ON HOW TO DRAW A GRAPH

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Abstract. We give an overview of Tutte's paper, "How to draw a graph", that contains: (i) a proof that every simple 3-connected planar graph admits a straight-line embedding in the plane such that each face boundary is a convex polygon, (ii) an elegant algorithm for finding such an embedding, (iii) an algorithm for testing planarity, and (iv) a proof of Kuratowski's theorem.

1. Introduction

We give an overview on Tutte's paper on "How to draw a graph". This exposition is based on Tutte's paper [1], as well as a treatment of the topic by Lovász [2], and a series of talks by Chris Godsil.

A *convex embedding* of a circuit C is an injective function $\phi : V(C) \to \mathbb{R}^2$ that maps C to a convex polygon with each vertex of C being a corner of the polygon. A *convex embedding* of a 2-connected planar graph G is an injective function $\phi : V \to \mathbb{R}^2$ such that (i) ϕ determines a straight-line plane embedding of G, and (ii) each facial circuit of the embedding is convex. (The condition that G is 2-connected ensures that faces are bounded by circuits.)

Tutte [1] proves that every simple 3-connected planar graph admits a convex embedding. He also gives an efficient and elegant algorithm for finding such an embedding, he gives and efficient planarity testing algorithm, and he proves Kuratowski's Theorem. There are faster planarity testing algorithms and shorter proofs of Kuratowski's Theorem, but the methods are very attractive.

Given a circuit C of a graph G, a *quasi-convex embedding* of (G, C) is a function $\phi : V \to \mathbb{R}^2$ such that:

- ϕ is a convex embedding of C, and
- for each $v \in V(G) - V(C)$, either v is in the relative interior of the convex hull of $\phi(N_G(v))$, or $\phi(N_G(v)) = \{\phi(v)\}$.

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Note that the definition of a quasi-convex embedding says nothing about planarity.

A circuit C of a graph G is *peripheral*, if G/C has one block. (Note that each loop is a block.)

Theorem 1.1 (Tutte, 1963)**.** *Let C be a peripheral circuit in a 3-connected graph $G = (V, E)$. If G is planar, then every quasi-convex embedding of (G, C) is a convex embedding of G.*

The *barycentre* of vectors $v_1, \dots, v_k \in \mathbb{R}^2$ is $\frac{1}{k}(v_1 + \cdots + v_k)$. The following lemma, which we will prove in Section 3, can be used to obtain a quasi-convex embedding.

Lemma 1.2. *Let C be a circuit in a simple connected graph $G = (V, E)$. If ϕ is a convex embedding of C, then ϕ extends uniquely to a function $V \to \mathbb{R}^2$ such that each $v \in V(G) - V(C)$ is embedded at the barycentre of its neighbours.*

We will refer to any embedding obtained from Lemma 1.2 as a *barycentric embedding.* Note that every barycentric embedding is quasi-convex. The problem of finding a barycentric embedding gives rise to a system of linear equations which can be solved efficiently.

Following Tutte, we prove a strengthening of Theorem 1.1 that contains Kuratowski's Theorem. The reader is primarily interested in Theorem 1.1 can skip directly to Section 4. Section 2 is related to barycentric embeddings and Section 3 is mostly concerned with partial results towards Kuratowski's Theorem.

2. Some linear algebra

Let $G = (V, E)$ be a simple graph. The *incidence matrix* of G is the matrix $B \in \{0, \pm 1\}^{V \times E}$ where $B_{ve} = 1$ if and only if v is and end of e. Thus there are exactly two ones in each column; a *signed incidence matrix* of G is any matrix obtained from B by replacing one of the ones in each column with -1. Thus the rows of any signed incidence matrix sum to zero, and any two signed incidence matrices of G are equivalent up to column scaling. All of the results in this section extend to non-simple graphs, but some care needs to be taken with loops.

