2$, then a particular solution can have the form $y_p = u_1y_1 + u_2y_2$, where the functions u_1 and u_2 can be determined by substituting y_p into the DE. This method is particularly helpful when the method of undetermined coefficients cannot be applied, either because the coefficient functions $P(x)$ and $Q(x)$ are not constant or because the right-side function $f(x)$ isn't compatible with that method.

Finally, we note that when we derived the "multiply by x" rule on page 24, we used a trial function of the form $u(x)e^{mx}$, where $u(x)$ was the "varying parameter".

Epilogue

Differential equations is a large subject, and in AMath 250 we have essentially examined the "tip of a large iceberg".

The subject of DEs splits into two branches:

I: Linear DEs
II: Non-linear DEs

"Solving a DE" means one of three things:

i) finding an *exact* (i.e. explicit) *solution* of the DE

ii) finding an *approximate numerical solution* of the DE using a computer, e.g. MATLAB,

iii) finding an *approximate analytic solution*, e.g. an expansion in terms of a small parameter, referred to as *perturbation methods*.

In all three approaches it is desirable to have a *graphical representation* of the solution.

In AMath 250, we have almost exclusively worked with linear DEs, in particular *linear DEs with constant coefficients*, and have shown how to solve them explicitly. For other DEs, there are no general solution algorithms, but one can find solutions in special cases by making use of

- tables of solutions and solution methods
  e.g. D. Zwillinger, Handbook of DEs, 2nd ed., Section II A, pp. 185-363.

- computer software, e.g. MAPLE, WolframAlpha, etc..

- artificial intelligence

As regards the other methods, numerical solutions are considered in **AMath 342** (see also the appendix) and perturbation methods are introduced in **AMath 351**.

A completely different approach, which complements solving a DE, is to study the properties of general classes of solutions, e.g. their long term behaviour. This approach, which dates back to Henri Poincaré at the turn of the 20th century, is called the *qualitative analysis of DEs*. We had a glimpse of this topic when we sketched orbits in state space in Chapter 5. To progress further one needs to study various theoretical issues, which begins in **AMath 351**. The emphasis in qualitative analysis is on *non-linear DEs*, and this subject is the main topic in **AMath 451**.

Appendix A

Appendix: Numerical solution of DEs

Our goal in this appendix is to give a brief introduction to the problem of computing numerical solutions of differential equations. In particular, we develop algorithms for calculating an approximate solution of the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{A.1}$$

on the interval $t_0 \le t \le t_0 + T$. Since a computer can only deal with problems involving finite sets of numbers, it is essential to replace the *continuum-based* problem (A.1) with a *discrete* problem. This process of replacement is called *discretization*. In addition, it is essential that the algorithm use only *arithmetic operations*.

In Section A.1 we use the so-called Euler method to introduce and illustrate the basic ideas associated with solving the initial value problem (A.1) numerically. Then in Section A.2 we discuss more accurate and efficient methods, the so-called *Runge-Kutta methods*, which are used in software supplied by MATLAB (see van Loan, pages 323-8).

A.1 Basic concepts

The principal idea underlying the numerical solution of differential equations (ordinary and partial) is that of discretization. In the case of the initial value problem (A.1), we represent the solution $x(t)$, defined on a finite interval $t_0 \le t \le t_0 + T$, by a finite sequence of real numbers, constructed as follows. Choose a partition $\{t_0, t_1, \ldots, t_N\}$ of the interval and use the *finite* array

$$X^{\text{exact}} = \{x(t_0), x(t_1), \ldots, x(t_N)\}. \tag{A.2}$$

For simplicity we consider uniform partitions, and let

$$h = \frac{T}{N} \tag{A.3}$$

denote the length of each subinterval, called the *step-size*. A numerical method of solution yields an approximate solution table, i.e. a finite array

$$X^{\text{approx}} = \{x_0, x_1, \ldots, x_N\}, \tag{A.4}$$

which approximates the exact array (A.2) with increasing accuracy as the step-size h tends to zero. One can then use the solution table to create an approximate solution function by the process of *interpolation*, e.g. join the discrete solution points by straight line segments.

We shall restrict our considerations to *explicit single-step methods*, which means that the entries of X^{approx} are computed recursively, starting with the given initial value $x(t_0) = x_0$, i.e. x_{n+1} is determined explicitly in terms of x_n and the step-size h. The only information at one's disposal is the slope of the solutions which is provided directly by the DE: *the expression $f(t, x)$ on the right side of the DE equals the slope of the solution through the point* (t, x). We shall refer to $f(t, x)$ as the *slope function*. One then uses various ideas from Calculus to give an algorithm for computing x_{n+1}, given x_n. In Section A.1.1 we consider the simplest of these ideas, namely, the linear approximation, on which Euler's method is based.

A.1.1 Euler's Method

The goal is to determine a numerical solution of the initial value problem

$$x' = f(t, x), \qquad x(t_0) = x_0, \tag{A.5}$$

on the interval $t_0 \le t \le t_0 + T$. We choose a uniform partition $\{t_0, t_1, \ldots, t_N\}$ with $t_N = t_0 + T$. The step-size h is given by (A.3). The exact solution array is given by (A.2). We construct the approximate solution array (A.4) as follows.

The linear approximation applied to $x(t)$ at t_0 gives

$$\begin{aligned} x(t_1) &\approx x(t_0) + (t_1 - t_0)x'(t_0) \\ &= x_0 + hf(t_0, x_0), \end{aligned}$$

where we have used the DE, the initial condition and the fact $t_1 - t_0 = h$. This approximation suggests that we define

$$x_1 = x_0 + hf(t_0, x_0). \tag{A.6}$$

It follows that x_1 can be computed from the given data, and will approximate $x(t_1)$, i.e.

$$x(t_1) \approx x_1, \tag{A.7}$$

provided the step-size h is sufficiently small. Continuing, we apply the linear approximation to $x(t)$ at $t = t_1$, obtaining

$$\begin{aligned} x(t_2) &\approx x(t_1) + (t_2 - t_1)x'(t_1) \\ &= x(t_1) + hf(t_1, x(t_1)). \end{aligned} \tag{A.8}$$

In this case, however, the right side is not computable, since we do not know the exact solution value $x(t_1)$. So we use the approximation $x(t_1) \approx x_1$ in (A.8), which suggests that we define

$$x_2 = x_1 + hf(t_1, x_1). \tag{A.9}$$

This step establishes the pattern, and leads to the recursive formula

$$x_{n+1} = x_n + hf(t_n, x_n), \tag{A.10}$$

with

$$t_n = t_0 + nh,$$

for $n = 0, 1, 2, \ldots, N - 1$. Equation (A.10) defines *Euler's method*. The points x_n provide an approximation for the exact solution:

$$x(t_n) \approx x_n, \tag{A.11}$$

for $n = 0, 1, \ldots, N$.

The method is illustrated in Figure A.1. It should be noted that the line segment joining (t_0, x_0) and (t_1, x_1) is tangent to the solution curve at (t_0, x_0). On the other hand the line segment joining (t_1, x_1) and (t_2, x_2) is tangent to a different solution curve, namely the solution curve that passes through the point (t_1, x_1). This aspect of the approximation is illustrated in greater detail in Figure A.2, in connection with the error analysis given in Section A.1.2.

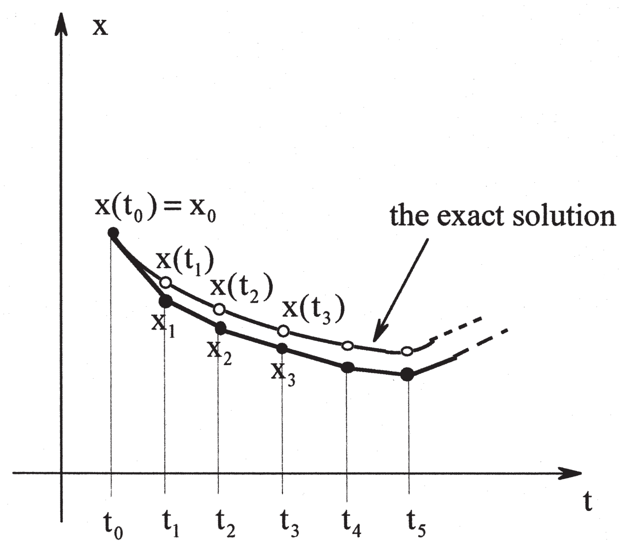


Figure A.1: The open circles are the values of the exact solution at the points of the partition, and the solid dots are the approximate values determined by Euler's method.

A.1.2 Local and cumulative error

In any numerical method, the analysis of error is of paramount importance. In solving initial value problems numerically there are two sources of error, namely

i) discretization error,

ii) round-off error.

The process of discretization, i.e. replacing the continuum-based variable t by the discrete variable n, and successively approximating the solution of the IVP at a finite sequence of points $t_1, t_2, \ldots, t_N$, inevitably introduces errors that are referred to as *discretization errors.*[1] The second type of error, *round-off error*, arises from the fact that computers can use only a finite number of decimal places to represent real numbers (finite precision arithmetic). The appropriate place to discuss this type of error is a numerical computing course, and so we shall not consider it further.

It is important to distinguish between two types of discretization error, namely[2]

i) local discretization error,

ii) cumulative discretization error.

[1] Also referred to as truncation errors.

[2] Cumulative error is also known as global error. For brevity we shall drop the word "discretization" error and simply refer to "local error" and "cumulative error".

We first define these concepts in a general setting and then apply them to Euler's method.

The local error is the error that arises in calculating x_{n+1}, given x_n. The matter is complicated by the fact that the computed value x_n only approximates the value of the solution $x(t)$ of the given IVP at $t = t_n$. So we have to consider the nearby solution through the point (t_{n-1}, x_{n-1}), which we shall denote by $\tilde{x}_{n-1}(t)$. We shall refer to $\tilde{x}_{n-1}(t)$ as the *local solution through* (t_{n-1}, x_{n-1}). The local error is then defined to be the difference

$$e_n = \tilde{x}_{n-1}(t_n) - x_n.$$

On the other hand, the global error E_n at t_n is the error in using x_n to approximate the value $x(t_n)$ of the true solution, i.e.

$$E_n = x(t_n) - x_n.$$

These errors are illustrated in Figure A.2.

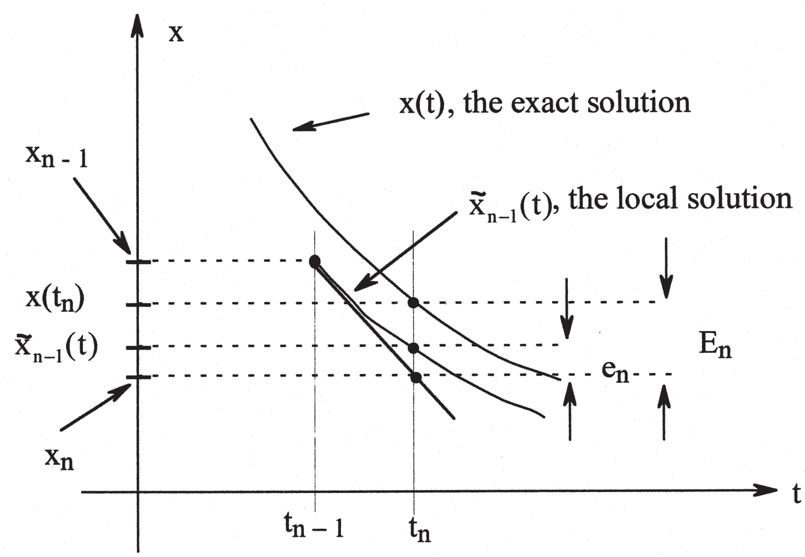


Figure A.2: The local error e_n and the global error up to the n^{th} step, E_n. The function $\tilde{x}_{n-1}(t)$ is the solution of the DE subject to the initial condition $\tilde{x}_{n-1}(t_{n-1}) = x_{n-1}$.

We now give the formal definitions.

Definition A.1 (Local & cumulative error):

Let $x(t)$ be the solution of the IVP

$$x' = f(t, x), \qquad x(t_0) = x_0.$$

Let x_n be the computed approximation at t_n, i.e.

$$x_n \approx x(t_n), \qquad n = 1, 2, \dots N.$$

Let $\tilde{x}_{n-1}(t)$ be the local solution through (t_{n-1}, x_{n-1}), i.e.,

$$\tilde{x}_{n-1}(t_{n-1}) = x_{n-1}.$$

The *local discretization error*, denoted by e_n, is defined by

$$e_n = \tilde{x}_{n-1}(t_n) - x_n. \tag{A.12}$$

The *cumulative discretization error* at t_N, denoted by E_N, is defined by

$$E_N = x(t_N) - x_N. \tag{A.13}$$

A fundamental concept in assessing the accuracy of a numerical method is its order.
Definition A.2 (order of a numerical method):

A numerical method for solving an IVP, with step-size h, is said to be *of order* p if there exists a number C such that the local error satisfies

$$\mid e_n \mid \leq C h^{p+1}. \tag{A.14}$$

Comments:

i) The inequality (A.14) reads

$$\mid e_n \mid = O(h^{p+1}) \quad \text{as} \quad h \to 0,$$

in terms of the big-oh notation.

ii) The constant C in (A.14) is independent of the step-size h and the step number n.

It is important to understand the relation between the cumulative error up to step N, and the sum of the local errors up to step N,

$$\text{i.e. how is} \qquad \mid E_N \mid \qquad \text{related to} \quad \sum_{n=1}^{N} \mid e_n \mid \ ?$$

One might hope that $\mid E_N \mid$ is less than $\sum_{n=1}^{N} \mid e_n \mid$, but this is not true in general. Nevertheless, there is a useful relation between local error and cumulative error, as follows. If the method is of order p, equation (A.14) implies that

$$\sum_{n=1}^{N} \mid e_n \mid \leq C N h^{p+1} = C T h^p,$$

since $h = T/N$, T being the length of the interval, i.e.

$$\sum_{n=1}^{N} \mid e_n \mid = O(h^p).$$

The following theorem shows that $\mid E_N \mid$ is of the same order of smallness, i.e. $\mid E_N \mid = O(h^p)$.

Theorem A.1:

Consider any one-step method for numerical solution of

$$x' = f(t, x), \qquad x(t_0) = x_0,$$

over an interval of length T, where f satisfies the Lipshitz condition

$$\mid f(t, x) - f(t, \overline{x}) \mid \leq L \mid x - \overline{x} \mid,$$

for $0 \leq t \leq T$ and for all $x, \overline{x} \in \mathbb{R}$. If the local error satisfies

$$| e_n | \leq \varepsilon h,$$

for $n = 1, 2, \ldots, N$, where h is the step-size, then the cumulative error over the interval of length T satisfies

$$| E_N | \leq \frac{1}{L}(e^{LT} - 1)\varepsilon.$$

Proof: We refer to Birkhoff & Rota (page 191, Theorem 9) or Brauer & Nohel (page 360, Theorem 8.3). □

If the method is of order p, then

$$\varepsilon = Ch^p,$$

for some constant C (see (A.14)), leading to the following Corollary.

Corollary: If the numerical method is of order p, i.e. the local error satisfies

$$| e_n | = O(h^{p+1}),$$

for $n = 1, \ldots, N$, then the cumulative error satisfies

$$| E_N | = O(h^p).$$

We can now analyze the error in the Euler method. The local error can be bounded using Taylor's theorem, subject to the reasonable assumption that the function $f(t, x)$ in the DE is of class C^1, which will ensure that the solutions are of class C^2.

Proposition A.1 (Error bound for Euler's method):

Consider the IVP

$$x' = f(t, x), \qquad x(t_0) = x_0.$$

If f is of class C^1, then Euler's method is of order $p = 1$.

Proof: The idea is to derive an inequality for the local error (A.12):

$$e_n = \tilde{x}_{n-1}(t_n) - x_n.$$

Applying Taylor's theorem to the local solution $\tilde{x}_{n-1}(t)$ at t_{n-1} gives

$$\tilde{x}_{n-1}(t_n) = \tilde{x}_{n-1}(t_{n-1}) + (t_n - t_{n-1})\tilde{x}'_{n-1}(t_{n-1}) + \tfrac{1}{2}(t_n - t_{n-1})^2\tilde{x}''_{n-1}(\xi), \tag{A.15}$$

where ξ satisfies $t_{n-1} < \xi < t_n$. The derivative of the local solution is given by

$$\tilde{x}'_{n-1}(t_{n-1}) = f(t_{n-1}, \tilde{x}_{n-1}(t_{n-1})),$$

and the initial condition for the local solution reads

$$\tilde{x}_{n-1}(t_{n-1}) = x_{n-1}.$$

Thus, using the fact that $t_n - t_{n-1} = h$, equation (A.15) gives

$$\tilde{x}_{n-1}(t_n) = x_{n-1} + hf(t_{n-1}, x_{n-1}) + O(h^2).$$

The recursion formula (A.10) for Euler's method is

$$x_n = x_{n-1} + hf(t_{n-1}, x_{n-1}).$$

Subtracting these two equations thus shows that

$$| \tilde{x}_{n-1}(t_n) - x_n |= O(h^2),$$

i.e., $| e_n |= O(h^2)$, establishing that the order is $p = 1$. □

We can now use the Corollary to Theorem A.1 to conclude that the cumulative error for Euler's method satisfies

$$| E_N |= O(h),$$

i.e. there exists a constant C such that

$$| E_N | \leq Ch, \tag{A.16}$$

where C is independent of h and N. The constant C is determined by the slope function $f(t, x)$ in the DE.

As a simple illustrative example, consider the IVP

$$x' = -\frac{2tx}{1+t^2}, \qquad x(0) = 1, \tag{A.17}$$

on the interval $0 \leq t \leq 1$. Euler's method, applied with various step-sizes, gives the values for $x(1)$ shown in Table A.1 (see Edwards & Penney, pages 106-7). The exact value[3] is $x(1) = 0.5$, leading to the constant value for $| E_N | /h$ of 0.11, i.e. for this IVP, the constant C in (A.16) can be taken to be 0.11 □

h	Euler approximation to $x(1)$	$\| E_N \| /h$
0.04	0.50451	0.11
0.02	0.50220	0.11
0.01	0.50109	0.11
0.005	0.50054	0.11
0.0025	0.50027	0.11
0.00125	0.50013	0.11

Table A.1: Values of $x(1)$ computed using Euler's method with step-size h, for the IVP (A.17), showing the cumulative error E_N over the interval $0 \leq x \leq 1$. The exact value is $x(1) = 0.5$.

