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Fidelity Decay Saturation Level for Initial Eigenstates

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We show that the fidelity decay between an initial eigenstate evolved under a unitary chaotic operator and the same eigenstate evolved under a perturbed operator saturates well before the $1/N$ limit expected for a generic initial state, where N is the dimension of the Hilbert space. We provide a theoretical argument and numerical evidence that, for perturbations of intermediate strength, the saturation level depends quadratically on the perturbation strength.

KEY WORDS: Quantum chaos; fidelity decay; local density of states.

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1. INTRODUCTION

Over the past twenty years different phenomena found in quantum systems that have chaotic classical analogs have been suggested as appropriate signatures of quantum chaos.[(1–6)] One of these conjectures, that of Peres,[(4)] is that the initial rate and behavior of a system's fidelity decay due to a small perturbation in the Hamiltonian will differentiate between chaotic and non-chaotic systems. This signature is analogous to the sensitivity to initial conditions which characterizes classical chaos but, as a consequence of strictly unitary evolution, cannot emerge in quantum systems. Recent insights[(7–9)] have led to a more detailed understanding of this signature.

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2. THEORETICAL FRAMEWORK

For a unitary map, U, the fidelity compares the evolution of an initial state under unperturbed and perturbed dynamics. The fidelity is given by

$$F(n) = |\langle\psi_o|\left(U^{\dagger}\right)^n (U_\delta)^n|\psi_o\rangle|^2 \tag{1}$$

where $U_\delta = U_p U$ and $U_p = \exp(-i\delta V)$ is the perturbation operator of strength δ. ψ_o is the initial state of the system. The fidelity decay behavior depends not only on whether the map is classically chaotic, but also on the initial state of the system and the strength of the perturbation. For chaotic systems, the fidelity eventually saturates. Here, we focus on the characteristics of this saturation level by studying

$$F_\infty = \lim_{T\to\infty} \frac{1}{T}\sum_{n=1}^{T} F(n) \tag{2}$$

For initial random states the fidelity saturates at about $1/N$ as shown in Refs. 7 and 10, and discussed below. For eigenstates, however, the saturation level, F_∞, is much larger and depends on the perturbation strength, δ. The study of initial eigenstate fidelity decay is of particular interest since it is equivalent to the survival probability of a system eigenstate under the influence of a perturbation. Below we provide theoretical arguments showing a region where F_∞ depends quadratically on the perturbation strength. We also test this prediction numerically on quantum chaotic maps.

In the following, we assume chaotic dynamics and identify the various regimes of initial fidelity decay behavior based on the strength of the perturbation. Perturbation strength is measured by σ, a typical off-diagonal matrix element of the perturbation Hamiltonian, δV, expressed in the ordered eigen basis of the system Hamiltonian. When σ is less than Δ, the average system level spacing, the initial fidelity decay behavior is Gaussian as expected from perturbation theory[4] and random matrix theory.[11] Perturbations strong enough that $\sigma > \Delta$ are said to be in the Fermi Golden Rule (FGR) regime[8] where the initial fidelity decay behavior is exponential. It has been shown that for some perturbations the rate of the exponential decay increases as δ^2, until saturating at a rate given by the corresponding classical system's Lyapunov exponent,[8,12,13] or the bandwidth of the system Hamiltonian.[8]

Following Jacquod[10] and Prosen,[7] we derive the FGR regime exponential fidelity decay rate and the saturation level of initial random states as follows. Let us define eigenvectors and eigenangles for the unperturbed operator $U|v_j\rangle = \exp(-i\phi_j)|v_j\rangle$, and the perturbed operator $U_\delta|v'_k\rangle = \exp(-i\phi'_k)|v'_k\rangle$. Then, for an initial random state, $|\psi_o\rangle = \sum_j c_j|v_j\rangle$, the fidelity can be written as

$$F(n) = \sum\nolimits_{ij} c_j^* c_i \langle v_j|U_\delta^n|v_i\rangle e^{-i\phi_i n} \sum\nolimits_{kl} c_k c_l^* \langle v_l|\left(U_\delta^n\right)^\dagger|v_k\rangle e^{i\phi_l n} \tag{3}$$