Lemma 2.1. *If B is a signed incidence matrix of a simple graph G, then rank$(B) = |V(G)| - c$, where c is the number of connected components of G.*

Proof. A vector $y \in \mathbb{R}^V$ satisfies $y^t B = 0$ if and only if y is constant on each component of G, so $\text{rank}(B) = |V(G)| - c$. □

The *adjacency matrix* of G is the symmetric matrix $A \in \mathbb{Z}^{V\times V}$ which has zeros on the diagonal and, for distinct u and v, the entry A_{uv} is the number of edges having both u and v as ends.

Lemma 2.2. *Let A be the adjacency matrix of a simple graph $G = (V,E)$, B be a signed incidence matrix of G, and let $\Delta \in \mathbb{Z}^{V\times V}$ be the diagonal matrix such that Δ_{vv} is the degree of v. Then*

$$BB^t = \Delta - A,$$

Proof. Easy exercise. $\square$

The matrix BB^t is called the *Laplacian* of G.

Lemma 2.3. *Let M be the Laplacian of a simple connected graph $G = (V,E)$. Then, for each proper subset X of V, the matrix $M[X,X]$ is nonsingular.*

Proof. Note that B has rank $|V| - 1$ and the rows of B sum to zero. Therefore, since $X \neq V$, $B[X,E]$ has full row-rank. Note that $M[X,X] = B[X,E]B[X,E]^t$. For any non-zero $y \in \mathbb{R}^X$, we have $y^tB[X,E] \neq 0$, so

$$y^tM[X,X]y = (y^tB[X,E])(B[X,E]^ty) = (y^tB[X,E])(y^tB[X,E])^t > 0.$$

Therefore $M[X,X]y \neq 0$, as required. $\square$

Finding a barycentric embedding. Let C be a circuit in a simple connected graph $G = (V,E)$, let $Y = V(C)$ and let $X = V - Y$. Suppose that we are given a convex embedding $\phi : Y \to \mathbb{R}^2$ of C. Consider the problem of extending ϕ to a barycentric embedding. For each $x \in X$ we require:

$$deg(x)\phi(x) - \sum_{v\in N(x)} \phi(v) = 0.$$

If M is the Laplacian of G, these equations can be written as:

$$M[X,V]\phi = 0.$$

We are given $\phi[Y]$ and want to solve for $\phi[X]$, so it is convenient to write the equations as:

$$M[X,X]\phi[X] = -M[X,Y]\phi[Y].$$

By Lemma 2.3, $M[X,X]$ is non-singular, so there is a unique solution. This proves Lemma 1.2.

3. Peripheral circuits

Let H be a connected subgraph of a loopless 2-connected graph G. A *bridge* of H is the subgraph of G induced by the edges in any block of $G/E(H)$. The vertices in $V(B) \cap V(H)$ are the *attachments* of the bridge B.

Lemma 3.1. *Let B_0 be a bridge of a circuit C in a simple 3-connected graph $G = (V, E)$. If $V(B_0) - V(C) \neq \emptyset$, then, for each edge $e \in E(C)$, there is a peripheral cycle C' of G such that $e \in E(C') \subseteq E(G) - E(B_0)$.*

Proof. We may assume that: (i) there is no circuit C' containing e such that one of the bridges of C' properly contains B_0, and (ii) C is chordless. We claim that C is peripheral. Suppose not, then there is a bridge B_1 of C other than B_0. Since G is 3-connected, B_1 has distinct attachments x, y, and z on C. Let w be an attactment of B_0 on C. By relabelling, we may assume that w is not on the (x, y)-path P of $C - z$. There is a circuit C' in $C \cup B_1$ such that $C \cap C' = P$. Let B_0' be the bridge of C' that contains w. Note that B_0 is a subgraph of B_0'; moreover B_0' is strictly larger than B_v since it contains the edges of C incident with w. This contradicts assumption (i). $\square$

If H_1 and H_2 are subgraphs of G then $H_1 \cap H_2$ denotes the subgraph of G with vertex set $V(H_1) \cap V(H_2)$ and edge set $E(H_1) \cap E(H_2)$. We define $H_1 \cup H_2$ similarly.