[3]The exact solution is $x(t) = \frac{1}{1+t^2}$.

A.2 Higher order one-step methods

Euler's method, discussed in Section A.1.1, is of little practical use, since it does not give sufficient accuracy. It is conceptually important, however, since the idea on which it is based — use the slope function $f(t, x)$ multiplied by the step-size h to calculate the increment in the solution — can be elaborated to produce more accurate and efficient numerical methods, for example the *fourth order Runge-Kutta method*, which is used in current software.

Before discussing these higher order methods, we digress to introduce the notion of a *difference equation*, which forms the basis of all numerical methods for DEs.

A.2.1 Difference Equations

A difference equation is an algebraic relation between the terms of a sequence $\{x_n\}$ that determines the terms recursively, once the first term x_0 is specified. An *explicit first order difference equation* has the form

$$x_{n+1} = f(n, x_n), \tag{A.18}$$

$n = 0, 1, 2, \ldots$, where f is a given function on $\mathbb{R}^2$. Thus, given x_0, one can successively compute

$$x_1 = f(0, x_0), \qquad x_2 = f(1, x_1), \text{ etc.}$$

One can think of a difference equation as a discrete version of a differential equation: imagine a physical system whose state is described by a scalar x that is measured at discrete times $t = 0, 1, 2, \ldots$. Then the difference equation determines the state at time $n+1$ in terms of the state at time n.

One can write the difference equation (A.18) in the form

$$x_{n+1} = x_n + F(n, x_n), \tag{A.19}$$

expressing the change $x_{n+1} - x_n$ in the state in terms of the state x_n at time n. In this form, the link with a DE is more apparent. Indeed, we have seen that Euler's method with step-size h, for the IVP

$$x' = f(t, x), \qquad x(t_0) = x_0, \tag{A.20}$$

has the form (A.10):

$$x_{n+1} = x_n + hf(t_n, x_n), \tag{A.21}$$

where

$$t_n = t_0 + nh.$$

and $n = 0, 1, 2, \ldots, N$. The idea is that *the solution of the difference equation* (A.21), denoted

$$\{x_0, x_1, \ldots, x_N\}$$

will approximate the values

$$\{x(t_0), x(t_1), \ldots, x(t_N)\}$$

of the exact solution of the IVP (A.20) at the points $t_n = t_0 + nh$.

We note that solving the difference equation requires repeated evaluation of the slope function $f(t, x)$, for which an algorithm must be provided. Euler's method is based on a single evaluation of $f(t, x)$ at each step, and, as we have seen, has a local error

$$| e_n |= O(h^2),$$

and a cumulative error

$$| E_N | = O(h),$$

i.e. the method is of order $p = 1$. By performing more than one evaluation of $f(t, x)$ at each step, one can increase the order of the method, thereby achieving greater accuracy and efficiency. The difference equation (A.21) is then replaced by one with a more complicated form, namely

$$x_{n+1} = x_n + h\, m(t_n, x_n, h). \tag{A.22}$$

We will present two methods that lead to a difference equation of the form (A.22), namely the improved Euler method (see Section A.2.2), and the fourth order Runge-Kutta method (see Section A.2.3).

A.2.2 The improved Euler method

The idea on which the improved Euler method is based is shown in Figure A.3. We let m_1 be the slope at the current point (t_n, x_n), i.e.

$$m_1 = f(t_n, x_n), \tag{A.23}$$

and let m_2 be the slope at the point calculated by Euler's formula (A.10) i.e.

$$m_2 = f(t_{n+1}, x_n + hm_1). \tag{A.24}$$

The idea is to use the average of these two slopes to calculate the next point (t_{n+1}, x_{n+1}):

$$x_{n+1} = x_n + \tfrac{1}{2}h(m_1 + m_2). \tag{A.25}$$

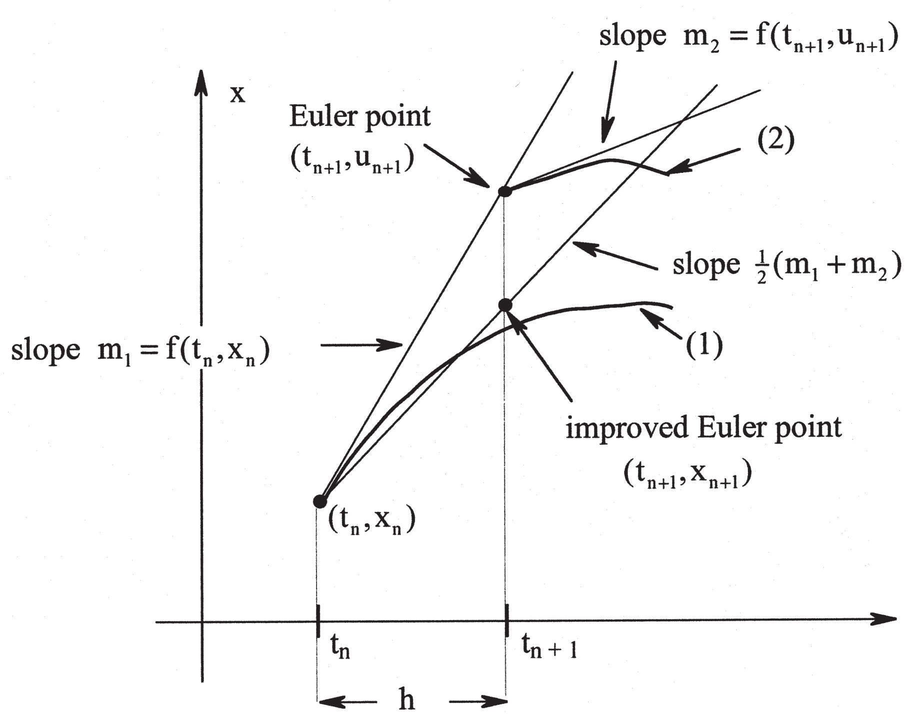


Figure A.3: This figure illustrates the improved Euler method, which is based on calculating two slopes m_1 and m_2. The curve (1) is the local solution through (t_n, y_n) and the curve (2) is the local solution through (t_{n+1}, u_{n+1}).

We now prove that the improved Euler method deserves its name — its cumulative error satisfies

$$| E_N |= O(h^2),$$

and hence it gives better accuracy than the Euler method for sufficiently small step-size h.

Proposition A.3 (Error in the improved Euler method):

Consider the IVP

$$x' = f(t,x), \qquad x(t_0) = x_0.$$

If f is of class C^2, then the improved Euler method is of order $p = 2$.

Proof: Let $u(t)$ denote the local solution through (t_n, x_n), i.e.

$$u'(t) = f(t, u(t)), \qquad u(t_n) = x_n. \tag{A.26}$$

Since $u(t)$ is of class C^3, Taylor's theorem implies that

$$u(t_{n+1}) = u(t_n) + hu'(t_n) + \tfrac{1}{2}h^2 u''(t_n) + O(h^3), \tag{A.27}$$

where

$$h = t_{n+1} - t_n.$$

Using the Chain Rule to differentiate the DE gives

$$u''(t) = f_t(t, u(t)) + f_x(t, u(t))u'(t). \tag{A.28}$$

It follows from (A.26)-(A.28) that

$$u(t_{n+1}) = x_n + hf(*) + \tfrac{1}{2}h^2 \left[f_t(*) + f_x(*)f(*)\right] + O(h^3), \tag{A.29}$$

where

$$(*) = (t_n, x_n).$$

We can also use Taylor's theorem and the Chain Rule to expand the slope m_2, given by (A.24), in powers of h:

$$\begin{aligned} m_2 &= f(t_n + h, x_n + hm_1) \\ &= f(*) + h\left[f_t(*) + m_1 f_x(*)\right] + O(h^2), \end{aligned} \tag{A.30}$$

It now follows from (A.23), (A.25) and (A.30) that

$$x_{n+1} = x_n + hf(*) + \tfrac{1}{2}h^2 \left[f_t(*) + f_x(*)f(*)\right] + O(h^3). \tag{A.31}$$

Subtracting (A.29) and (A.31) gives

$$u(t_{n+1}) - x_{n+1} = O(h^3).$$

Since $u(t) = \tilde{x}_n(t)$, the local solution through (t_n, x_n), it follows from the definition (A.12) of local error that

$$| e_{n+1} |= O(h^3),$$

i.e. the method is of order $p = 2$. $\square$

It now follows from the corollary to Theorem A.1 that the cumulative error for the improved Euler method satisfies

$$| E_N | = O(h^2),$$

i.e. there exists a constant C such that

$$| E_N | \leq Ch^2, \tag{A.32}$$

where C is independent of h and N. As a simple illustration we again consider the IVP (A.17). The improved Euler method, applied with various step sizes, gives the values of $x(1)$ shown in Table A.2 (see Edwards & Penney, page 111). The exact value is $x(1) = 0.5$, leading to the constant value for the ratio $| E_N | /h^2$ of 0.12, i.e. for this IVP the constant C in (A.32) can be taken to be 0.12.

h	Improved Euler approximation to $x(1)$	$\vert E_N \vert /h^2$
0.04	0.500195903	0.12
0.02	0.500049494	0.12
0.01	0.500012437	0.12
0.005	0.500003117	0.12
0.0025	0.500000780	0.12
0.00125	0.500000195	0.12

Table A.2: Values of $x(1)$ calculated using the improved Euler method with step-size h, for the IVP (A.12), showing the cumulative error $| E_N |$.

In summary, the *improved Euler method* for computing a numerical solution of the initial value problem

$$x' = f(t, x), \qquad x(t_0) = x_0,$$

is given by

$$x_{n+1} = x_n + \tfrac{1}{2}h(m_1 + m_2), \tag{A.33}$$

with

$$m_1 = f(t_n, x_n),$$
$$m_2 = f(t_n + h, x_n + hm_1).$$

Note that (A.33) is a difference equation of the form (A.22), i.e.

$$x_{n+1} = x_n + h\, m(t_n, x_n, h),$$

where

$$m(t_n, x_n, h) = \tfrac{1}{2}(m_1 + m_2).$$

Thus, as in Euler's method, the discretization process replaces a continuum-based evolution equation (the differential equation) by a discrete evolution equation (the difference equation (A.33)).

A.2.3 The fourth order Runge-Kutta method

We now discuss the famous fourth order Runge-Kutta method, which uses four function evaluations at each step, and, as the name suggests, is of order $p = 4$. The idea behind the method is illustrated schematically in Figure A.4.

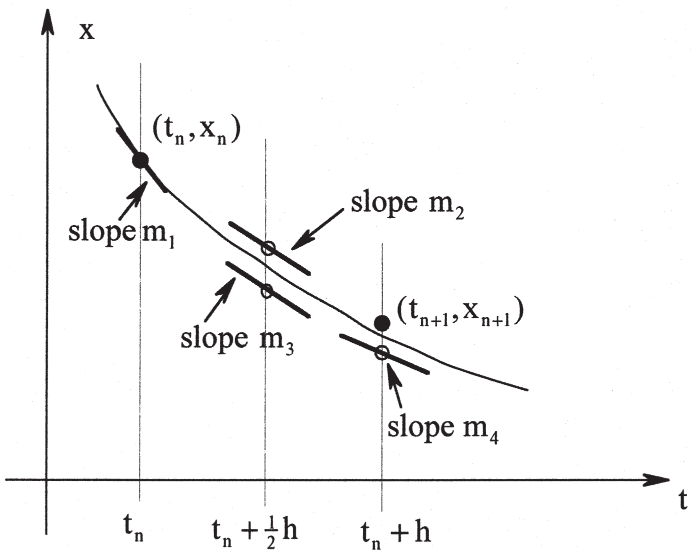


Figure A.4: This figure illustrates the fourth order Runge-Kutta method schematically, which is based on calculating four slopes. The curve is the local solution through (t_n, x_n).

Given the initial value problem

$$x' = f(t, x), \qquad x(t_0) = x_0,$$

and a uniform partition $\{t_0, t_1, \ldots, t_N\}$ of the interval $t_0 \leq t \leq t_N$, we define the following four slopes at the n^{th} stage:

i)
$$m_1 = f(t_n, x_n), \tag{A.34}$$
the Euler method slope at $(t_n x_n)$,

ii)
$$m_2 = f\left(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}hm_1\right), \tag{A.35}$$
an estimate of the slope at the midpoint of the interval $t_n \leq t \leq t_{n+1}$, using the Euler method to give the x-value,

iii)
$$m_3 = f\left(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}hm_2\right), \tag{A.36}$$
a second estimate of the slope at the midpoint of the interval, using m_2 to give the x-value,

iv)
$$m_4 = f(t_n + h, x_n + hm_3), \tag{A.37}$$
an estimate of the slope at the endpoint of the interval, using the slope m_3 to give the x-value.

In terms of these slopes the difference equation for the fourth order Runge-Kutta method is chosen to be

$$x_{n+1} = x_n + hm(t_n, x_n, h), \tag{A.38}$$

where

$$m(t_n, x_n, h) = \tfrac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4). \tag{A.39}$$

The key property of this difference equation is that it provides an approximation method of order $p = 4$.

Proposition A.3 (Error in the fourth order Runge-Kutta method):

Consider the initial value problem

$$x' = f(t, x), \qquad x(t_0) = x_0.$$

If f is of class C^4, then the Runge-Kutta method defined by (A.34)-(A.39) is of order $p = 4$.

Proof: The proof is of a similar nature as the proof of Proposition A.2, but is considerably more complicated. We refer to Birkhoff & Rota (pages 218-9) for details. □

We can now use the Corollary to Theorem A.1 to conclude that the fourth-order Runge-Kutta method satisfies

$$| E_N | = O(h^4),$$

i.e. there exists a constant C such that

$$| E_N | \leq Ch^4, \tag{A.40}$$

where C is independent of h and N.

A.2.4 Overview

Figure A.5 gives an overview of the process of computing a numerical solution of the IVP

$$x' = f(t, x), \qquad x(t_0) = x_0.$$

The final result of the process is a continuous function $x_P(t)$, obtained by interpolation using the points in the approximate solution table $\{x_0, x_1, \ldots, x_N\}$. The subscript P indicates that this function depends on the partition P.

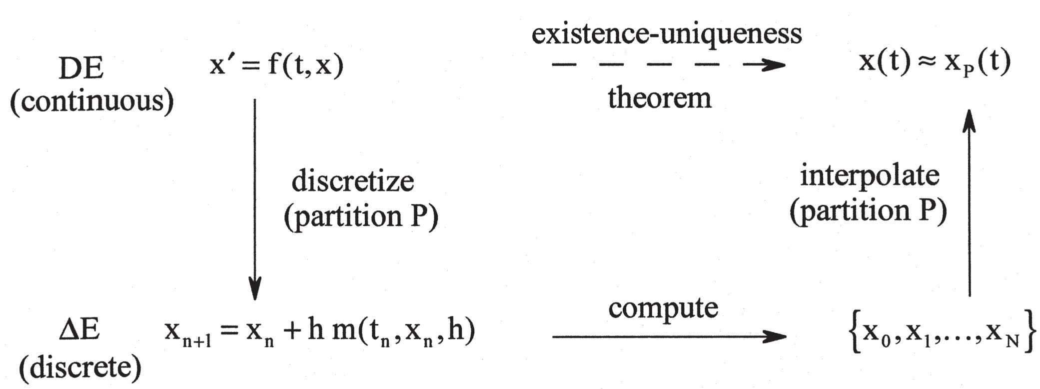


Figure A.5: The process of computing a numerical solution of an initial value problem. The abbreviation ΔE stands for difference equation ($\Delta \equiv$ difference).

The heart of the process is the choice of difference equation,

$$x_{n+1} = x_n + h\, m(t_n, x_n, h).$$

The different explicit one-step methods differ in the choice of the function $m(t_n, x_n, h)$, which is an approximation to the mean slope of the local solution over the interval $t_n \leq t \leq t_{n+1}$.

The geometric interpretation of the difference equation is illustrated in Figure A.6.

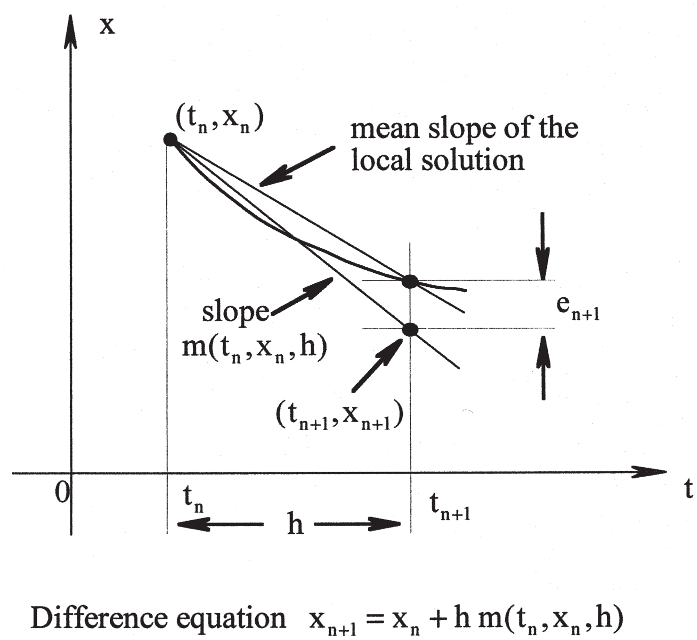


Figure A.6: The geometric interpretation of the difference equation that governs a one-step method for numerically solving an initial value problem.

We conclude this section with some comments on the control of error from a practical standpoint. In practice, the constant C in the error bounds (A.16), (A.32) and (A.40) cannot be determined because the exact solution is not known. So how can one tell whether one has chosen a small enough step-size h? One way used in practice is to compute the numerical solution several times, halving the step-size each time. When the results no longer change within the desired numbers of significant figures, it is a rule of thumb (although not an infallible test) that h is small enough and that the approximate solution is accurate within the precision desired. The second concern is whether round-off error is affecting the results. One technique is to repeat a calculation with extended precision. If the numerical results change in a significant way, it is almost certain that round-off errors are occurring.

References

Borelli, R.L. and Coleman, C.S., 1987, *Differential Equations: A Modeling Approach*, Prentice-Hall.

Goldberg, J. and Potter, M.C., 1998, *Differential Equations: A Systems Approach*, Prentice-Hall.