We are interested in characterizing the fidelity averaged over all random states. To do this, the above equation is broken into three parts. $F_A(n)$ includes the terms $\delta_{ij}\delta_{kl}$ with $i \neq k$, and $\delta_{ik}\delta_{jl}$ with $i \neq j$. $F_A(n)$ leads to the initial exponential fidelity decay for chaotic systems in the FGR regime, however, it does not contribute to the saturation level. F_B, consists of terms where $\delta_{il}\delta_{jk}$ including the term $i = j$. F_B will lead to the $\simeq 1/N$ saturation level for random states. The remaining terms in Eq. (3) will vanish when averaged over all random states.[10]

The $\delta_{ij}\delta_{kl}$ plus the $\delta_{ik}\delta_{jl}$ terms can be written as

$$F_A(n) = 2 \sum_{ik, i\neq k} |c_i|^2 |c_k|^2 \langle v_i|U_\delta^n|v_i\rangle\langle v_k|\left(U_\delta^n\right)^\dagger|v_k\rangle e^{i(\phi_i - \phi_k)n} \tag{4}$$

The initial exponential fidelity decay for chaotic systems is derived from Eq. (4) by replacing U_δ with $\sum_q e^{i\phi'_q}|v'_q\rangle\langle v'_q|$. F_A can then be approximated by[10]

$$\langle F_A(n)\rangle \simeq \frac{2}{N^2}\left|\sum_{iq} |\langle v_i|v'_q\rangle|^2 e^{i\left(\phi'_q - \phi_i\right)n}\right|^2 \tag{5}$$

Equation (5) is recognized as the Fourier transform of the local density of states (LDOS). The LDOS is the spectral density of the original system under transition rules given by the perturbation. Hence, it is a measure of the overlap between perturbed and unperturbed eigenstates separated by an angle $(\phi_j - \phi'_k)$

$$\eta\left(\phi_j - \phi'_k\right) = |\langle v_j|v'_k\rangle|^2 \tag{6}$$

The connection between the LDOS and the fidelity decay was first pointed out by Jacquod and coworkers in Ref. 8.

Previous studies suggest that the LDOS of a complex system in the regime of strong perturbation is Lorentzian[14–16]

$$\eta\left(\phi_j - \phi'_k\right) \propto \frac{\Gamma}{\left(\phi_j - \phi'_k\right)^2 + (\Gamma/2)^2} \tag{7}$$

with a width of $\Gamma = 2\pi\sigma^2/\Delta$, where, as before, σ is a typical off diagonal element of the perturbation operator and Δ is the average system level spacing. Thus, using the Fourier transform relation, the initial fidelity decay is exponential with a rate Γ

$$\langle F_A(n)\rangle \simeq \exp(-\Gamma n) \tag{8}$$

An alternative derivation of the exponential decay in the FGR regime is discussed in Ref. 7.

Assuming that the eigenvalues of U are non-degenerate, each term of F_A, as given by Eq. (4), is of the form $Ce^{i\phi n}$, where C is positive constant. Hence, each term has a time average equal to zero and does not contribute to the fidelity decay saturation level.

F_B includes the $\delta_{il}\delta_{jk}$ and $\delta_{il}\delta_{jk}\delta_{ij}$ terms of Eq. (3). It can be written as

$$F_B(n) = \sum_{ij} |c_i|^2|c_j|^2\langle v_j|U_\delta^n|v_i\rangle\langle v_i|\left(U_\delta^n\right)^\dagger|v_j\rangle \tag{9}$$

Again replacing U_δ with $\sum_q e^{-i\phi'_q}|v'_q\rangle\langle v'_q|$ and averaging over time (denoted by an overline) yields

$$\overline{F_B} = \sum_{ijq} |c_i|^2|c_j|^2|a_{iq}|^2|a_{jq}|^2 \tag{10}$$

where $a_{iq} = \langle v_i|v'_q\rangle$ is the overlap between a perturbed and unperturbed operator eigenvector. When averaging over all random states $\langle|c_i|^2|c_j|^2|\rangle = 1/(N^2)$ for $i \neq j$ and $\langle|c_i|^4\rangle = (4-\beta)/(N^2)$. $\beta = 1$ when the $|c_i|^2$ refer to magnitudes of eigenvector elements of matrices from the circular orthogonal ensemble (COE), and $\beta = 2$ when the $|c_i|^2$ refer to the magnitude of eigenvector elements of matrices from the circular unitary ensemble (CUE).[7,17] Hence, when averaged over all states F_B becomes

$$\overline{\langle F_B\rangle} = \frac{(4-\beta)}{N^2}\sum_{iq}|a_{iq}|^4 + \frac{1}{N^2}\sum_{ijq}|a_{iq}|^2|a_{jq}|^2 \tag{11}$$