Lemma 3.2. *Let $e = ab$ be an edge of a simple 3-connected graph. Then there exist two peripheral circuits C_1 and C_2 such that $C_1 \cap C_2 = G[\{e\}]$*

Proof. Since G is 3-connected, there exists two circuits C_1 and C_2 with $C_1 \cap C_2 = G[\{e\}]$. Let B_0 be the bridge of C_1 containing $C_2 - \{a, b\}$. Then, by Lemma 3.1, there is a peripheral circuit C_1' with $C_1' \cap C_2 = G[\{e\}]$. Similarly, we can find a peripheral circuit C_2' with $C_1' \cap C_2' = G[\{e\}]$. $\square$

Lemma 3.3. *Let $e = ab$ be an edge in a graph G, let C_1 and C_2 be peripheral circuits of a graph G with $C_1 \cap C_2 = G[\{e\}]$, and let P be a path from $V(C_1) - \{a, b\}$ to $V(C_2) - \{a, b\}$. If there is a path Q from a to b in $G - e$ that is disjoint from P, then G has a minor isomorphic to $K_{3,3}$ or K_5.*

Proof. Let $C = (C_1 \cup C_2) - e$. We may assume that P is a minimal path from $V(C_1) - \{a, b\}$ to $V(C_2) - \{a, b\}$; let c and d be the ends of P in $V(C_1) - \{a, b\}$ and $V(C_2) - \{a, b\}$ respectively. Let Q' be a minimal subpath of Q connecting the two components of $C-c-d$ and let B be the bridge of C that contains Q'.

Suppose that B contains P. Take a minimal path P' connecting $V(Q') - V(C)$ to $V(P) - V(C)$ in $G - V(C)$. Then $C \cup P \cup P' \cup Q$ is a subdivision of $K_{3,3}$. So we may assume that B does not contain P. Let G' be obtained from G by contracting $B - X$ to a single vertex x and contracting P to a path. Note that x has neighbours in both components of $C - c - d$ and, since C_1 and C_2 are peripheral circuits in G, x has neighbours in both components of $C - a - b$. Then, up to symmetry, we will find one of the minors in Figure 1 and each of those has an obvious K_5- or $K_{3,3}$-minor. $\square$

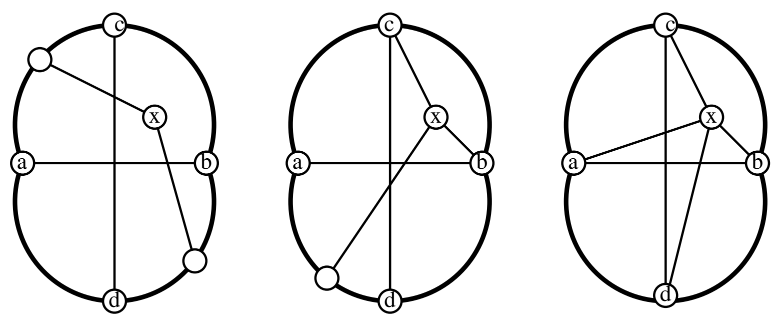

FIGURE 1. Possible neighbours of x

The next lemma follows easily from Lemma 3.3.

Lemma 3.4. *If G is a simple 3-connected graph with no $K_{3,3}$- or K_5-minor, then every edge is in exactly to peripheral circuits. Moreover, if C_1 and C_2 are peripheral circuits using an edge e, then $C_1 \cap C_2 = G[\{e\}]$.*

Proof. By Lemma 3.2, there exist peripheral circuits C_1 and C_2 such that $C_1 \cap C_2 = G[\{e\}]$. Suppose there is a third peripheral circuit C using e. Let Q be the path in $C - e$ connecting the ends, a and b, of the edge e. Since C is peripheral, there is a path P connecting $V(C_1) - \{a, b\}$ to $V(C_2) - \{a, b\}$. Then, by Lemma 3.3, G has a K_5- or $K_{3,3}$-minor. $\square$

Note that Lemma 3.4 comes very close to proving Kuratowski's Theorem, it only remains to prove that the peripheral cycles determine the face boundaries in a planar embedding.