Simmons, G.F., 1972, *Differential Equations – with applications and historical notes*, McGraw-Hill.

Reiss, E.L., Callegari, A.J., Ahluwalia, D.S., 1976, *Ordinary differential equations with applications*, Holt, Rinehart & Winston.

Brauer, F. and Nohel, J.A., 1967, *Ordinary Differential Equations*, W.A. Benjamin.

Braun, M., 1983, *Differential Equations and their Applications*, Springer-Verlag.

Boyce, W.E. and DiPrima, R.C., 1997, *Elementary Differential Equations and Boundary Value Problems*, 6th edition, J. Wiley & Sons.

Van Loan C F 1997 *Introduction to Scientific Computing.* Prentice Hall. (See Chapter 9 for a light introduction to numerical methods for DEs.)

Edwards C H & Penney D E 1996 *Differential Equations & Boundary Value Problems.* Prentic Hall. (Less advanced than this course.)

Birkhoff and Rota GC *Differential Equations* (contains many proofs including those for Existence and Uniqueness of solutions).

Problem Sets

*Problems marked with an asterisk (*) are considered challenging.

Review Problem Set

1. Express the following antiderivatives in elementary terms. Check your answers by differentiation.

(i)	$\int te^{-t}\,dt$	(ii)	$\int te^{-t^2}\,dt$	(iii)	$\int \frac{1}{1+t}\,dt$
(iv)	$\int \frac{t}{1-t}\,dt$	(v)	$\int \frac{1-t}{t}\,dt$	(vi)	$\int \frac{1}{t(1-t)}\,dt$
(vii)	$\int \frac{1}{4+t^2}\,dt$	(viii)	$\int \frac{1}{4-t^2}\,dt$	(ix)	$\int t\sin(t^2)\,dt$
(x)	$\int \sin^2\,t\,dt$	(xi)	$\int t\sin t\,dt$	(xii)	$\int t^3e^{t^2}\,dt$

Hint for (xii): Make a change of variable $u = t^2$.

2. Give a qualitative sketch of the graphs of the following functions. The goal is to give a sketch of the graph, not drawn accurately to scale, but which shows the essential properties of the function. Think about symmetry, asymptotes and the behaviour as $x \to \pm\infty$, where appropriate.

(i)	$f(x) = \frac{1}{1+x^2}$	(ii)	$f(x) = \frac{1}{1-x^2}$	(iii)	$f(x) = x + \frac{1}{x}$
(iv)	$f(x) = x - \frac{1}{x}$	(v)	$f(x) = e^{-x^2}$	(vi)	$f(x) = e^{-\lvert x\rvert}$
(vii)	$f(x) = x^3 + x$	(viii)	$f(x) = x^3 - x$	(ix)	$f(x) = \sin^2 x$
(x)	$f(x) = e^{-x} + e^{2x}$	(xi)	$f(x) = e^{-x} - e^{2x}$	(xii)	$f(x) = \frac{e^x}{2+e^x}$

3. Sketch a graph of the given function, indicating clearly how it was obtained from the graph of a 'basic' function by scaling, translation, and/or reflection, or 'flip'. (You should show the basic graph and intermediate steps as lightly dotted or dashed curves, appropriately labelled, or show a sequence of separate graphs, indicating the operations.)

   a) $2 - \sin x$ b) $\sin^2 x$

4. (i) Simplify the expression

$$e^{a\ln b - b\ln a},$$

   where a and b are positive constants i.e. rewrite the expression so as to eliminate the logs and exponentials.

   (ii) Find all solutions of the equation $e^x - e^{-x} = 2$.
   Hint: Let $u = e^x$.

(iii) Suppose that

$$T(t) = T_0 e^{-kt}$$

and

$$T(t_1) = T_1, \qquad T(t_2) = T_2,$$

where k, t_1 and t_2 are positive. Verify that

$$t_2 = t_1 \frac{\ln(T_2/T_0)}{\ln(T_1/T_0)}.$$

5. Use Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

to derive the cosine and sine addition identities:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

Hint: Let $\theta = A + B$ and use $e^{i(A+B)} = e^{iA}e^{iB}$.

6. What does the graph of $f(x) = \sin^{2n} x$ look like for large n? Make a conjecture about the value of $\int_0^\pi \sin^{2n} x\, dx$ for large n.

7. We shall find that functions such as

$$f(t) = Ae^{-kt}\sin(\omega t + \beta)\,,$$

where $A, k > 0, \omega > 0$ and β are constants, describe *damped oscillations*.

a) (i) Sketch the graph of $f(t)$ for $A = 1, k = 1, \omega = 1$ and $\beta = 0$.
   (ii) Describe how the graph changes as ω increases, and as k increases.

b) Show that $f(t)$ satisfies

$$f'' + 2kf' + (\omega^2 + k^2)f = 0\,.$$

8. Exponential growth of a population i.e. $\hat{N}(t) = N_0\, e^{rt}$, $r > 0$, can only occur if the resources are essentially unlimited. If there are limited resources one encounters functions such as

$$N(t) = \frac{N_0\, e^{rt}}{1 - \frac{N_0}{M} + \frac{N_0}{M}\, e^{rt}}\,,$$

where r, N_0, and M are positive constants.

(i) Evaluate $\lim_{t\to+\infty} N(t)$.

(ii) Sketch the graph of $N(t)$ for $N_0 = \frac{1}{2}M$, $r = 1$. How does the shape of the graph change as r increases?

(iii) You will notice that $N(t)$ in (ii) is close to $\hat{N}(t) = N_0 e^t$ over a subinterval of the t-axis. Find the restriction on t that will ensure that

$$\frac{\hat{N}(t) - N(t)}{\hat{N}(t)} < \frac{1}{100}.$$

(iv) In the general case show that if $0 < t < \frac{1}{r}\ln\left(\frac{\epsilon M}{N_0} + 1\right)$, then $0 < \frac{\hat{N}(t)-N(t)}{\hat{N}(t)} < \epsilon$.

Note: In the section of the course dealing with Laplace transforms we shall work with *improper integrals*. We shall use the following results.

9. Show that if $s > 0$ and a is constant, then

$$\int_0^\infty e^{-st}\sin at\,dt = \frac{a}{s^2+a^2}, \qquad \int_0^\infty e^{-st}\cos at\,dt = \frac{s}{s^2+a^2}.$$

Hint: Evaluate the complex antiderivative

$$\int e^{(-s+ia)t}dt,$$

and use Euler's formula.

10. (i) Derive the reduction formula

$$\int t^n e^{-st}dt = -\frac{1}{s}e^{-st}t^n + \frac{n}{s}\int t^{n-1}e^{-st}\,dt, \quad \text{for} \quad s > 0.$$

(ii) Let $I_n(s)$ denote the improper integral

$$I_n(s) = \int_0^\infty t^n e^{-st}\,dt, \quad s > 0.$$

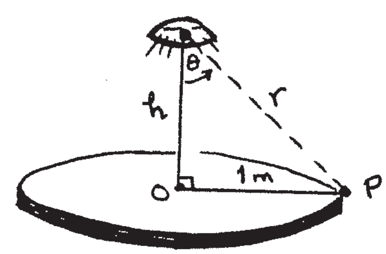
Show that

$$I_n(s) = \frac{n}{s}\,I_{n-1}(s), \quad n = 1, 2, \cdots$$

Hence show that

$$I_n(s) = \frac{n!}{s^{n+1}}.$$

11. A light is suspended a height h above a table of radius 1m. The illumination $I(\theta)$ is proportional to $\cos\theta$, and inversely proportional to r^2, where r is the distance from the light to a point P on the edge of the table. How far above the table should the light hang to maximize I at P?



12.* **Integral of an inverse.** Find $\int f^{-1}(x)\,dx$ in terms of x, $f^{-1}(x)$ and $\int f(u)\,du$ where u is to be specified by YOU.
Suggestion: Warm up by doing a few integrals of inverses of some common functions.

*Denotes a challenging question.

Problem Set 1
First Order Differential Equations

1. Each equation below defines a one-parameter family of curves. The parameter k assumes all real values.

   (i) $y = e^x + k$ (ii) $y = 2 + ke^{-2x}$ (iii) $y = x + ke^{-x}$

   (iv) $y = x + \frac{k}{x}$ (v) $y = \ln(e^x + k)$ (vi) $k(x^2 + y^2) = 2y$

   a) Derive the DE that is satisfied by the family of curves. State whether the DE is separable, linear or neither.
   b) Use information deduced from the DE and from the given equation to give a qualitative sketch of each family. For part (v), include an analysis of concavity and find any vertical asymptotes.

2. Consider the following first order DEs:

   (i) $\frac{dy}{dx} = -2y + e^{-x}$ (ii) $\frac{dy}{dx} = y \sin x$ (iii) $\frac{dy}{dx} = x(1 - y)$

   (iv) $\frac{dy}{dx} = y(1 - y)$ (v) $\frac{dy}{dx} = \frac{y}{1+x^2}$ (vi) $x^2 \frac{dy}{dx} + 3xy = 1$

   (vii) $\frac{dy}{dx} = -y - x$ (viii) $\frac{dy}{dx} = -2xy + 2x^3$ (ix) $\frac{dy}{dx} = 1 - y^2$

   a) Find the general solution of each DE.

      Hint: For (viii), see the Review Set, #1 (xii).
   b) By using the form of the DE and the solution, give a qualitative sketch of the family of solutions of each DE, showing some typical solutions and all exceptional solutions. In your diagram, indicate as a dotted curve the set of all points at which the slope $\frac{dy}{dx}$ is zero.

3. Repeat question 2 for the DE
$$x\frac{dy}{dx} + (x + 1)y = e^{-x}.$$
   Note: The sketch of the family of solution curves has an intricate structure, and it will take some effort to discover it.

4. Solve the DE $\dfrac{dy}{dx} = e^{x^2} - \dfrac{y}{x}$ and give a qualitative sketch of the solution curves. (The sketch is a little tricky.)

5. Solve each initial value problem. Specify the interval in which your solution is valid. Sketch your solution.

   (i) $\frac{dy}{dx} = y^2 \cos x, \quad y(0) = \frac{1}{2}$ (ii) $\frac{dy}{dx} = y^2 \cos x, \quad y(0) = 2$

   (iii) $\frac{dy}{dx} = \frac{2y}{x} + x, \quad y(1) = e$ (iv) $\frac{dy}{dx} = e^{y-x}, \quad y(0) = \ln 2$

6. Suppose that $y = y(x)$ is a solution of the DE $\frac{dy}{dx} + 2xy = 2e^{-x^2}$.
   If $y(0) = 2$, find $y(1)$. Evaluate $\lim_{x \to +\infty} y(x)$, if the limit exists. Does $y(x)$ attain a maximum value for $x \geq 0$? Sketch the graph of $y(x)$.

7. Use the method of undetermined coefficients to find the general solution for each DE *where applicable*. Give the reason if not applicable, but do not solve. *Hint for (vii):* the method *does* apply; it may take a couple of tries to get the right trial function.

(i) $\dfrac{dy}{dx} + 2y = 3 - 2x$ (ii) $\dfrac{dy}{dx} - 2y = 2 + e^{-x}$

(iii) $\dfrac{dy}{dx} + y = \sin 2x$ (iv) $\dfrac{dy}{dx} + 2xy = x^2$

(v) $\dfrac{dy}{dx} + y = e^{-x}$ (vi) $\dfrac{dy}{dx} + 3y = y^2$

(vii) $\dfrac{dy}{dx} + 2y = xe^{x}$ (viii) $\dfrac{dy}{dx} + y = \tan x$

8. Find the general solution to the DE $\dfrac{dy}{dx} = ay + b$ (where a and b are constants) using any method you like.

9. For each DE, indicate which type(s) it is (Separable and/or Linear) and which methods apply (Separation of Variables, Integrating Factor, and/or Undetermined Coefficients) by placing a check mark in the appropriate box(es). The first one is done for you.

	Type:		Method:		
Differential Equation	Separable	Linear	Sep. Vars	Int. Factor	Undet. Coeffs.
$\frac{dy}{dx} = y + 1$	✓	✓	✓	✓	✓
$\frac{dy}{dx} - \cos x = y^2$					
$\frac{dy}{dx} - x = xy^2$					
$t\frac{dy}{dt} + y = t^3$					
$m\frac{dv}{dt} = mg - \alpha v$					

10. Suggest an 'undetermined coefficients' trial function for a particular solution $y_p(x)$ for each of the following DEs where applicable. Write "not applicable" if undetermined coefficients can't be used. The first one is done for you.

DE:	Trial function $y_p(x)$
$\frac{dy}{dx} + 2y = x^2$	$Ax^2 + Bx + C$
$\frac{dy}{dx} + 2y = \cos x$	
$\frac{dy}{dx} + 2y = x\cos x$	
$\frac{dy}{dx} + 2y = e^{-2x}$	
$\frac{dy}{dx} + 3y^2 = \sin x$	

Hints: Keep in mind the solution to the homogeneous DE. Also, at most one of the DEs is "not applicable".

11. Solve the DE

$$\frac{dy}{dt} + ky = A\sin\omega t,$$

with initial condition $y(0) = y_0$, where k, A and ω are positive constants, with $[\omega] = T^{-1}$ $[k] = T^{-1}$, $[A] = [y]T^{-1}$.

a) Are all/any of the solutions periodic in t?

b) Evaluate $\lim_{t \to +\infty} y(t)$, if the limit exists.

c) Write the solution $y(t)$ as the sum of a *transient term* and a *steady state term.*

12. a) When a coil of steel is removed from an annealing furnace its temperature is 684° C. Four minutes later its temperature is 246°C. How long will it take to reach 100° C? Assume that *Newton's law of cooling* holds, which states that the time rate of change of temperature of a cooling body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium. Assume that room temperature is 27°.

    b) You will find it quite tedious to solve part a) because of all the numbers. The problem can be solved efficiently by formulating it more generally, as follows.

    Let T_A be the temperature of the surrounding medium (called the ambient temperature). Let T_0 be the temperature of the coil when it is removed from the furnace at time $t = 0$. The temperature is measured to be T_1 at time t_1. The problem is to find the time t_2 at which the temperature is T_2. So the given quantities are T_A, T_0, T_1, T_2 and t_1 and the unknown is t_2. The idea is to solve the DE for the temperature function $T(t)$ and show that

$$t_2 = t_1 \left[\frac{\ln\left(\frac{T_2 - T_A}{T_0 - T_A}\right)}{\ln\left(\frac{T_1 - T_A}{T_0 - T_A}\right)} \right].$$

13. The velocity $v(t)$ of a skydiver falling towards the earth's surface satisfies the DE

$$m\frac{dv}{dt} = mg - \alpha v,$$

where m is the mass, g the acceleration due to gravity (assumed constant), and α is the drag coefficient (see the notes).

a) Find $v(t)$ assuming an initial velocity $v(0) = v_0 > 0$.

b) Show that as time passes, $v(t)$ approaches the terminal velocity $v_{\text{term}} = mg/\alpha$. Does $v(t)$ ever equal v_{term}?

c) Find the distance y fallen as a function of time t and verify that your formula is dimensionally consistent.

d) Find the distance y fallen as a function of velocity v, and verify that your formula is dimensionally consistent.

14. The velocity v of a projectile fired vertically up from the surface of a planet and travelling only under the influence of gravity, satisfies the DE

$$v\frac{dv}{dr} = -\frac{gR^2}{r^2},$$

where r is the distance of the projectile from the centre of the planet, R is the radius of the planet and g is the acceleration due to gravity on the surface of the planet (see the notes).

a) Find v as a function of r, assuming the initial condition $v(R) = v_{\text{init}}$.

b) Find the escape velocity for the planet. Verify that your formula is dimensionally consistent.

15. A student carrying a flu virus returns to an isolated college campus of 1,000 students. The rate at which the virus spreads is proportional to the product of the number of infected students and the number not infected. Predict the number of infected students after 7 days, if it is found that after 3 days 40 students are infected. When is the infection spreading most rapidly? Illustrate your solution by graphing the number of infected students as a function of time t.

    Suggestion: Formulate and solve this problem more generally, as in #12b).

16. Suppose that a corpse with temperature 27°C is discovered at midnight, and that the ambient temperature is a constant 17°C. The body is moved quickly to a morgue where the ambient temperature is 7°. After one hour the body temperature is 20°C. Estimate the time of death.

    Suggestion: Formulate and solve this problem more generally, as in #12b).

17. A tank contains 200 litres of water in which 300 grams of salt is dissolved. Brine containing 1 gram of salt per litre is then pumped into the tank at a rate of 4 litres per minute; the well-mixed solution is pumped out at the same rate. Find the mass $m(t)$ (in grams) of salt in the tank at time t.

18. A tank is used in certain hydrodynamic experiments. After one experiment, the tank contains 200 litres of a dye solution with a concentration of 1 g/litre. To prepare for the next experiment, the tank is to be rinsed with clear water flowing in at a rate of 2 litres/min, the well-stirred mixture flowing out at the same rate. How long will it take to reduce the concentration of dye to 1% of its original value?

19. A 300-litre tank initially contains 200 litres of a dye solution with a concentration of 1 gram/litre. It is to be rinsed with clear water flowing in at a rate of 3 litres/minute, with the well-stirred solution being drained from the tank at a rate of 2 litres/minute. What will the concentration be when the tank is full?

20. The DE
$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right) N - h,$$
    where r, K and h are positive constants, describes a population of fish with natural growth rate coefficient r, carrying capacity K and a constant harvest rate h.

    a) Show that if $h < \frac{1}{4}rK$ there are two equilibrium solutions $N(t) = N_1$ and $N(t) = N_2$ where N_1 and N_2 are constants with $0 < N_1 < N_2$. Find N_1 and N_2.

    b) Give a qualitative sketch of the solution curves in the case $h < \frac{1}{4}rK$. Discuss the long-term behaviour of $N(t)$ in the two cases (i) $N(0) > N_1$ and (ii) $N(0) < N_1$.

    Hint: There's no need to solve the DE. Don't pay much attention to the restriction on h at first; it will become clear later.

21. A projectile of mass m is fired straight up from the earth's surface with initial velocity v_0. If v_0 is small compared to the escape velocity it is reasonable to assume that the acceleration due to gravity g is constant throughout the motion. After rising to a certain height, the projectile will momentarily stop before returning to the earth's surface. The ascent time t_a is the time taken to reach maximum height. Calculate t_a in two cases:

a) neglecting air resistance,

b) assuming that the force due to air drag is proportional to the velocity.

