CO749 **Assignment 4** **Due: November 16**

Problem 1: Let A be a set of vertices in a graph $G = (V, E)$.

(a) Show that, if $\nu(G - v, A - \{v\}) = \nu(G, A)$ for each vertex $v \in V$ and G is connected, then $2\nu(G, A) + 1 = |A|$. (Hint: You may use the Edmonds-Gallai Theorem: If $D = \{v \in V : \nu(G - v) = \nu(G)\}$, $X = N_G(D)$, and k denotes the number of odd components of $G - X$, then $\nu(G) = \frac{|V-X|-k}{2} + |X|$.)

(b) Prove that there exists $X \subseteq V$ with $|X| \leq 2\nu(G, A)$ such that each A-path intersects X.

Problem 2: Let $\Theta = \Sigma(0, 0; 2)$ be a cylinder with holes h_1 and h_2.

(a) Let G be a graph embedded in Θ, let $S = V(G \cap h_1)$, and let $T = V(G \cap h_2)$. Show that, if there exist k disjoint (S, T)-paths in G and there exist n disjoint cycles in G each separating h_1 from h_2 in Θ, then there exist disjoint (S, T)-paths $P_1, \ldots, P_k$ in G and there exist disjoint cycles $C_1, \ldots, C_n$ in G each separating h_1 from h_2 in Θ such that $P_i \cap C_j$ is a path for each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$.

(b) Let (G_1, G_2) be a separation in a graph G where G_1 is embedded in Θ with $V(G_1) \cap V(G_2)$ is embedded in h_2, let $S = V(G_1 \cap h_1)$ and let $T \subseteq V(G_2)$. Show that, if there exist k disjoint (S, T)-paths in G and there exist $n + 2k$ disjoint cycles in G_1 each separating h_1 from h_2 in Θ, then there exist disjoint (S, T)-paths $P_1, \ldots, P_k$ in G and there exist disjoint cycles $C_1, \ldots, C_n$ in G_1 each separating h_1 from h_2 in Θ such that $P_i \cap C_j$ is a path for each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$. (Hint: Consider the proof of the Key Lemma in the proof of the Grid Theorem.)

Problem 3: Let $\Sigma = \Sigma(h, c)$ where $h + c > 0$ and let G be a graph embedded in Σ with representativity $3k$. Let $\mathcal{T}$ denote the set of all $(k-1)$-separations (H_1, H_2) in G such that there is a non-contractible curve C in Σ such that $G \cap C = H_2 \cap C$. Prove that $\mathcal{T}$ is a tangle of order k in G.