4. Quasi-convex embeddings

For the reader who has skipped Section 3; the peripheral circuits of a 3-connected plane graph are precisely its face boundaries. So, for the remainder of this section suppose that you are given a plane graph and substitue "facial circuit" for "peripheral circuit".

If L is a (straight) line in $\mathbb{R}^2$, then the points in $\mathbb{R}^2 - L$ lie in two connected regions that we refer to as *open half-planes*.

Lemma 4.1. *Let C be a circuit in a connected graph $G = (V, E)$ and let ϕ be a quasi-convex embedding of (G, C). If H is a half-plane and X is the set of all vertices of G that are embedded in H, then $G[X]$ is connected.*

Proof. Since ϕ embeds all points in the convex hull of $\phi(V(C))$, we may assume that $H \cap V(C) \neq \emptyset$. Since C is embedded as a convex polygon, $G[X \cap V(C)]$ is connected. It suffices to find a path in $G[X]$ from any given vertex $v \in X$ to a vertex in C.

We can write $H = \{x \in \mathbb{R}^2 : a^t x > b\}$ where $a \in \mathbb{R}^2$ and $b \in \mathbb{R}$. Let $v \in X$, let $L = \{x \in \mathbb{R}^2 : a^t x = a^t \phi(v)\}$, and let X' be the set of all vertices that are embedded in L. Since G is connected, there is a path from v to a vertex w in $G[X']$ that has a neighbour outside L. We may assume that $w \notin V(C)$. Then, by the definition of a quasi-convex embedding, w has a neighbour w' such that $a^T\phi(w') > a^T\phi(w) = a^T\phi(v)$. Inductively we will find a path from w' to $V(C)$, and hence also from v to $V(C)$, in $G[X]$. $\square$

A quasi-convex embedding ϕ of (G, C) is *degenerate* if there is a vertex v such that the points in $\phi(N_G(v))$ are collinear. Figure 2 shows two ways that degeneracy can arise. Consider any quasi convex embedding of the first graph (where the outer circuit is C). Since $\{1, 3\}$ is a cut-set, both u and v will necessarily be embedded on the line spanned by 1 and 3. For the second graph, consider a barycentric embedding in which the outer circuit is embedded as a square. Then a, b, and v will all embed on the line connecting 1 and 3.

Lemma 4.2. *Let C be a peripheral circuit in a 3-connected graph G. If (G, C) has a degenerate quasi-convex embedding, then G has a $K_{3,3}$-minor.*

Proof. Let L be a line in $\mathbb{R}^2$ that contains all neighbours of a vertex $v \in V(G)$. Let H_1 and H_2 be the two open half-planes in $\mathbb{R}^2$ separated by l. Now let G_1 and G_2 be the subgraphs of G induced by the vertices embedded in H_1 and H_2 respectively; by Lemma 4.1, G_1 and G_2 are both connected.

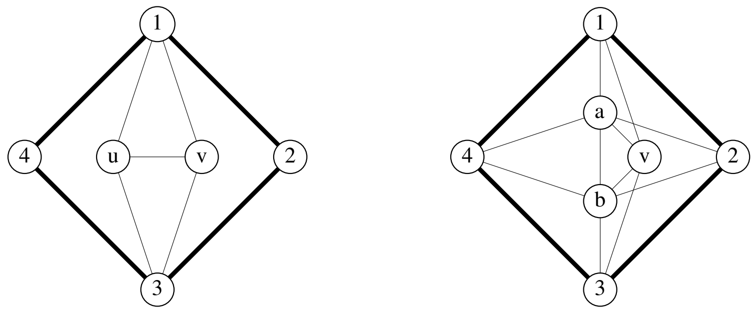

FIGURE 2. Causes of degeneracy

Since C is embedded as a convex polygon, $v \notin V(C)$. Therefore v is embedded on L. Let D_0 be the set of vertices that are embedded on L and that have a neighbour embedded off of L. By the definition of a quasi-convex embedding, each vertex in D_0 is adjacent to a vertex in $V(G_1)$ and to a vertex in $V(G_2)$.