22. An object of mass m is released from rest at a height of h metres above the earth's surface, and strikes the ground after falling for t_h seconds. Assume that the force due to air drag is proportional to the velocity. It is possible to determine the drag coefficient and knowing m, h and t_h? If not, under what conditions is it possible?

    Hint: Use your result from # 13(c) with $v_0 = 0$. Note that you don't need to find the actual value of α. You will get one (nonlinear) equation with one unknown, but this does not imply that there is a solution. For example, the equation $\alpha^2 + 1 = 0$ does not have a (real) solution.

23. Consider the problem of a falling raindrop. Let's assume that the raindrop is spherical, with initial radius $r_0 > 0$. As the drop falls through mist, it accumulates moisture and becomes larger.

    a) Assuming that the rate of change of the mass of the raindrop is proportional to its surface area, show that the radius of the raindrop is given by

    $$r(t) = kt + r_0$$

    where $k > 0$ is a constant. Note that the surface area of a sphere with radius r is $4\pi r^2$ and the mass m is $\frac{4}{3}\pi r^3 \rho$ where ρ is the density of water, and is assumed to be constant.

    b)* Using Newton's second law, derive and solve a DE for the velocity $v(t)$ of the raindrop, assuming that the only forces are gravity and air drag, and that air drag is proportional to velocity **and** surface area. *Hint:* Be careful; mass is not constant, so the left-hand side of Newton's second law is $\frac{d}{dt}(mv)$, **not** $m\frac{dv}{dt}$. Saying that "A is proportional to B and C" means that $A = kBC$ where k is a constant. Also, feel free to 'absorb' constants as you go for simplicity.

24. Experiments show that the rate of decrease of atmospheric pressure p with height h is proportional to the product of the acceleration due to gravity (assumed constant) and the pressure.

    (a) Express the above result as a differential equation for the unknown function $p(h)$.

    (b) Let p_0 denote the pressure at sea-level, p_1 denote the pressure at a reference altitude of h_1. Show that the dependence of height h on pressure p can be written in the form

    $$h = h_1 \frac{\ln[p/p_0]}{\ln[p_1/p_0]}.$$

    (c) Suppose that the pressure at sea level is 104 kPa and at an altitude of 3000 m is 70 kPa. Most people will lose consciousness if the pressure falls below 50 kPa. At what altitude does this occur? (kPa means kilopascals, and 1 Pa = 1 N/m^2).

25. Consider a population of size $N(t)$ which grows exponentially at a rate r ($[r] = T^{-1}$), but is harvested at a constant rate H per day. Then $N(t)$ satisfies the DE

    $$\frac{dN}{dt} = rN - H.$$

    a) Suppose that $r = 0.01$ days^{-1} and $N(0) = 4000$. Show that if $H > 40$ per day, then the population becomes extinct in a finite time, but if $H < 40$, the population will increase without bound.

b) Referring to a), if $H > 40$ per day, find the time of extinction.

c) In a), $H = 40$ represents the critical harvest rate. Find the critical harvest rate for arbitrary $r > 0$ and arbitrary $N(0)$.

26. **Radiocarbon Dating.**

An important tool in archelogical research is radiocarbon dating. This is a means of determining the age of certain wood and plant remains, hence of animal or human bones or artifacts found buried at the same levels. The procedure was developed by the American chemist Willard Libby (1908-1980) in the early 1950s and resulted in his winning the Nobel prize for chemistry in 1960. Radiocarbon dating is based on the fact that some wood or plant remains contain residual amounts of carbon-14, a radioactive isotope of carbon. This isotope is accumulated during the lifetime of the plant and begins to decay at its death. Since the half-life of carbon-14 is long (approximately 5568 years), measureable amounts of carbon-14 remain after many thousands of years. Libby showed that even if a tiny fraction of the original amount of carbon-14 is still present, then by appropriate laboratory measurements the *proportion* of the original amount of carbon-14 that remains can be accurately determined. In other words, if $Q(t)$ is the amount of carbon-14 at time t and Q_0 is the original amount, then the ratio $Q(t)/Q_0$ can be determined, at least if this quantity is not too small. Present measurement techniques permit the use of this method for time periods up to about 100,000 years, after which the amount of carbon-14 remaining is only about 4×10^{-6} of the original amount. To model the phenomenon of radioactive decay, it is assumed that the rate dQ/dt at which carbon-14 decays is proportional to the amount (more precisely, the number of nuclei) $Q(t)$ of carbon-14, with proportionality constant k, remaining at time t.

a) Express this relationship as a DE for $Q(t)$, but set it up in such a way that $k > 0$ (you may need to insert a minus sign somewhere—think carefully about whether you have growth or decay).

b) Using the DE, sketch typical solutions as you did in # 2.

c) Solve the DE for $Q(t)$, and find the solution satisfying $Q(0) = Q_0$. Note that k is still an unknown constant.

d) Given the half-life of carbon-14 (see paragraph above), find k.

e) Suppose that certain remains are discovered in which the current residual amount of carbon-14 is 2 percent of the original amount. Determine the age of these remains.

27. **'Another decay problem.'**

a) Suppose that a given radioactive element A decomposes into a second radioactive element B, and that B in turn decomposes into a third element C. If the amount of A present initially is x_0, if the amounts of A and B present at a later time t are x and y, respectively, and if k_1 and k_2 are the rate constants of these two reactions, find y as a function of t.

Note: Consider k_1 and k_2 to be positive, so, for example, one of the equations you'll need is $\frac{dx}{dt} = -k_1 x$ (the other equation is slightly less simple).

b) Radon is an intensely radioactive gas (with a half-life of 3.8 days) that is produced as the immediate product of the decay of radium (which has a half-life of 1600 years). The atmosphere contains traces of radon near the ground as a result of seepage from soil and rocks, all of which contain minute quantities of radium. In recent years there has been concern in some parts of Canada about possibly dangerous accumulations of radon in the enclosed basements of houses whose concrete foundations and underlying ground contain appreciably greater quantities of radium than normal because of nearby uranium mining. If the rate constants (fractional losses per unit time, in years) for the decay of radium and radon are $k_1 = 0.00043$ and $k_2 = 66$, use the result of part (a) to determine how long after the completion of a basement the amount of radon will be at a maximum.

28.* Joe and Dave sit down for a cup of coffee. Joe adds the cream right away and Dave waits 5 minutes before adding the cream. Who has the hotter coffee? Use Newton's law of cooling. (Assume that the cream was kept at a constant temperature, which is less than room temperature, for that 5 minutes. Also, you may ignore any chemical reactions, and treat the thermal properties of the cream as being the same as coffee; i.e. they both have the same k value.)

29.* Dr. Van Nostrand would like to make a drink for his infant son. The recipe: a half cup of hot tea (weakly brewed) and a half cup of cold milk. (Sugar is also added, but we'll ignore that here.)
Now suppose that the hot tea is poured into the cup first. Then, due to an impending diaper change, Dr. Van Nostrand must decide whether to add the cold milk immediately or in 5 minutes. Which method will yield the *coolest* drink?
Use Newton's law of cooling, assuming the same k value for the tea, milk, and tea-milk mixture. Also assume that mixing occurs instantly, and that the milk is kept in the fridge until used.

30.* H.E. Pennypacker, a wealthy American industrialist, has just returned home with two ready-made pies from the grocery store. The pies are at room temperature (20°C). He would like to heat them up in the oven (150°C), but only one pie can fit in the oven at a time. He puts the first pie in the oven at time $t = 0$, then puts the second pie in at time $t = t_1$, taking the first pie out and sitting it on the counter. At what time $t = t_2$ should he take the second pie out so that both pies have the same temperature? Put your answer in terms of t_1 and the proportionality constant k. (Assume that Newton's law of cooling is valid.)
To think about later: what if he had n pies?

Problem Set 2
Dimensional Analysis

1. Calculate the dimensions of

(i)	pressure gradient	(iv)	velocity gradient
(ii)	density gradient	(v)	shear stress
(iii)	surface tension	(vi)	viscosity

Note: Force = mass × acceleration, pressure = force/area, work = force × distance, gradient means "rate of change with respect to distance" (e.g. $\frac{dT}{dx}$ is the temperature gradient.) "Surface tension" is the work done in creating unit area of a fluid surface. A velocity gradient in a fluid (called the 'shear rate" by engineers) has associated with it a shear stress (a force per unit area). The shear stress is proportional to the shear rate, and the constant of proportionality is called the "viscosity" of the fluid.

2. The *Duffing oscillator* with mass m is described by the non-linear second order DE

$$m\frac{d^2x}{dt^2} + \alpha x + \beta x^3 = F_0 \sin \omega t.$$

where $[x] = L$ and $[t] = T$. Calculate the dimensions of α, β, F_0 and ω.

3. The diffusion of heat in a slab of material is governed by the partial differential equation (the *heat equation*)

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},$$

where the temperature T varies only in the x-direction. The constant κ is called the *thermal diffusivity*, and depends on the nature of the material. Find the dimensions of κ.

4. The one-dimensional version of a Navier-Stokes equation is given by the partial differential equation (PDE)

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + f(x,t)$$

where $u(x,t)$ is the velocity of the fluid particle at location x and time t, $\rho(x,t)$ is the density of the fluid, $p(x,t)$ is the pressure (force per unit area), μ is the viscosity of the fluid, and $f(x,t)$ is some external force per unit volume. Find the dimensions of μ.

5. A mixing tank of capacity $V_{\max}$ litres initially contains V_0 litres of pure water. A salt solution of constant concentration c_{in} gm/litres flows in at a rate $2f$ litres/min and the contents of the tank flow out at f litres/min. For this physical situation it is reasonable to define a characteristic time by $t_c = \frac{V_0}{f}$ (the time to drain the initial volume, with zero inflow) and a characteristic mass by $m_c = V_{\max}c_{in}$ (the mass of salt in the tank if it were completely filled by the inflow). Then one can define a dimensionless time τ and dimensionless mass $\mathcal{M}$ by

$$\tau = \frac{t}{t_c}, \quad \mathcal{M} = \frac{m}{m_c},$$

where m is the mass of salt in the tank at time t.

a) Show that $\mathcal{M}$ satisfies the DE

$$\frac{d\mathcal{M}}{d\tau} = -\frac{\mathcal{M}}{1+\tau} + 2b,$$

where $b = \frac{V_0}{V_{\max}}$. That is, derive a DE for the system then non-dimensionalize it to get the dimensionless DE shown.

b) Show that when the tank is filled it contains $(1 - b^2)V_{\max}c_{\text{in}}$ grams of salt.

6. Consider a skydiver who falls from rest under the influence of gravity, with air drag proportional to velocity.

a) Introduce a characteristic time t_c and length ℓ_c, and use them to define dimensionless variables.

b) Write the DE for the velocity as a function of time, using dimensionless variables, and solve to find the velocity as a function of time. Then find distance as a function of time, also using dimensionless variables.

c) Derive a DE for the velocity regarded as a function of distance fallen, using dimensionless variables. Solve to obtain distance fallen as a function of velocity.

d) Determine how far the skydiver has fallen and for how long, at the instant when his velocity is $\frac{9}{10}$ of the terminal velocity. Give your answer in two forms,

(i) in terms of dimensionless distance and time, and

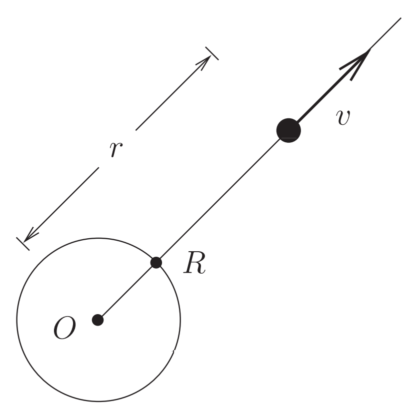
(ii) in terms of actual distance and time, assuming the skydiver has a mass of 80 kg and a drag coefficient $\alpha = 10$ kg s^{-1}.

e) Use b) to write asymptotic forms for the physical velocity and distance fallen in the limiting cases (i) $0 < t \ll t_c$ and (ii) $t \gg t_c$.

7. The velocity v of a projectile fired vertically up from the surface of a planet and travelling only under the influence of gravity, satisfies the DE.

$$v\frac{dv}{dr} = -\frac{gR^2}{r^2},$$

where r is the distance of the projectile from the centre of the planet, R is the radius of the planet and g is the acceleration due to gravity on the surface of the planet.



(a) Using the two parameters R and g, introduce a characteristic velocity v_c and a characteristic distance r_c.

(b) Define dimensionless velocity $V = v/v_c$ and dimensionless distance $\hat{r} = r/r_c$ and non-dimensionalize the DE.

(c) Solve the DE in (b); use the initial condition $v(R) = v_{\text{init}}$, which will need to be written in terms of the dimensionless variables.

(d) Re-dimensionalize your solution in (c) in order to find $v(r)$.

8. Consider the more realistic skydiver DE:

$$m\frac{dv}{dt} = mg - \beta v^2$$

where $v > 0$ is the velocity of the skydiver (downward direction) at time t, m is her mass, g is the acceleration due to gravity, and β is the drag coefficient. m, g, and β are constants.

   (a) By introducing a charateristic time and velocity, non-dimensionalize the DE. *Hint:* You should get $\frac{dV}{d\tau} = 1 - V^2$.

   (b) Find the general solution to the dimensionless DE and give a quick qualitative sketch of typical $V(\tau)$ where $V > 0$ is dimensionless velocity and $\tau > 0$ is dimensionless time. Don't worry about the scale on the τ axis.

   (c) Re-dimensionalize your solution, and give a qualitative sketch of $v(t)$.

   (d) Explain briefly why the DE is not valid for negative v. For example, if a pilot ejects from a fighter jet, she will have negative initial velocity. Why will the above DE not work? Suggest a modified DE, but don't solve.

9. Consider the logistic model for population growth

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

where $N(t)$ is the population at time t, and r and K are positive constants.

   (a) Let $N_c = K$ be the characteristic population. Define an appropriate characteristic time and non-dimensionalize the DE. *Comment:* Population has no units, so it is already dimensionless; nonetheless, define $\mathcal{N} = N/N_c$ to be a new 'scaled' population. This will help to simplify the DE. The new DE won't have any parameters in it.

   (b) Give a qualitative sketch of solutions to your dimensionless DE in the $\tau\mathcal{N}$-plane. You need not solve the DE. (If you can't get part (a), just sketch solutions to the original DE in the tN-plane.)

   (c) Use the substitution $u = 1/\mathcal{N}$ to transform the non-dimensionalized DE into a linear DE, but do not solve.

10. Consider the IVP

$$\frac{dy}{dx} = y(1-y), \quad y(0) = y_0.$$

   (a) Solve the IVP.

   (b) Based on this solution and your work in the previous question (logistic model), write down the solution to the logistic model with initial condition $N(0) = N_0$ (your solution will depend on N_0). *Hint:* This should take only a few lines of work.

   (c)* Does your function $N(t)$ have any vertical asymptotes? If so, give a physical interpretation. If the asymptotes are 'unphysical', explain why.
   Suggestion: The dimensionless solution $\mathcal{N}(\tau)$ has the same qualitative properties, so look at that function instead for simplicity.

11. Recall Newton's law of cooling, which is mathematically written as

$$\frac{dT}{dt} = -k(T - T_A)$$

where $T(t)$ is the temperature of an object at time t, $k > 0$ is a constant, and T_A is the ambient temperature, assumed constant. Let the initial temperature of the object be $T(0) = T_0$.

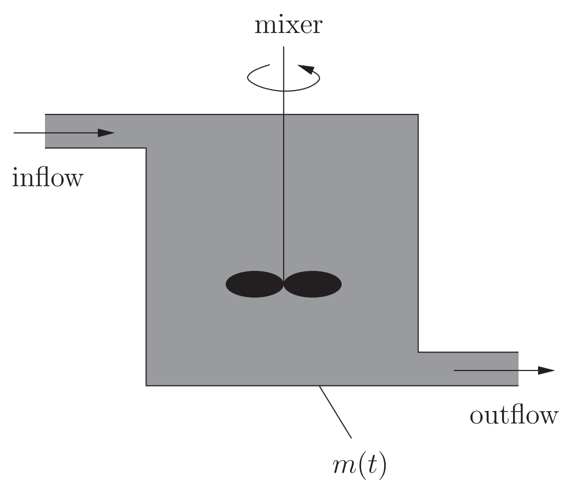
(a) Determine the dimensions of k using the principle of dimensional homogeneity. Note that temperature is a fundamental unit, denoted θ. (Other fundamental units include mass (M), length (L) and time(T), not all of which are involved in this question.)

(b) By introducing a characteristic time and temperature, define a dimensionless temperature $\hat{T}$ and a dimensionless time τ.

(c) Non-dimensionalize the above DE.

(d) Solve your dimensionless DE for $\hat{T}(\tau)$, and hence write down the 'dimensionful' solution $T(t)$.

12. Consider the mixing tank shown with saltwater inflow rate f litres per minute, inflow concentration c_{in} grams per litre, and outflow rate f (same as inflow rate). Initially, there are V_0 litres of salt water in the tank containing m_0 grams of salt.



(a) Write down a differential equation for $m(t)$, the mass of salt in the tank at time t, but DO NOT SOLVE.

(b) By introducing a characteristic mass and characteristic time, show that the DE may be written as

$$\frac{d\mathcal{M}}{d\tau} = 1 - \mathcal{M}$$

where $\mathcal{M}$ and τ are dimensionless variables. *Reminder:* when choosing characteristic values, it's best to use parameters which appear in the DE itself.

(c) Solve your dimensionless DE, using the appropriate initial condition, and give a quick sketch of the solution curves for various m_0.

(d) Based on (c), find the 'dimensionful' mass $m(t)$. What do the solution curves look like? *Hint:* not much work is required!

13. "The wine cask problem."

Torricelli's Law states that the water flowing from a hole in a container exits at the speed that it would have achieved had it fallen from the level of the surface of the water in the container.

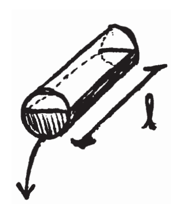
This can be reformulated in several ways. For example, one rule derived from this is that the water level in a container being drained from the bottom decreases at a rate proportional to the square root of the height, divided by the area of the surface of the water: $\frac{dh}{dt} = \frac{-k\sqrt{h}}{A(h)}$, or $A(h)\frac{dh}{dt} = -k\sqrt{h}$ (the value of k depends on factors such as the size and shape of the hole).

a) Suppose we have water in a cylindrical container, standing upright and being drained from the bottom.

i) Define a characteristic time and length, and nondimensionalize the DE (use τ and $\hbar$ for your dimensionless variables). *Reminder:* The only parameters in the DE itself are r and k. Other parameters, such as h_0 and l, are not in the DE and should not be used to create the characteristic time and length.

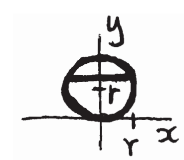
ii) Solve the nondimensionalized DE, subject to the condition that $h(0) = h_0$ (which will need to be rewritten in terms of $\hbar$), and then express your solution in terms of the original variables. How long does it take the container to drain? How long does it take if the barrel is full to begin with (call this time t_{vert})?

b) Now suppose that the container is placed on its side. In this orientation, the area of the surface depends on h, r, and l:



$$A(h) = 2l\sqrt{2rh - h^2}$$

(Why? Consider the equation of a circle of radius r, situated with its lowest point at the origin, and you should be able to find the length of the horizontal secant lines as a function of y).



When we have many parameters, nondimensionalization is less helpful. It may be impossible to simplify our equations significantly, or, if it is possible, it may be difficult to identify the appropriate scale factors without solving the equation first (would you use r or l as your length scale here?).

i) Solve the DE, in dimensional form, with $h(0) = h_0$. Note: it will be more convenient to use the diameter instead of the radius here, so solve $2l\sqrt{hd - h^2}\,\frac{dh}{dt} = -k\sqrt{h}$, $h(0) = h_0$.

ii) How long does it take the container to drain this way? Verify that your result is dimensionally consistent. Find t_{horiz}, the time required for a full barrel to drain while lying on its side.

iii) Define $\tau = \frac{t}{t_{horiz}}$ and show that if we could have identified this as our characteristic time to begin with, and identified the characteristic height as $h_c = d$ as well (so

$\hbar = \frac{h}{d}$), we could indeed have simplified the DE significantly (you don't need to solve the DE again; just nondimensionalize it).

c) From comparison of t_{vert} and t_{horiz}, you should be able to see that if $l \gg d$ then a full barrel will drain fastest if stood upright, while if $d \gg l$ it will drain faster on its side. Find the ratio l/d which will allow a full barrel to drain equally quickly in either orientation.

14. The goal is to obtain information about the period of a satellite in a circular orbit around a planet. The relevant physical quantities are the period P, the planetary mass $\mathcal{M}$, the satellite mass m, the radius of the orbit r, and, since the process is governed by gravity, by the gravitational constant G.

    (a) Set up the dimensional matrix and determine its rank.

    (b) Find a complete set of independent dimensionless quantities.

    (c) Deduce the relation between P and r for fixed masses m and $\mathcal{M}$. This result is one of Kepler's laws for planetary motion.

15. a) The three physical quantities involved in the escape velocity problem are the initial velocity $v(0) = v_{\text{esc}}$, the radius of the planet R, and the acceleration due to gravity g. Set up the dimensional matrix to show that there is only one dimensionless variable. Hence write down an expression for v_{esc}^2.

    b) Suppose that planet X has radius 6 times that of earth and that the acceleration due to gravity on the surface is twice that on earth. How does the escape velocity for planet X compare to that for the earth?

16. An explosive charge releases energy E in a soil medium of mass density ρ. Assuming that the only other important quantity is the acceleration g of gravity, investigate the length scale ℓ of the resulting crater. If it is desired to double the length scale of the crater, by how many factors must the energy be increased?

17. Consider a star as being a fluid sphere held together by its own gravity. A star may undergo vibrations, the most important being those in which all particles of the star execute simple harmonic motions which are in phase with each other, with frequency ν. Assume that ν could depend on only the mass density ρ of the star, its radius r and the gravitational constant G (the viscosity of the star is not important here). Investigate ν as a function of ρ, r and G.

18. A little cooking problem ... how does the cooking time of a roast depend on the size of the roast?

    The physical quantities are the cooking time Δt, the volume of the roast V, the thermal diffusivity κ of the roast (see #3), the initial temperature difference $\Delta T_i = T_i - T_0$ and the final temperature difference $\Delta T_f = T_f - T_0$. Here T_0 is the oven temperature, and T_i, T_f denote the initial and final temperature of the centre of the roast, respectively. Show that Δt is proportional to $V^{2/3}$.

19. Consider the motion of gravity waves in deep water (the "ocean swell"). The physical parameters are the speed of the wave v, the wavelength of the wave λ, the density of water ρ and the acceleration g due to gravity. Show that v is proportional to $\sqrt{\lambda g}$.

20. Referring to #19, if the water is shallow, we must add the depth h to the list of essential parameters. Show that in this case

$$v = f\left(\frac{h}{\lambda}\right) \sqrt{\lambda g},$$

where f is an arbitrary function of one variable.

21. Consider a lake with average diameter D and kinematic viscosity ν (units: $\frac{L^2}{T}$). Assume that D is large enough so that the effect of the Earth's rotation (the "Coriolis force") cannot be ignored, and that the frequency of the Coriolis force is Ω. We are interested in determining the velocity scales U of the lake. Use the Buckingham Pi theorem to investigate U as a function of D, Ω, and ν. *Comment:* The *Ekman number*, defined by $E_k = \frac{\nu}{\Omega D^2}$ should be one of your dimensionless quantities.

22. Suppose we are interested in the power P necessary to keep a ship of length L moving at constant velocity U. Energy must be supplied to replace energy wasted in making waves, which is lost because of water viscosity (friction), and a worker in the field may be led to assume that

$$P = P(L,\, U,\, g,\, \rho,\, \eta)\,, \tag{1}$$

where g is the acceleration of gravity, ρ is the mass density of water, and η is its viscosity. Assuming that (1) is correct, prove that

$$P = \rho\, L^2\, U^3\; f(Fr, Re),$$

where f is an unknown function of two variables, Fr is the "Froude number", and Re is the "Reynolds number":

$$Fr \equiv U^2/(Lg)\;, \quad Re \equiv U\, L\, \rho/\eta\,.$$

23. Consider steady nonturbulent incompressible flow of a fluid, of mass density ρ and viscosity η, in a cylindrical pipe, of length L and radius R. The pressure difference ΔP between the ends should depend on only L, R, ρ, η, and the maximum speed U of the flow. If this is so, show that

$$\Delta P = \tfrac{1}{2}\,\rho\, U^2 f(Re_L,\, Re_R)\,, \tag{2}$$

where f is an unknown function of two variables, and the two Reynolds numbers are defined by

$$Re_L \equiv U\, L\, \rho/\eta\;, \quad Re_R \equiv U\, R\, \rho/\eta\,.$$

Suggest a possible great simplification of the result (2) if L/R is sufficiently large.

24. Consider the lift force F produced by the wings of an airplane during flight. Suppose that

$$F = f(\rho, v, S)$$

where ρ is the density (mass per unit volume) of the air, v is the speed of the airplane, and S is the surface area of the wings. Use a suitable theorem to find the function f, up to a constant. Be sure to explain your logic.

25. Suppose that the energy E released by an atomic bomb depends on the radius R of the explosion at time t and the average density ρ of the air in the area; that is, suppose that

$$E = f(R, t, \rho).$$

where f is some function.
(a) Use the Buckingham Pi Theorem to investigate the relationship between the quantities involved.

Hint: Energy has the same dimensions as force × distance and [force]=[mass] × [acceleration]. Density is mass per unit volume.
Comment: For some interesting historical background, search "Trinity atomic bomb test".

(b) For a blast with fixed energy E, how does the radius grow as a function of time? (Assume that E and ρ are constants.) Give a sketch.

26.* Consider the problem of a falling object with cross-sectional area A. If the cross-sectional area is quadrupled, what happens to the terminal velocity? Assume that the important quantities are the terminal velocity v_{term}, the object's mass m, the acceleration due to gravity g, the air density ρ, and of course the cross-sectional area A of the object. *Hint:* Applying the 'Pi' theorem in the usual way will not give enough information. Think about how drag force relates to terminal velocity. *Comment:* Earlier, drag force was assumed to be proportional to the velocity of the object. **However**, this is just an approximation and shouldn't be assumed here.

27.* Consider the problem of a melting ice cube, which is assumed to have the shape of a cube with side length $h(t)$. Ignoring change-of-state energy requirements, one might suppose that the rate of change of temperature is proportional to (i) the temperature difference $T(t) - T_A$ between the ice cube and the surroundings, and (ii) the ratio of the surface area to volume of the cube. Call the proportionality constant k_1. Also suppose that the rate of change of the volume of the ice cube is proportional to the rate of change of the temperature of the cube; call this proportionality constant k_2.

a) Derive, but DO NOT attempt to solve, a nonlinear DE for $T(t)$ based on the above suppositions. The unknown length $h(t)$ should not appear in the final version of your DE.

b) Consider the time it takes for the ice cube to melt. Suppose that the important constants are k_1, k_2 (see part a)), the initial temperature difference $\Delta T_i = T(0) - T_A$, the initial side length $h_0 = h(0)$, and of course the time it takes to melt t_{melt}. Show that t_{melt} is inversely proportional to k_1. Can you conclude anything about how t_{melt} depends on h_0?

Acknowledgement: Thanks to F.O. Goodman & G. Tenti for some of these problems.

Problem Set 3
Second Order Linear Differential Equations

1. Find the general solution of each *homogeneous* linear DE:

   (i) $y'' + y' - 2y = 0$ (ii) $y'' + 4y' + 4y = 0$

   (iii) $y'' + 2y' + 5y = 0$ (iv) $y'' - 6y' + 10y = 0$.

2. Find the general solution of the following *inhomogeneous* linear DEs.

   (i) $y'' + y' - 6y = 6t$

   (ii) $y'' + y' - 6y = 5\cos t$

   (iii) $y'' + y' - 6y = 5e^{4t}$

   (iv) $y'' + y' - 6y = 2e^{-3t}$

3. a) Find the unique solution of the initial value problems

   (i) $y'' + 4y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$

   (ii) $y'' + 2y' + 5y = 0, \quad y(0) = 0\ , \quad y'(0) = 1.$

   b) Give a qualitative sketch of the solutions in a).

   c) Solve the IVP $y'' + y' - 6y = 2e^{-3t}$, $y(0) = 0$, $y'(0) = 1$.

4. Solve the following IVP: $\left\{ \begin{array}{l} y'' + 5y' + 4y = e^{-x} + 2 \\ y(0) = 1, y'(0) = 0 \end{array} \right\}$.

5. Write the general solution of the linear DE

$$y'' + \omega^2 y = b,$$

   where ω and b are constants, *by inspection.*

6. A particle undergoes simple harmonic motion (SHM) described by

$$y'' + \omega^2\, y = 0,$$

   with initial conditions $y(0) = y_0$, $y'(0) = v_0$. Show that the amplitude of the SHM is

$$y_{\max} = \sqrt{y_0^2 + \tfrac{v_0^2}{\omega^2}}.$$

7. Find the general solution of the DE

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + b^2 y = 0,$$

   where k and b are constants.

   Note: There are a number of different cases, depending on the values of k and b.

8. Find the general solution of the DE

$$y'' + y' - 6y = \alpha e^{rt},$$

where r and α are constants. Which values of r have to be treated as special cases?

9. Find the general solution of the DE

$$y'' + 2y' + 2y = \alpha \cos \omega t$$

where α and ω are constants.

10. Consider an undamped, sinusoidally-forced oscillator modelled by the DE

$$my'' + ky = f_0 \cos(\omega t) \quad \text{which becomes} \quad y'' + \omega_0^2 y = \frac{f_0}{m} \cos(\omega t).$$

where $y(t)$ is the position of the oscillator at time t, and the constants are the mass m, the natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$, and the applied force's amplitude, f_0.

(a) Non-dimensionalize the DE above (the one on the right). *Comment:* The natural frequency ω_0 is more 'basic' than the forcing frequency ω, so use the former when constructing your characteristic values.

(b) Find the *form* of the general solution to your dimensionless DE in two cases: (i) $\omega \neq \omega_0$ and (ii) $\omega = \omega_0$. *Clarification:* Your solution will have 4 unknown constants which you need not find. That is, do everything you would do to solve the DE *except* finding those constants.

(c) What value of ω is the resonant forcing frequency for the above oscillator?

11. The homogeneous linear DE

$$y'' + py' + qy = 0 \tag{1}$$

has $y(t) = 0$ as an equilibrium solution. In studying a physical system one is interested under what conditions the general solution will approach the equilibrium solution as time $t \to \infty$. Show that the general solution of (1) approaches the equilibrium solution as $t \to +\infty$ if and only if $p > 0$ and $q > 0$.

12. a) An object of mass 2 kg is hung vertically and extends a spring 0.1 m from its equilibrium position. Calculate the spring constant k in Newtons per metre. Assuming that the mass moves with no damping and no driving force, calculate the period of the resulting Simple Harmonic Motion (SHM).

b) A resistive medium exerts a damping force of 200 Newtons when acting on the above object moving with a velocity of 10 metres per second. Calculate the damping constant in kilograms per second.

c) Consider a damped mass–spring system with spring as in a) and damping as in b). Will the system undergo damped oscillations?

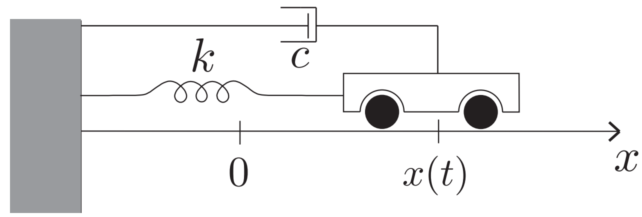
Note: A force of 1 Newton will give a mass of 1 kilogram an acceleration of 1 metres/sec^2.

13. A mechanical system undergoes damped oscillations governed by the DE

$$y'' + 2\lambda y' + w_0^2 y = 0.$$

Suppose that $\zeta = \frac{\lambda}{w_0} = 10^{-5}$. How many oscillations will take place in the time interval during which the amplitude decays by 1%? In this situation the motion of the system approximates SHM over a restricted time interval.

14. Consider the mass-spring system shown to the right. The mass is $m = 1$ kg, the spring constant is k, and the damping force is proportional to the speed, with proportionality constant $c = 10$ kg/second.



(a)-(c) Find *and sketch* the position $x(t)$ of the mass at time t, if:

(a) the spring constant is $k = 16$ N/m and the mass is released from rest from a point 1 metre to the *right* of the equilibrium position.

(b) same as (a), except that the initial velocity of 12 m/s to the *left*.

(c) the constants are $k = 22.5$ N/m and $c = 3$ kg/s, and the mass is released *from rest*, 1 m to the right of the equilibrium position. In this case, how many oscillations take place while the amplitude decays by 90%? How about 99%?

Note: If oscillations occur, be sure to label the quasi-period on your sketch. If oscillations do not occur, find the time, if any, that the mass crosses the equilibrium position. Be sure to make your sketch consistent with this information and with the initial conditions.

(d) Suppose now that the trolley is subject to a sinusoidal force of the form $f(t) = 10\cos\omega t$. Using your notes as a reference, find the frequency ω that results in resonance, and the amplitude of the resulting oscillations. Write down the equation of motion $x(t)$, identifying the transient and steady-state terms. (Don't bother finding c_1 and c_2, but assume the same value of k and c as in the previous part). Finally give a rough sketch of both terms separately, then put them together to get a rough sketch of $x(t)$. (For the sketch, assume the same initial conditions as in part (c).)

15. Solve the following dimensionless IVP

$$\begin{cases} Y'' + \frac{1}{2}Y' + Y = 0 \\ Y(0) = 1, Y'(0) = 0 \end{cases}$$

Is the motion underdamped, overdamped, or critically damped?

16. Consider the dimensionless DE

$$Y'' + 2\zeta Y' + Y = 0,$$

with initial conditions $Y(0) = 1,\ Y'(0) = -V_0$, where $'$ is differentiation with respect to a dimensionless time variable τ, Y is a dimensionless displacement, and ζ is a dimensionless constant satisfying $\zeta \geq 1$, so that oscillations do not occur.

(i) One expects that if V_0 is sufficiently large, then the mass will pass through the equilibrium position before coming to rest. Find the restriction on V_0 that will ensure that this happens, in the case of critical damping $\zeta = 1$. Also find the time τ_{zero} at which Y is zero. (You may assume that $\zeta = 1$ for the remainder of this question.)

(ii) Find the time τ_{crit} at which Y attains its minimum value $Y_{\min}$. Show that τ_{crit} satisfies

$$\tau_{\text{crit}} = \tau_{\text{zero}} + 1,$$

and that

$$Y_{\min} = -(V_0 - 1)e^{-\tau_{\text{crit}}}.$$

(iii) Sketch the graphs of the displacement $Y(\tau)$ and the velocity $Y'(\tau)$, and label τ_{zero}, τ_{crit}, V_0 and $Y_{\min}$.

17. Referring to part (i) of the previous question, show that in the case $\zeta > 1$, the restriction on V_0 is

$$V_0 > \zeta + \sqrt{\zeta^2 - 1}$$

18.* Recall that the solutions for the damped, unforced oscillator can take one of three forms:

i) $y = e^{-\zeta\omega_0 t}\left[c_1 \cos\left(\sqrt{1-\zeta^2}\omega_0 t\right) + c_2 \sin\left(\sqrt{1-\zeta^2}\omega_0 t\right)\right]$, if $\zeta < 1$

ii) $y = (c_1 + c_2 t)e^{-\omega_0 t}$, if $\zeta = 1$

iii) $y = e^{-\zeta\omega_0 t}\left[c_1 e^{\sqrt{\zeta^2-1}\,\omega_0 t} + c_2 e^{-\sqrt{\zeta^2-1}\,\omega_0 t}\right]$, if $\zeta > 1$.

In many practical applications, one would like to design a system which will rapidly return to equilibrium.

It is clear that $\lim\limits_{t\to\infty} y(t) = 0$ in all cases, but in which of the three cases does this occur most rapidly (as time tends to infinity)? Explain, by considering the dominant term of each case.