Let G_3 be the component of $G - D_0$ containing v. Note that G_1, G_2 and G_3 are disjoint connected subgraphs of G; let G' be the graph obained from G by contracting each G_i, $i \in \{1, 2, 3\}$, to a single vertex v_i. Let $D_1 \subseteq D_0$ be the neighbours of v_3 in G'. Since G is 3-connected, $|D_1| \geq 3$. Moreover, each vertex in D_1 is adjacent to v_1, v_2, and v_3, so G has a $K_{3,3}$-minor. □

We say that a line $L \in \mathbb{R}^2$ *separates* a set $X \subseteq \mathbb{R}^2$ from a set $Y \subseteq \mathbb{R}^2$ if $L \cap (X \cup Y) = \emptyset$ and neither of the half-planes determined by L contains both a point in X and a point in Y.

Lemma 4.3. *Let C be a peripheral circuit in a 3-connected graph G with no minor isomorphic to $K_{3,3}$ or K_5, let ϕ be a quasi-convex embedding of (G, C), and let $e = ab \in E - E(C)$, and let C_1 and C_2 be peripheral circuits of G containing e. Then the line through $\phi(a)$ and $\phi(b)$ separates $V(C_1) - \{a, b\}$ from $V(C_2) - \{a, b\}$.*

Proof. Suppose not. Let $L = \{x \in \mathbb{R}^2 : \alpha^t x = \beta\}$, where $\alpha \in \mathbb{R}^2$ and $\beta \in \mathbb{R}$ be the line through $\phi(a)$ and $\phi(b)$. Let $H^+ = \{x \in \mathbb{R}^2 : \alpha x \geq \beta\}$ and $H^- = \{x \in \mathbb{R}^2 : \alpha x \leq \beta\}$. Let X^+ and X^- be the set of all vertices that embed in the open half planes $H^+ - L$ and $H^- - L$ and let X_L be the vertices that embed on L.

4.3.1. *If $x, y \in X_0 \cup X^+$ then there is a path P in $G[X_0 \cup X^+]$ whose internal vertices are all in X^+. Moreover, if $x, y \in X_0$, then there exists such a path with length at least 2.*

Proof of Claim. By Lemma 4.1 it suffices to show that each vertex in X_0 has a neighbour in X^+, which follows from Lemma 4.2 and the definition of quasi-convex embedding. □

By Lemma 3.4, $C_1 \cap C_2 = G[\{e\}]$. Since L does not separate $V(C_1) - \{a,b\}$ from $V(C_2) - \{a,b\}$, by symmetry we may assume that there exists $c \in V(C_1) - \{a,b\}$ and $d \in V(C_2) - \{a,b\}$ that are both contained in $X_0 \cup X^+$. By the claim, there is a path P from c to d such that $V(P) \subseteq X_0 \cup X^+$ and $V(P) - \{c,d\} \subseteq X^+$. Similarly, there is a path Q from a to b in $G - e$ such that $V(Q) \subseteq X_0 \cup X^-$ and $V(Q) - \{a,b\} \subseteq X^-$. Now P and Q are disjoint. Then, by Lemma 3.4, G has a minor isomorphic to K_5 or $K_{3,3}$. □

Now we can prove the main result.

Theorem 4.4. *Let C be a peripheral circuit in a simple 3-connected graph G with no $K_{3,3}$- or K_5-minor. Then every quasi-convex embedding of (G,C) is a convex embedding.*

Proof. Let ϕ be a quasi-convex embedding. Let $\mathcal{P}$ be the set of all peripheral circuits other than C. By Lemma 3.4, each edge is in exactly two circuits in $\mathcal{P} \cup \{C\}$. Consider a circuit $C' \in \mathcal{P}$. Applying Lemma 4.3 to each edge of C', we deduce that ϕ is a convex embedding of C'; let $S(C')$ be the interior of this convex polygon.