19. An electric charge q of mass m moves with velocity $\mathbf{v} = (v_1, v_2, v_3)$ in a region of space where there is a magnetic field $\mathbf{B} = (0, 0, B)$ of constant strength. According to the laws of physics, the motion of the charge is given by the following equations:

$$m\frac{dv_1}{dt} = \frac{qB}{c}v_2, \quad m\frac{dv_2}{dt} = -\frac{qB}{c}v_1, \quad m\frac{dv_3}{dt} = 0.$$

where c is the velocity of light in a vacuum. Assume an initial condition of the form $\mathbf{v}(0) = (u_1, u_2, u_3)$.

a) Eliminate v_2 from the first two equations and get a single second order DE for v_1.

b) Show that the quantity $\omega_c = \frac{qB}{mc}$ has the dimensions of a frequency, thus justifying its name, which is the "*cyclotron frequency*" (or "*gyrofrequency*").

c) Solve the resulting DE for v_1, and then find v_2.

d) Identify the curve representing the path of the particle.

20. We wish to find a particular solution of the DE

$$Y'' + 2\zeta Y' + Y = \cos(\Omega\tau). \tag{2}$$

Consider the complex DE

$$Z'' + 2\zeta Z' + Z = e^{i\Omega\tau} \tag{3}$$

The DE (2) is the real part of this DE. Consider a trial function

$$Z = Re^{i(\Omega\tau - \phi)}, \tag{4}$$

where R and ϕ are real constants.

(i) Show that (4) is a solution of (3) if and only if

$$R = [(1-\Omega^2)^2 + 4\zeta^2\Omega^2]^{-1/2}, \tag{5}$$

and

$$\cos\phi = R(1-\Omega^2), \quad \sin\phi = 2R\zeta\Omega \tag{6}$$

(ii) Hence conclude that

$$Y = R\cos(\Omega\tau - \phi),$$

where R and ϕ are given by (5) and (6), is a particular solution of (2).

21. a) Find the general solution of the DE

$$y'' + \omega_0^2 y = \alpha\cos\omega t,$$

where ω_0, α and ω are constants. Which values(s) of ω have to be treated as special cases?

b) Find the unique solution of the DE in a) with $\omega^2 \neq \omega_0^2$, subject to the initial condition $y(0) = 0$, $y'(0) = 0$,

c) Show that the solution in b) can be written in the form

$$y = A(t)\sin\left[\tfrac{1}{2}(\omega_0 + \omega)t\right],$$

where

$$A(t) = \tfrac{2\alpha}{\omega_0^2 - \omega^2}\sin\left[\frac{1}{2}(\omega_0 - \omega)t\right].$$

Give a qualitative sketch of the graph of $y(t)$ in the case where $\omega_0 - \omega$ is small compared to ω_0. If you want to use specific values, use $\omega_0 = 11, \omega = 9$, but don't try to sketch the curve exactly.

d) Show that the unique solution of the DE in a) with $\omega^2 = \omega_0^2$, subject to the initial condition $y(0) = 0$, $y'(0) = 0$ is

$$y = \frac{\alpha}{2\omega_0}t\sin\omega_0 t.$$

Give a qualitative sketch of the graph.

22. Consider a mechanical oscillator with equation of motion

$$y'' + cy' + 4y = \cos\omega t$$

where initial conditions aren't specified and won't matter for this question.

(a) Write the form of the general solution assuming $0 < c < 4$; that is, do everything that you would do to solve the DE but don't solve for any of the coefficients. Then identify the transient and steady-state solutions.

(b) If $c = 0$, write down the form of the general solution and give a sketch of the particular solution $y_p(t)$ showing long-term behaviour. Find and label on your sketch the time per oscillation. (Hint: You must consider two cases, depending on what ω is.)

(c) If $c = 1$ the particular solution may be written as

$$y_p(t) = \frac{4}{\sqrt{(\omega^2-4)^2+\omega^2}}\cos(\omega t - \phi).$$

Find the value of ω, if any, which causes resonance. Then give a rough sketch of $y_p(t)$ using this ω (or a general, unspecified ω if no resonant frequency exists). Be sure to find and label the period on your sketch.

23. The 'Tunnel in the Earth' problem: Suppose that a straight tunnel is drilled from one side of the Earth, through the middle, to the opposite side, and that the Earth is perfectly spherical, has constant density, and that the tunnel itself does not affect the gravitational pull. Then it can be shown using a law from vector calculus (e.g. AMath 231) that the force of gravity at any point in the tunnel is directed toward the centre of the Earth and has magnitude $mg\frac{r}{R}$ where m is the mass of the object in the tunnel, $g \approx 9.81$ (m/s^2) is the acceleration due to gravity at the surface of the Earth, r is the distance from the centre of the Earth to the object, and $R \approx 6.37 \times 10^6$ (m) is the radius of the Earth.

a) Suppose that an object is dropped into the tunnel, and that the only force acting on it is gravity. By letting $r(t)$ be the (signed) position of the object in the tunnel ($r(t) = 0$ corresponds to the centre of the Earth, with positive r on one side and negative on the other), show by deriving and solving an IVP that the object undergoes simple harmonic motion with amplitude R and frequency $\omega = \sqrt{\frac{g}{R}}$. (Remarkably, this does not depend on the mass of the object!)

b) Show that it would take about 42 minutes or so for an object dropped from one side of the Earth to reach the other side of the Earth. What is the maximum speed of the object as it falls? *(In reality, however, tunelling costs, friction, and the molten core of the Earth would make this application impractical.)*

Comment: One can also show that it would take the same time to travel between **any** *two points on the Earth using a straight tunnel between those points.* [4]

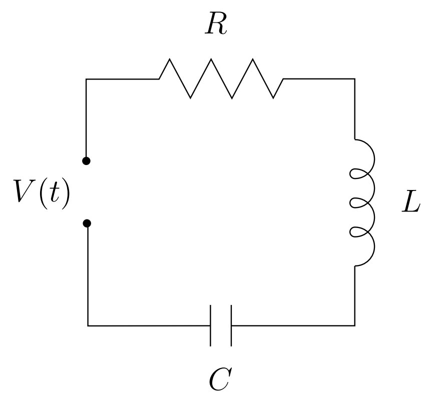
24. The charge $Q(t)$ on the capacitor in the electrical circuit shown satisfies

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t),$$

$L = 0.5$ henrys is the coil's inductance
$R = 6$ ohms is the resistor's resistance
$C = 0.02$ farads is the capacitor's capacitance.

and $V(t)$ is the applied voltage.

[4] This requires a small amount of high school physics knowledge (namely, taking components of a force.)

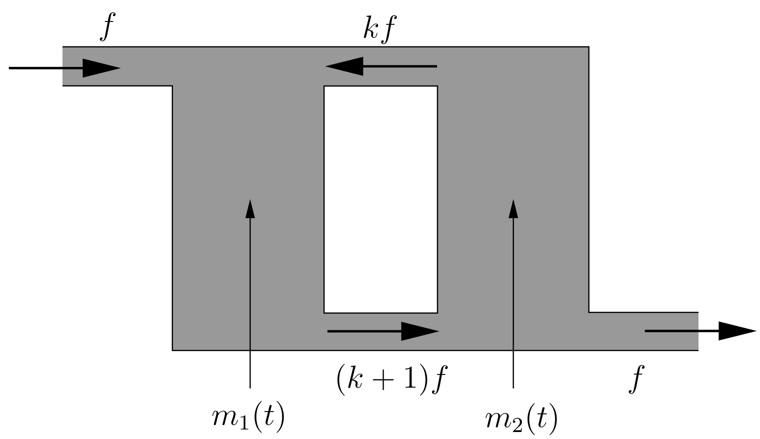


(i) Is the circuit oscillatory? If so, calculate the natural frequency, the unforced (damped) frequency, and the resonant frequency, using the notes as a reference.

(ii) If $V(t) = 24 \sin 10t$ volts and $Q(0) = 0 = Q'(0)$, find $Q(t)$.

(iii) Sketch the transient solution, the steady state solution and the full solution $Q(t)$.

25. Two mixing tanks of volume V are coupled together with flow rates as shown. Let $m_1(t)$ and $m_2(t)$ denote the mass of chemical in each tank at time t.



(i) Write down the first order differential equations satisfied by m_1 and m_2. Use a dimensionless time variable τ defined in the usual way in terms of the characteristic time

$$t_c = \frac{1}{\sqrt{k(1+k)}} \frac{V}{f}.$$

(ii) Show that both m_1 and m_2 satisfy the second order DE

$$m'' + 2qm' + (q^2 - 1)m = (q^2 - 1)Vc_{in}$$

where

$$q = \sqrt{\frac{1+k}{k}},$$

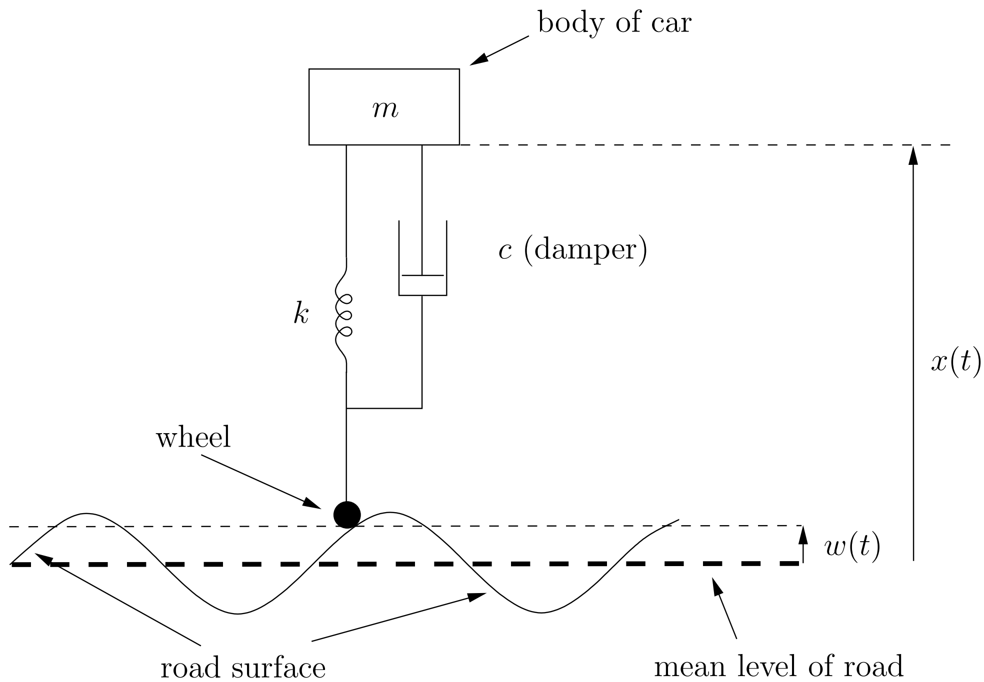
c_{in} is the inflow concentration, and $'$ denotes differentiation with respect to τ.

(iii) Assume that both tanks are initially filled with pure water. Show that

$$m_1(\tau) - m_2(\tau) = \frac{(q^2-1)Vc_{in}}{2q}\left[e^{-(q-1)\tau} - e^{-(q+1)\tau}\right]$$

(iv) What is the long term behaviour of the physical system?

26.* A car suspension system may be modelled by a mass-spring-damper model as shown; the wheel is assumed to ride on a sinusoidally-undulating road, that is the wheel's position $w(t)$ is assumed to be given by $w(t) = w_0\cos(\omega t)$ at time t. *Caution:* the letter w (in our alphabet) represents position, and this is not the same as the frequency ω (omega).



a) Show, using Newton's Second Law, that a suitable differential equation for describing the motion of the mass m is

$$m\ddot{x} + c(\dot{x} - \dot{w}) + k(x - w - \ell) = 0, \qquad (*)$$

where ℓ is the equilibrium length of the spring, and where $\dot{\xi} \equiv d\xi/dt$.

b) Show that the problem may be solved by considering it to be the same as that discussed in the lectures, but with a modified applied force; that is, show that $(*)$ may be written as

$$m\ddot{z} + c\dot{z} + kz = F(t),$$

with suitable choices of z and F. Find $F(t)$ in the form $F(t) = F_0\cos(\omega t - \psi)$.

Problem Set 4
Laplace Transforms and Differential Equations

1. Calculate the Laplace transform $Y = \mathcal{L}(y)$ of each function y, using the definition. Give the domain of validity. *Hint for (iii) and (iv):* Consider $te^{i\pi t}$.

   (i) $y(t) = t^2$

   (ii) $y(t) = te^{-2t}$

   (iii) $y(t) = t\cos \pi t$

   (iv) $y(t) = t\sin \pi t$

   (v) $y(t) = e^{2t}$

   (vi) $y(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ e^{3t}, & t > 1 \end{cases}$

   (vii) $y(t) = te^{4t}$

2. Use the fact that the Laplace transform operator $\mathcal{L}$ is *linear*, and the results from #1(i)-(iii) (or the table at the end of this problem set) to write down the Laplace transform of $y(t) = t(2t + 3e^{-2t} + \pi \sin \pi t)$.

3. Use the fact that the Laplace transform operator $\mathcal{L}$ is *linear*, and the results from #1(v)-(vii) to write down the Laplace transform of $y(t) = 2te^{4t} - 3e^{2t}$.

4. Knowing $\mathcal{L}(e^{\alpha t}) = \frac{1}{s-\alpha}$ for $s > Re(\alpha)$, find the Laplace transforms of

$$\mathcal{L}(e^{at}\cos bt) \quad \text{and} \quad \mathcal{L}(e^{at}\sin bt).$$

   (Don't use a shift theorem.)

5. Show that $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$, for $s > 0$, where n is a non-negative integer.
   Hint: See #10 on the Review Problem Set.

6. Use the table at the end of this problem set to find the inverse Laplace transforms of

$$\text{(i)} \quad Y(s) = \frac{1}{(s+a)(s+b)} \qquad \text{(ii)} \quad Y(s) = \frac{1}{(s^2+a^2)(s^2+b^2)}, \qquad \text{(iii)} \quad Y(s) = \frac{1}{(s+a)(s^2+b^2)},$$

   where a and b are real constants with $a \neq b$.

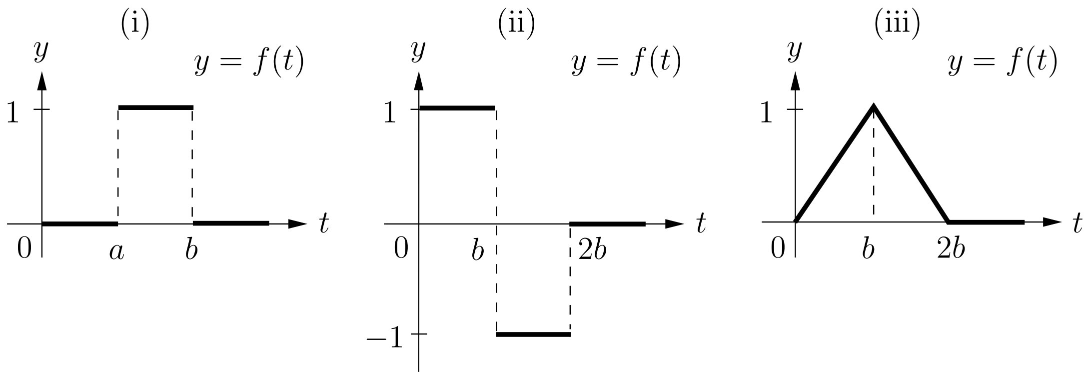
7. Find each inverse Laplace transform:

$$\text{(i)} \ \mathcal{L}^{-1}\left[\frac{5}{s(s^2+4s+5)}\right] \qquad \text{(ii)} \quad \mathcal{L}^{-1}\left[\frac{s-1}{2s^2+5s+2}\right] \qquad \text{(iii)} \quad \mathcal{L}^{-1}\left[\frac{1}{s^2-4s+8}\right]$$

8. Find the Laplace transform of each piecewise defined function (see figure) $f(t)$ in two ways:

   (a) by directly using the definition,

   (b) by expressing $f(t)$ in terms of the Heaviside step function, $H(t)$.



9. Find the inverse Laplace transform using the second shift theorem, and sketch the graph of the resulting function:

$$\text{If} \quad \mathcal{L}^{-1}[F(s)] = f(t), \quad \text{then} \quad \mathcal{L}^{-1}[e^{-cs}F(s)] = H(t-c)f(t-c).$$

(i) $\mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{s+2}\right]$ (ii) $\mathcal{L}^{-1}\left[\dfrac{e^{-\pi s}}{s^2+1}\right]$ (iii) $\mathcal{L}^{-1}\left[\dfrac{e^{-\pi s}}{s^2-1}\right]$.

10. Find the inverse Laplace transform using the second shift theorem, and sketch the graph of the resulting function:

(i) $\mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{s+4}\right]$ (ii) $\mathcal{L}^{-1}\left[\dfrac{e^{-\pi s}}{s^2+4}\right]$ (iii) $\mathcal{L}^{-1}\left[\dfrac{e^{-\pi s}}{s^2-9}\right]$.

11. Derive the equation

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0)$$

for a suitably restricted function $f(t)$. State the hypothesis on $f(t)$.
Hint: Very little work is required; use the formula for the first derivative recursively.

12. Solve the following initial value problems using the Formula Sheet.

(i) $y' + 2y = e^{-3t}$, $y(0) = 5$ (ii) $y' + 2y = 4\cos 2t$; $y(0) = 1$.
(iii) $y'' - y' - 2y = 0$; $y(0) = 1$, $y'(0) = 0$.
(iv) $y'' + y = 3\sin 2t$; $y(0) = 6$, $y'(0) = 1$.

13. Solve the following initial value problems:

(a) $2y''' + 3y'' - 3y' - 2y = e^{-t}$, $y(0) = 0, y'(0) = 0, y''(0) = 1$
(b) $y''' + 2y'' - y' - 2y = \sin 3t$, $y(0) = 0, y'(0) = 0, y''(0) = 1$
(c) $y' + y = e^{-3t}\cos 2t$, $y(0) = 0$

14. A mixing tank with constant volume V_0 and flow rate k is initially filled with pure water. If the inflow concentration is a constant c_{in} for $0 \le t < T$, and is then zero afterwards, calculate the mass of chemical in the tank at time $2T$. *Hint:* use the Heaviside step function.

15. Consider a population $y(t)$ which undergoes exponential growth with growth factor k. Starting at time $t = 0$, the population is harvested at a constant rate h (number per unit time), for a period of time T.

(i) By introducing a dimensionless time τ and a suitably scaled population function $z(\tau)$, show that this system is governed by the DE

$$z' - z = -1 + H(\tau - b),$$

where $b = kT$ and H is the Heaviside step function.

(ii) If the initial population is y_0, find the scaled population z at time τ. Let z_0 be the initial value of z.

(iii) Sketch the family of dimensionless solutions with z_0 as parameter. What happens if $z_0 < 1 - e^{-b}$? What is the physical interpretation?

(iv) For what values of y_0 does the population y fall below y_0 during the interval $0 < t < T$? In this case, at what time does y recover to its initial value?

16. A frozen tofurky (0 degrees C) is placed into a 300-degree oven. Forty minutes later, it is taken out of the oven and placed on the counter. Room temperature is 20 degrees.

a) Use Newton's Law of "cooling" to find the temperature $T(t)$ of the tofurkey for any time $t \geq 0$. Use Laplace transforms.

b) Give a qualitative sketch of $T(t)$.

c) Your solution will have an unknown constant in the exponential. In reality, how can one determine the value of it? (Briefly explain).

17. Consider the system described by the dimensionless DE

$$y' + y = g(t),$$

where the input function $g(t)$ is the saw-tooth function in #8(iii), and the initial condition is $y(0) = 0$.

(i) Based on the graph of the input function, make an educated guess as to the graph of the response $y(t)$.

(ii) Show that the response at time t is

$$y(t) = \tfrac{1}{b}[f(t) - 2H(t-b)f(t-b) + H(t-2b)f(t-2b)],$$

where $f(t) = t - 1 + e^{-t}$.

(iii) Confirm your "educated guess" in (i) by appropriate analysis. In particular show that the maximum response occurs at time $t_{\max} = \ln(2e^b - 1)$, and is given by $y_{\max} = \frac{1}{b}(2b - t_{\max})$.

18. Consider the undamped oscillator DE with a driving force that is a rectangular pulse of duration b:

$$y'' + y = 1 - H(\tau - b),$$

where H is the Heaviside step function.

(i) Use the Laplace transform to show that the unique solution satisfying the initial conditions $y(0) = 0 = y'(0)$ is

$$y(\tau) = 1 - \cos\tau - H(\tau - b)[1 - \cos(\tau - b)],$$

(ii) Show that for $\tau > b$, the response $y(\tau)$ can be expressed as

$$y(\tau) = A\sin(\tau - \delta),$$

i.e. simple harmonic motion. Express the amplitude A and phase δ in terms of the pulse duration b.

(iii) Sketch the full response for $b = 2\pi, 3\pi$ and $2\pi + \epsilon$, where $0 < \epsilon \ll 1$. *Hint:* use the identity on page 77.

(iv) For what values(s) of b is the amplitude of the long-term response a maximum? zero?

19. Consider the undamped oscillator DE with a driving force that happens to be discontinuous:

$$y'' + y = 2 - H(\tau - b),$$

where H is the Heaviside step function.

(i) Use the Laplace transform to find the unique solution satisfying the initial conditions $y(0) = 0 = y'(0)$.

(ii) Show that for $\tau > b$, the response $y(\tau)$ can be written in the form

$$y(\tau) = A\cos(\tau - \delta) + c$$

i.e. simple harmonic motion. Express A and $\tan\delta$ in terms of b.

20. Consider the modified skydiver DE

$$m\frac{dv}{dt} = mg - \alpha v + f(t).$$

Suppose that the skydiver begins firing her rocket straight down at some time $t = T$, so that the extra force $f(t)$ is given by

$$f(t) = \begin{cases} 0, & \text{if} \quad 0 \le t < T \\ -\frac{1}{2}mg, & \text{if} \quad t \ge T. \end{cases}$$

a) Write down an expression for $f(t)$ using the Heaviside step function.

b) Substitute your expression for $f(t)$ into the modified skydiver DE above, and by introducing *the usual* characteristic velocity and time, non-dimensionalize the resulting DE to get

$$\frac{dV}{d\tau} = 1 - V - \frac{1}{2}H(\tau - b)$$

where b is a convenient constant (you should state what it equals). *(Comment: do NOT use T as the characteristic time).*

c) Solve the dimensionless DE for $V(\tau)$, assuming that the skydiver starts from rest.

d) Sketch $V(\tau)$, then give a physical interpretation of $\lim_{\tau\to\infty} V(\tau)$.

21. Suppose that a cake mixture is at room temperature (70° Farenheit) and is placed into an oven at $t = 0$. The oven is *not* preheated, but is turned on at $t = 0$ and its temperature increases linearly until $t = 4$ minutes at which point it reaches 300°, and remains at that temperature thereafter. Find the temperature $T(t)$ of the cake at any time t, using Newton's law of "cooling" and the Heaviside step function.
Comment: Your answer will depend on the unknown constant k.

22.* **Existence of a Laplace Transform**
Certain conditions were given which guarantee the existence of the Laplace transform. There are, however, functions which don't satisfy these conditions but still have a Laplace transform. In fact, one of the following exists:

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} \quad or \quad \mathcal{L}\left\{\frac{1}{t}\right\}$$

(a) Which one? A heuristic argument is sufficient here.

(b) Given that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ (showing this is a Math 237 problem), show that one of the transforms above equals $\sqrt{\frac{\pi}{s}}$.

23.* **Transform of a periodic function.** Prove that if f is periodic with period T and piecewise continuous for $t \geq 0$, then

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)dt.$$

Problems related to Convolution, Transfer Function, and Dirac Delta:

24. Let $*$ denote the convolution operation. Show that

i) $e^{\alpha t} * e^{\alpha t} = te^{\alpha t}$ for any $\alpha \in \mathbb{C}$.

ii) $e^{\alpha t} * e^{\beta t} = \dfrac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta}$ for any $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$.

25. Use complex functions to verify that

i) $2\cos at * \sin at = t\sin at$.

ii) $\cos at * \cos at - \sin at * \sin at = t\cos at$.

iii) $\cos at * \cos at + \sin at * \sin at = \frac{1}{a}\sin at$.

26. Prove that the convolution operation is commutative, i.e. $f * g = g * f$.

27. Use the convolution theorem to find the inverse Laplace transform of

i) $G(s) = \dfrac{1}{(s+1)^2(s+2)}$ ii) $G(s) = \dfrac{s}{(s^2+1)(s+2)}$ iii) $G(s) = \dfrac{1}{(s^2+1)(s+2)}$.

Hint: Use complex functions for ii) and iii).

28. Use the Convolution Theorem to prove the *time integration formula*:

$$\text{if} \quad \mathcal{L}[f(t)] = F(s), \quad \text{then} \quad \mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s).$$

29. Derive the *frequency integration formula*:

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\sigma)d\sigma.$$

Hint: Let $\mathcal{L}\left[\frac{f(t)}{t}\right] = G(s)$ and $\mathcal{L}[f(t)] = F(s)$. Show that $G'(\sigma) = -F(\sigma)$, integrate from $\sigma = s$ to $\sigma = a$ and take the limit as $a \to +\infty$.

30. i) Derive the *frequency differentiation formula*:

$$\text{if} \quad \mathcal{L}[f(t)] = F(s), \quad \text{then} \quad \mathcal{L}[-tf(t)] = F'(s), \quad \text{for} \quad s > a.$$

It is assumed that $f(t) = O(e^{ct})$ as $t \to +\infty$. You may assume that differentiating inside the integral is valid.

ii) Generalize i) to give a formula for the n^{th} derivative $F^{(n)}(s)$.

31. *A link between the Laplace transform and the gamma function.*

Use the definition of the Laplace transform to show that

$$\mathcal{L}[\, t^p \,] = \frac{\Gamma(p+1)}{s^{p+1}},$$

for all $p > -1$.

Hint: Make the change of variable $st = u$ in the integral.

32. Prove that if f is periodic of period T, and piecewise continuous for $t \geq 0$, then

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t) dt.$$

33. Solve the integral equation

$$\int_0^y \frac{f(\tau)}{\sqrt{y - \tau}} d\tau = \sqrt{2g}\, T,$$

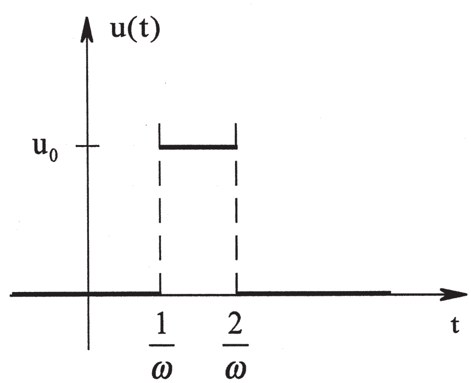
for the unknown function $f(y)$, where g and T are constants.[5]

34. i) Express the solution of the initial value problem

$$y'' - \omega^2 y = u(t),$$
$$y(0) = 0, \quad y'(0) = 0$$

as a convolution. Give the answer in terms of hyperbolic functions.

ii) Find the asymptotic form of the response as $t \to +\infty$ to the given input $u(t)$.



35. The current $y(t)$ in an RLC circuit with applied voltage $e(t)$ satisfies the DE

$$y'' + \frac{R}{L} y' + \frac{1}{LC} y = \frac{1}{L} e'(t).$$

Find the transfer function $G(s)$, regarding $e(t)$ as the input and $y(t)$ as the response. Assume that $e(t) = 0$ for $t \leq 0$.

[5]This integral equation arises in solving Abel's mechanical problem: consider a wire bent into a smooth curve and let a bead start from rest and slide without friction down the wire to the origin under its own weight. For what curve will the time of descent be a constant T, irrespective of the starting height?

36. A linear time-invariant system is of the form

$$y'' + a_1 y' + a_0 y = u(f(t), f'(t), \ldots).$$

Given that the transfer function of the system is $G(s) = \frac{2+5}{s^2+4s+3}$, find the DE relating $y(t)$ and $f(t)$.

37. i) Consider the linear time invariant system described by the scalar DE

$$y' + ky = 0,$$

where k is a positive constant.

At time $t = 0$, the state is $y(0) = y_0$. At time $t_1 > 0$, an instantaneous impulse of magnitude p is applied to the system. Find the response $y(t)$ for $t > 0$, and sketch its graph.

ii) Suppose that an idealized impulse of magnitude p is applied periodically, starting at $t = 0$, with period T, to the system in i). Find the periodic steady state response $y_{\text{per}}(t)$, and sketch its graph. In particular, give the maximum and minimum values of y for this response.

38. Consider the linear time-invariant system described by the DE

$$y'' + \omega^2 y = 0,$$

with initial condition $y(0) = 0$, $y'(0) = v_0$. At time $t = t_0$ an idealized impulse $p\delta(t - t_0)$ is applied which brings the system momentarily to rest.

i) Find the value of p that will achieve this.

ii) Find the response $y(t)$ for all $t > 0$.

iii) Sketch the graphs of $y(t)$ and $y'(t)$, for $t > 0$.

39. Suppose that the transfer function of a linear time-invariant system is

$$G(s) = \frac{s}{(s^2+1)(s+2)}.$$

Find the response of the system in the time domain to

i) an idealized unit impulse at time $t_0 > 0$,

ii) a unit step input at time $t_0 > 0$.

40. i) Calculate $\mathcal{L}\left[\delta_\varepsilon(t-a)\right]$, where δ_ε is the pulse function, defined by

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & \text{if} \quad 0 \le t < \varepsilon \\ 0, & \text{otherwise,} \end{cases}$$

and $a \ge 0$.

ii) Show that $\lim\limits_{\varepsilon \to 0^+} \mathcal{L}\left[\delta_\varepsilon(t-a)\right] = e^{-as}$, thereby providing further justification for the definition

$$\mathcal{L}\left[\delta(t-a)\right] = e^{-as}.$$

41. Use the Laplace transform to show that the two initial value problems

    i) $my'' + cy' + ky = p\delta(t), \qquad y(0) = 0, \quad y'(0) = 0,$

    ii) $my'' + cy' + ky = 0, \qquad y(0) = 0, \quad y'(0) = \frac{p}{m},$

    have the same solution for $t > 0$.

    Comment: This result shows that an idealized impulse of magnitude p imparts an initial velocity of $v_0 = p/m$ to a mass-spring system that is initially at rest.

42. An undamped mass-spring system of natural frequency ω is initially at rest. At each time $t = 2\pi n/\omega$, $n = 0, 1, 2, \ldots$ the mass is struck with a hammer which imparts an impulse of magnitude p per unit mass in the positive direction. Determine the resulting motion. Comment on the long term behaviour. Sketch the graphs of $y(t)$ and $y'(t)$. How smooth is $y(t)$?

Formula Sheet:

Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\sin kt$	$\dfrac{k}{s^2+k^2}$
$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f'(t)$	$sF(s) - f(0)$
$e^{at}f(t)$	$F(s-a)$ (First Shift Theorem)
$f(t-a)H(t-a)$	$e^{-as}F(s)$ (Second Shift Theorem)
$\delta(t-a)$	e^{-as}

Problem Set 5
Linear Vector DEs

1. In each case verify that the given vector-valued function satisfies the vector DE $\mathbf{x}' = A\mathbf{x}$.

   i) $A = \begin{pmatrix} 2 & 6 \\ -2 & -5 \end{pmatrix}$; $\quad \mathbf{x}(t) = e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

   ii) $A = \begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix}$; $\quad \mathbf{x}(t) = e^{-t}\begin{pmatrix} 5\cos t \\ -2\cos t - \sin t \end{pmatrix}$

   iii) $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$; $\quad \mathbf{x}(t) = e^{t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right]$.

2. a) Give a description of the *eigenvalue method* for solving a vector DE.

   b) Find the general solution of each vector DE using the eigenvalue method, referring to your explanation in a).

   i) $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}\mathbf{x}$ $\qquad$ ii) $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$

   iii) $\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}\mathbf{x}$ $\qquad$ iv) $\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\mathbf{x}$.

3. Give a qualitative sketch of the orbits for each DE in #2.

4. Find the general solution of each vector DE using the eigenvalue method. (For (c), use any method you like — there is a faster way!)

   (a) $\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -1 & -4 \end{pmatrix}\mathbf{x}$

   (b) $\mathbf{x}' = \begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix}\mathbf{x}$

   (c) $\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\mathbf{x}$

5. Give a qualitative sketch of the orbits for each DE in #4.

6. Find the general solution of each vector DE using the eigenvalue method.

   (a) $\mathbf{x}' = \begin{pmatrix} 10 & -5 \\ 8 & -12 \end{pmatrix}\mathbf{x}$

   (b) $\mathbf{x}' = \begin{pmatrix} 1 & -8 \\ 1 & -3 \end{pmatrix}\mathbf{x}$

   (c) $\mathbf{x}' = \begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix}\mathbf{x}$

7. Give a qualitative sketch of the orbits for each DE in #6.

8. Suppose a mechanical oscillator obeys the IVP

$$\begin{cases} y'' + 5y' + 4y = 0 \\ y(0) = 0, \quad y'(0) = 1 \end{cases} \tag{1}$$

a) Write this as a first-order vector IVP, that is, an IVP of the form

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{a} \end{cases} \tag{2}$$

where A is a constant 2×2 matrix and $\mathbf{a}$ is a constant vector.

b) Find the solution to (2), hence write down the solution to (1).

c) Sketch the phase orbits of the general solution to the vector DE in (2), highlighting the solution curve to the IVP (2) for $t \geq 0$.

d) Is the motion under-, over-, or critically damped?

9. Suppose a mechanical oscillator obeys the IVP

$$\begin{cases} y'' + 4y' + 5y = 0 \\ y(0) = 0, \quad y'(0) = 1 \end{cases} \tag{1}$$

(a) Write this as a first-order vector IVP, that is, an IVP of the form

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{a} \end{cases} \tag{2}$$

where A is a constant 2×2 matrix and $\mathbf{a}$ is a constant vector.

(b) Find the solution to (2), hence write down the solution to (1).

(c) Sketch the phase orbits of the general solution to the vector DE in (2), highlighting the solution curve to the IVP (2) for $t \geq 0$.

(d) Based on your phase portrait, is the motion under-, over-, or critically damped?

(e) Is the speed of the mass at a maximum just *before, after,* or *as* it crosses the equilibrium position for the first time? *[Hint: speed* $= |y'|$*; use (c).]*

10. Suppose a mechanical oscillator obeys the IVP

$$\begin{cases} y'' + 4y' + 4y = 0 \\ y(0) = 0, \quad y'(0) = 1 \end{cases} \tag{1}$$

(a) Write this as a first-order vector IVP, that is, an IVP of the form

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{a} \end{cases} \tag{2}$$

where A is a constant 2×2 matrix and $\mathbf{a}$ is a constant vector.

(b) Find the solution to (2), hence write down the solution to (1).

(c) Sketch the phase orbits of the general solution to the vector DE in (2), highlighting the solution curve to the IVP (2) for $t \geq 0$.

(d) Is the motion under-, over-, or critically damped?

11. (a) Give a description of the Laplace transform method for finding the *fundamental matrix* of a linear vector DE.

(b) Find the fundamental matrix for each DE in #2, referring to your description in part (a).

(c) Verify that $\Phi(t)$ has the properties

$$\Phi(0) = I, \qquad \Phi(-t) = [\Phi(t)]^{-1}.$$

(d) Find the unique solution for each DE in #2 that satisfies the initial condition $\mathbf{x}(0) = \begin{pmatrix}1\\2\end{pmatrix}$.

12. (a) Find the general solution to the homogeneous vector DE

$$\mathbf{x}' = \begin{pmatrix}-1 & 2\\ -1 & -4\end{pmatrix}\mathbf{x}$$

using any method you like.

(b) Write the solution to the DE in (a) in the form $\mathbf{x}(t) = \Phi(t)\mathbf{a}$ where $\mathbf{a} = \mathbf{x}(0)$. What is the name of the matrix $\Phi(t)$? Did you have fun finding it?

(c) Give a detailed sketch of the phase portrait of $\mathbf{x}(t)$.

(d) Explain, using a phase orbit, why the matrix inverse of $\Phi(t)$ is just $\Phi(-t)$; that is, use a geometrical argument to show that $\Phi(t)\Phi(-t) = I$ where $I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is the identity matrix.