It remains to prove that the straight-line embedding of G given by ϕ is planar. It suffices to show that, for distinct circuits $C_1, C_2 \in \mathcal{P}$, the sets $S(C_1)$ and $S(C_2)$ are disjoint. If this is not the case, then, since these sets are open, there exists $p \in S(C_1) \cap S(C_2)$ that is not on a line spanned by any two vertices. Now choose a line L containing P such that no point on L is on a vertex of G or at the intersection of two edges. Let α and β be the points on L on the boundary of C. For a point x on the line segment of L between α and β but not on any edge, let $n(x)$ be the number sets $(S(C') : C' \in \mathcal{P})$ that contain x. Note that, if we move x along the line segment, $n(x)$ cannot change unless we cross over an edge. Moreover, by Lemma 4.3, does not change when we cross over an edge since we move from on set into another. Finally observe that $n(x) = 1$ for points close to α, which contradicts that $n(p) \geq 2$. □

5. Planarity testing

We have proved the following result.

Theorem 5.1. *If C is a peripheral circuit C in a 3-connected graph $G = (V,E)$ and ϕ is a barycentric embedding of (G,C), then G is planar if and only if ϕ gives a straight-line plane embedding of G.*

This provides an efficient algorithm for planarity testing provided that:

(i) the input graph G is simple and 3-connected, and
(ii) we can efficiently find a peripheral circuit in G.

The reduction of the planarity testing problem to simple 3-connected instances is routine and well understood, so we will ignore (i) and assume that our input graph is simple and 3-connected. Below we sketch three different ways to overcome (ii).

Finding a peripheral circuit. The proof of Lemma 3.1 is constructive and hence can be used to find a peripheral circuit. There is a faster and easier algorithm due to Lovász for finding a peripheral circuit. Find a depth-first tree T of G from a root vertex r. A consequence of depth-first search is that, if $e = uv$ is a non-tree edge where v comes after u in the depth-first order, then u is contained on the (r, v)-path in T. Among all such edges $e = uv$, choose u as late as possible in the depth-first ordering and, subject to this choose v as early as possible. It is an easy exercise to show that, if G is 3-connected, then the unique circuit in $T + e$ is peripheral in G.

Finding three vertices on a peripheral circuit. Suppose that x, y, and z are three vertices on a peripheral circuit. By making these vertices pairwise adjacent we construct a peripheral triangle, and, if G were planar, adding these edges would preserve planarity. We can change the definition of barycentric embedding so that we only specify the embedding on x, y, and z. Naively one could try all $\binom{|V|}{3}$ triples (x, y, z), then G is planar if and only if one of the barycentric embeddings is planar. However, we can do a lot better. We may assume that G has a vertex x with degree at most 5, otherwise G is not planar. Then let y be a neighbour of x. By Lemma 3.2, the edge xy is in two peripheral circuits, so, to choose z, we need only try three of the remaining four neighbours of x.

Using a non-peripheral circuit. Take an arbitrary circuit C in G. Let B be the set of bridges of C and let P be the set of pairs of overlapping bridges. We may assume that (B, P) determines a bipartite graph since otherwise G is non-planar. Then, use the bipartition to construct two subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$, $C = G_1 \cap G_2$, and such that no two bridges of C in either G_1 or G_2 overlap. Now G is planar if and only if barycentric embeddings of (G_1, C) and (G_2, C) are planar. (Remark: The graphs G_1 and G_2 need not be 3-connected, but if $\{x, y\}$ is a vertex cut-set in G_i then $x, y \in V(C)$ and each component of $G_i - x - y$ contains a vertex of C. Using this, you can show that a barycentric embedding of (G_i, C) is planar if and only if there is a planar embedding of G_i in which C bounds a face.)

References

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