13. a) Give a description of the *variation of parameters method* for solving an inhomogeneous linear vector DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t). \qquad (*)$$

Explain how to find a particular solution, and how to find the solution which satisfies an arbitrary initial condition $\mathbf{x}(0) = \mathbf{a}$.

b) Find a particular solution of the DE (*) for the 2×2 matrices given in #2, and the input functions given below:

(i) $\mathbf{f}(t) = e^{-3t}\begin{pmatrix}1\\1\end{pmatrix}$ (ii) $\mathbf{f}(t) = \begin{pmatrix}2\cos 2t\\ \sin 2t\end{pmatrix}$

(iii) $\mathbf{f}(t) = e^{-t}\begin{pmatrix}0\\1\end{pmatrix}$ (iv) $\mathbf{f}(t) = \cos t\begin{pmatrix}1\\-1\end{pmatrix}$.

Note: (i) corresponds to the DE in #2(i), (ii) corresponds to (ii), etc.

14. Find the general solution to the vector DE

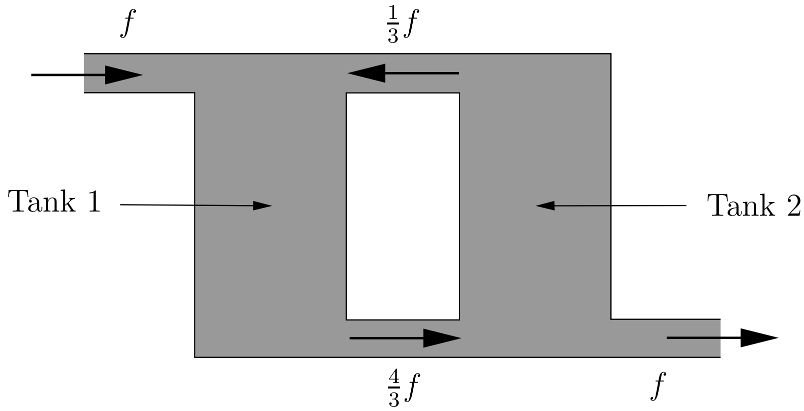
$$\mathbf{x}' = \begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}1\\2\end{pmatrix}$$

using the method of variation of parameters. (For extra practice later, consider repeating the problem using undetermined coefficients and/or Laplace transforms.)

15. Consider the coupled constant volume mixing tank system as shown with inflow concentration $c_{in}(t)$, with state vector

$$\mathbf{x} = \begin{pmatrix}m_1\\ m_2\end{pmatrix},$$

where m_1 and m_2 denote the mass of chemical in tanks 1 and 2 respectively. Let V be the volume of each tank.



a) Show that a vector DE governing the state of the system is

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}, \qquad A = \begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 3c_{in}V \\ 0 \end{pmatrix},$$

where $'$ denotes differentiation with respect to $\tau = \frac{f}{3V}t$. (The factor "3" is included to simplify the numbers.)

b) Find the fundamental matrix for the homogeneous DE.

c) Find the solution for the following initial conditions, assuming $c_{in}(t) = 0$:

i) $m_1(0) = M$, $m_2(0) = 0$ ii) $m_1(0) = \frac{1}{3}M$, $m_2(0) = \frac{2}{3}M$ iii) $m_1(0) = 0$, $m_2(0) = M$.

In each case give a qualitative sketch of the mass functions $m_1(\tau)$ and $m_2(\tau)$ on the same axes. Use the graphs to give a physical interpretation of the behaviour of the system, discussing whether the mass of chemical in each tank is increasing or decreasing and whether the masses are ever equal.

d) Referring to c), in which case does the system flush most rapidly, i.e. in which case does the total mass in the system tend to zero most rapidly? First make an "educated guess", and then give a mathematical analysis.

e) Sketch typical orbits of the DE in $\mathbb{R}^2$, subject to the restriction $m_1 \geq 0$, $m_2 \geq 0$. (You may still assume that $c_{\text{in}}(t) = 0$.)

(i) Mark the orbits corresponding to the three solutions in part c) on your sketch.

(ii) Consider an initial state with $m_2(0) < m_1(0)$. Use the sketch to describe the future evolution of the system.

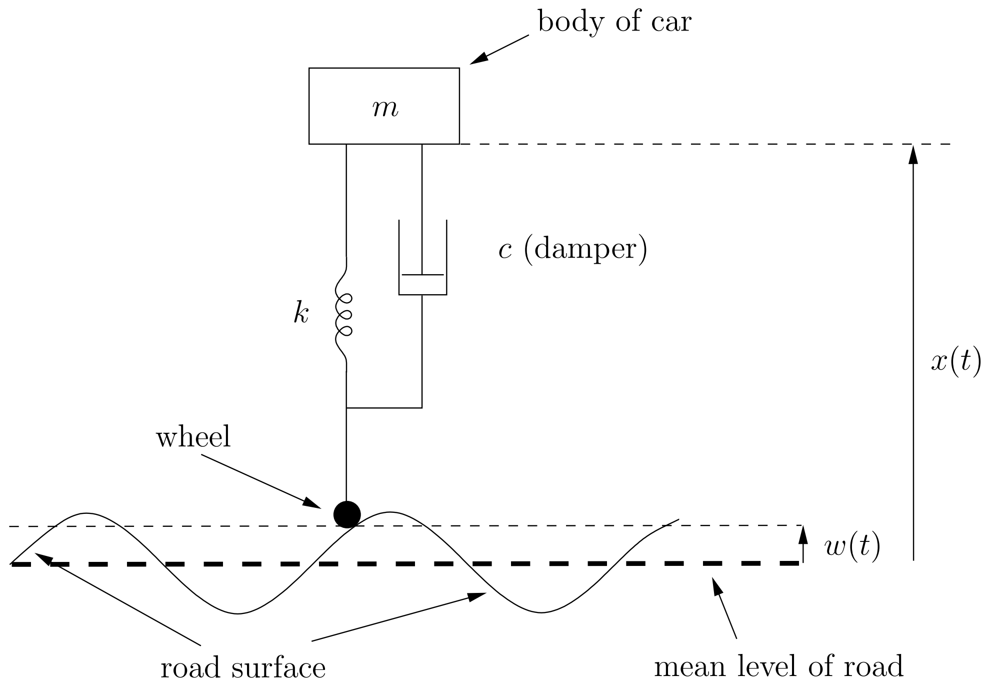
(iii) Do the same for an initial state with $m_2(0) > 4m_1(0)$.

f) Find the solution of the non-homogeneous DE assuming $c_{in}(t) = c$, a constant, and an arbitrary initial state $\mathbf{x}(0) = \mathbf{a}$. What is the asymptotic behaviour as $t \to +\infty$?

16.* A car suspension system may be modelled by a mass-spring-damper model as shown; the wheel is assumed to ride on a sinusoidally-undulating road, that is the wheel's position $w(t)$ is assumed to be given by $w(t) = w_0 \cos(\omega t)$ at time t. The equilibrium length of the spring (taking into account gravity already) is ℓ.

Show that the system may be modelled by a first-order vector DE of the form

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$$



and find the 2×2 matrix A and the forcing function $\mathbf{f}(t)$. Be sure to specify what each component of $\mathbf{x}(t)$ is, and show your work. *Hint:* see problem set 3.

Answers to Selected Odd-Numbered Problems

Purpose: to allow you to verify your answer AFTER you have tried a problem.

Note: Answers to challenging (*) questions are not included — ask your instructor or a TA about these. Other unlisted problems are typically sketches or "show that..."-type questions which of course include the answer in the question. Please report any errors/typos to Joe West at jjwest@uwaterloo.ca if this is the current edition of the course notes.

Review Problem Set

1. Verify your answer by differentiation.
11. $1/\sqrt{2}$ m.

Problem Set 1

1.(a) (i) $y' = e^x$ (ii) $y' = -2y+4$ (iii) $y' = 1-y+x$ (iv) $y' = 2-y/x$ (v) $y' = e^{x-y}$ (vi) $\frac{2y}{x^2+y^2}(x+yy') = y'$
3. Solution is $y = e^{-x}(1 + C/x)$.
5. (i) $y = 1/(2 - \sin x)$ for all x
(ii) $y = 1/(-\sin x + 0.5))$, $\frac{-7\pi}{6} < x < \frac{\pi}{6}$ (iii) $y = x^2(\ln x + e)$, $x > 0$ (iv) $y = -\ln(e^{-x} - 0.5)$, $x < \ln 2$
7. (Plug in your answer to verify.)
9. None, Separable & Sep. Vars., Linear and Int. Factor, All.
11. $y = Ce^{-kt} + \frac{A}{k^2+\omega^2}(k \sin \omega t - \omega \cos \omega t)$ (a) Only sol'n with $C = 0$ is periodic. (b) $y(t) \to y_p(t)$ as $t \to \infty$ but no unique limit exists. (c) Transient term is Ce^{-kt}, Steady-state term is $y_p(t)$.
13. (a) $v(t) = \frac{mg}{\alpha} + (v_0 - \frac{mg}{\alpha})e^{\frac{-\alpha}{m}t}$. (b) No. (c) $y(t) = \frac{mg}{\alpha}t + \frac{m}{\alpha}(v_0 - \frac{mg}{\alpha})(1 - e^{\frac{\alpha}{m}t})$.
15. Approx. 857 students; most rapid at 5.6 days.
17. $m(t) = 200 + 100e^{\frac{-1}{50}t}$.
19. $\frac{89}{300}$ grams per litre.
21. (a) $-v_0/g$ (b) $\frac{m}{\alpha}\ln(1 - \frac{\alpha v_0}{mg})$ using $v_0 < 0$.
25. (b) $100\ln(H/(H-40))$ (c) $H = rN_0$.
27. (a) $y(t) = \frac{k_1 x_0}{k_2-k_1}e^{-k_1 t} + Ce^{-k_2 t}$ (b) About 66 days.

Problem Set 2

1. (i) $M/(L^2T^2)$ (ii) ML^{-4} (iii) MT^{-2} (iv) $1/T$ (v) $ML^{-1}T^{-2}$ (vi) $M/(LT)$
3. L^2/T.
7. $v(r) = \sqrt{Rg}\sqrt{\frac{2R}{r} + \frac{v_{\text{init}}^2}{Rg} - 2}$. 9. (a) $t_c = 1/r$, $\frac{d\mathcal{N}}{d\tau} = \mathcal{N}(1-\mathcal{N})$. (c) $\frac{du}{d\tau} = -(u-1)$.
11. (a) $1/T$ (b) $\hat{T} = T/T_A$, $\tau = t/(1/k) = kt$ (c) $\frac{d\hat{T}}{d\tau} = -(\hat{T} - 1)$ (d) $T = T_A + (T_0 - T_A)e^{-kt}$.
13. (a) $t_{\text{vert}} = \frac{2\pi r^2\sqrt{\ell}}{k}$ (b) $t_{\text{horiz}} = \frac{4\ell d^{3/2}}{3k}$ (c) $\frac{9\pi^2}{64}$.
15. (a) $v_{\text{esc}}^2 = CRg$ for some constant C. (b) $2\sqrt{3}$ times as large.
17. $\nu = C\sqrt{\rho G}$.
21. $U = D\Omega f(\nu/(\Omega D^2))$.
25. (a) $E = C\frac{\rho R^5}{t^2}$ (b) $R \propto t^{2/5}$.

Problem Set 3

1, 3, 5: Check answer by plugging back in.

7. $y(t) = \begin{cases} c_1 e^{m_1 t} + c_2 e^{m_2 t} & \text{if } k \neq 0, k^2 > b^2 \\ c_1 \cos bt + c_2 \sin bt & \text{if } k = 0 \\ c_1 e^{-kt}\cos \alpha t + c_2 e^{-kt}\sin \alpha t & \text{if } k \neq 0, k^2 < b^2 \end{cases}$ where $\alpha = \sqrt{b^2 - k^2}$.

9. $y(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + \frac{\alpha(2-\omega^2)}{\omega^4+4}\cos\omega t + \frac{2\alpha\omega}{\omega^4+4}\sin\omega t$.
13. Approx. 160 oscillations.
15. (Check by pluggling in.) Underdamped.
19. (a) $v_1'' + \left(\frac{qB}{mc}\right)^2 v_1 = 0$ (c) $v_1(t) = c_1\cos\left(\frac{qB}{mc}t\right) + c_2\sin\left(\frac{qB}{mc}t\right)$; $v_2(t) = \left(\frac{qB}{mc}\right)^2\left(c_1\sin\left(\frac{qB}{mc}t\right) - c_2\cos\left(\frac{qB}{mc}t\right)\right) + C_3$ (d) Closed curves in horizontal plane.
21. (a) $y(t) = \begin{cases} c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{\alpha}{\omega_0^2-\omega^2}\cos\omega t & \text{if } \omega \neq \omega_0 \\ c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{\alpha}{2\omega_0}t\sin\omega_0 t & \text{if } \omega = \omega_0 \end{cases}$ (b) $c_1 = \frac{-\alpha}{\omega_0^2-\omega^2}$, $c_2 = 0$.
23(b) Max speed is $R\omega \approx 7900$ m/s.
25. Long term behaviour: concentration in tanks approach input concentration.

Problem Set 4
1. (Verify answer using table.)
3. $\frac{2}{(s-4)^2} - \frac{3}{s-2}$.
7. (i) $1 - e^{-2t}\cos t - 2e^{-2t}\sin t$ (ii) $-\frac{1}{2}e^{-\frac{1}{2}t} + e^{-2t}$ (iii) $\frac{1}{2}e^{2t}\sin 2t$.
9. (i) $e^{-2(t-3)H(t-3)}$ (ii) $\sin(t-\pi)H(t-\pi)$ (iii) $\sinh(t-\pi)H(t-\pi)$ where sinh is the hyperbolic sine.
13. (a) $\frac{1}{2}e^{-t} + \frac{5}{18}e^{t} - \frac{8}{9}e^{-t/2} + \frac{1}{9}e^{-2t}$ (b) $\frac{13}{60}e^{t} - \frac{13}{20}e^{-t} + \frac{16}{39}e^{-2t} + \frac{3}{130}\cos 3t - \frac{1}{65}\sin 3t$ (c) $\frac{1}{4}e^{-t} - \frac{1}{4}e^{-3t}\cos 2t + \frac{1}{4}e^{-3t}\sin 2t$.
15. (ii) $z(\tau) = z_0 e^{\tau} + 1 - e^{\tau} - (1 - e^{\tau-b})H(\tau - b)$ (iii) Extinction. (iv) $\frac{h}{r} - \frac{h}{r}e^{-rT} < y_0 < \frac{h}{r}$.
19. (i) $y(\tau) = 2 - 2\cos\tau - (1 - \cos(\tau - b))H(\tau - b))$.
21. $T(t) = 70 + 57.5(1/k + t - (1/k)e^{kt}) - 57.5(1/k + t - 4 - (1/k)e^{k(t-4)})H(t-4)$.
27. (i) $(t-1)e^{-t} + e^{-2t}$ (ii) $\frac{2}{5}\cos t + \frac{1}{5}\sin t - \frac{2}{5}e^{-2t}$ (iii) $\frac{-1}{5}\cos t + \frac{2}{5}\sin t + \frac{1}{5}e^{-2t}$.
33. $f(y) = \frac{\sqrt{2g}T}{\pi\sqrt{y}}$.
35: $G(s) = \frac{s}{Ls^2+Rs+\frac{1}{C}}$.
37: $y(t) = y_0 e^{-kt}$ for $t < t_1$ and $y_0 e^{-kt} + pe^{-k(t-t_1)}$ for $t \geq t_1$. (ii) $y(t) = y_0 e^{-kt} + pe^{-kt} + pe^{-k(t-T)} + pe^{-k(t-2T)} + ...$, no max, but supremum is $\frac{p}{1-e^{kT}}$.
39: (i) $y(t) = \frac{2}{5}\cos(t-t_0)H(t-t_0) + \frac{1}{5}\sin(t-t_0)H(t-t_0) - \frac{2}{5}e^{-2(t-t_0)}H(t-t_0)$. (ii) $y(t) = \frac{-1}{5}\cos(t-t_0)H(t-t_0) + \frac{2}{5}\sin(t-t_0)H(t-t_0) + \frac{1}{5}e^{-2(t-t_0)}H(t-t_0)$.

Problem Set 5
9. (a) $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix}\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where $\mathbf{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$. (b) $\mathbf{x}(t) = \begin{pmatrix} e^{-2t}\sin t \\ -2e^{-2t}\sin t + e^{-2t}\cos t \end{pmatrix}$, $y(t) = e^{-2t}\sin t$. (d) underdamped (e) before.
11. (b) (i) $\begin{pmatrix} (1/2)e^{-t} + (1/2)e^{-5t} & (1/2)e^{-t} - (1/2)e^{-5t} \\ (1/2)e^{-t} - (1/2)e^{-5t} & (1/2)e^{-t} + (1/2)e^{-5t} \end{pmatrix}$ (ii) $\begin{pmatrix} e^{-t}\cos 2t & -2e^{-t}\sin 2t \\ (1/2)e^{-t}\sin 2t & e^{-t}\cos 2t \end{pmatrix}$ (iii) $\begin{pmatrix} e^{-t} & 2te^{-t} \\ 0 & e^{-t} \end{pmatrix}$ (iv) $\begin{pmatrix} 1/2 + (1/2)e^{-2t} & -1/2 + (1/2)e^{-2t} \\ -1/2 + (1/2)e^{-2t} & 1/2 + (1/2)e^{-2t} \end{pmatrix}$ (d) Just multiply fun. matrix with vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
13. (b) (i) $\begin{pmatrix} (-1/2)e^{-2t} \\ (-1/2)e^{-2t} \end{pmatrix}$ (ii) $\begin{pmatrix} 2\cos 2t - 2e^{-t}\cos 2t \\ \sin 2t - e^{-t}\sin 2t \end{pmatrix}$ (iii) $\begin{pmatrix} t^2 e^{-t} \\ te^{-t} \end{pmatrix}$ (iv) $\begin{pmatrix} \sin t \\ -\sin t \end{pmatrix}$
15. (b) $\begin{pmatrix} (1/2)e^{-2\tau} + (1/2)e^{-6\tau} & (1/4)e^{-2\tau} - (1/4)e^{-6\tau} \\ e^{-2\tau} - e^{-6\tau} & (1/2)e^{-2\tau} + (1/2)e^{-6\tau} \end{pmatrix}$ (c) Just multiply fun. matrix by IC. (d) Case (iii) flushes most rapidly. (e) $\mathbf{x}(\tau) \to \begin{pmatrix} c_{\text{in}}V \\ c_{\text{in}}V \end{pmatrix}$ as $\tau \to \infty$.