The $|a_{iq}|^2$ depend on the perturbation strength. In the limit of strong perturbation, the $|a_{iq}|^2$ are the same as the overlap between random states, $\langle|a_{iq}|^2\rangle = 1/N$ and $\langle|a_{iq}|^4\rangle = (4-\beta)/(N^2)$. It follows that $\langle F_\infty\rangle$ is equal to $1/N + O(1/N^2)$. In the limit of weak perturbation, a_{iq} is simply a δ-function and $\langle F_\infty\rangle = 2/N$ for CUE.[7]

That $\langle F_\infty\rangle$ is of order $1/N$ is not surprising since after a certain amount of time we would expect the initial state to become evenly spread out over a complete set of states. The important point is that for random states $\langle F_\infty\rangle$ is only weakly dependent on perturbation strength. In contrast, for initial states that are eigenstates of the unperturbed system $\{F_\infty\}$ (where the curly braces indicate average over eigenstates) depends strongly on the perturbation strength. Prosen[7] has already noted that for initial eigenstates $\{F_\infty\} \to 1$ in the limit of weak perturbation and $\{F_\infty\} \to (4-\beta)/N$ for strong perturbation. Here, we estimate $\{F_\infty\}$ for intermediate perturbation strengths and provide a theoretical argument and numerical evidence for the quadratic behavior for $\{F_\infty\}$ vs. perturbation strength. To evaluate the dependence of the saturation level on perturbation strength we express the fidelity for an initial eigenstate, $|\psi_o\rangle = |v_m\rangle$, as

$$F(n) = |\langle v_m| \sum_l a_{lm} e^{-in(\phi'_l - \phi_m)} |v_m\rangle|^2 \tag{12}$$

The above equation can be separated into a time independent term plus a time dependent term

$$F(n) = \sum_l |a_{lm}|^4 + \sum_{lk} |a_{lm}|^2 |a_{km}|^2 \cos\left[\left(\phi'_l - \phi'_k\right)n\right] \tag{13}$$

The time average of the second term goes to zero while the first term shows that $\{F_\infty\}$ is an inverse participation ratio of the overlap between perturbed and unperturbed eigenvectors.[7] In other words, the fidelity saturation level is simply the sum of the squared elements of the LDOS.

The number of perturbed operator eigenvectors $|v'_l\rangle$ contributing to the initial eigenstate, $|v_m\rangle$, can be estimated by the width of the LDOS, Eq. (6). We assume that the (Γ/Δ) contributing $|a_{lm}|^2$ terms within the width, Γ, of the Lorentzian shaped LDOS each have a weight $1/\Gamma$. With this approximation and noting that the average level spacing Δ is equal to $2\pi/N$

$$\{F_\infty\} \propto 2\pi/(\Gamma N) \tag{14}$$

where $2\pi/N \leq \Gamma \leq 2\pi$. A similar result is mentioned in Ref. 18.

$\{F_\infty\}$ can now be related to the perturbation strength by evaluating Γ as follows: $\sigma^2 = \delta^2 \overline{V^2_{mn}}$ where $\overline{V^2_{mn}}$ is the second moment of the matrix elements V_{mn}. $\overline{V^2_{mn}}$ may be estimated by noting that for chaotic systems the eigenvectors are random, and, therefore, $\overline{V^2_{mn}} = \overline{\lambda^2}/N$[9] where $\overline{\lambda^2} = N^{-1}\sum_{i=1}^{N} \lambda_i^2$ is the variance of the eigenvalues of V. The rate of the exponential decay, Γ, can now be evaluated as $\Gamma = \delta^2\overline{\lambda^2}$ and $\{F_\infty\}$ is seen to have a quadratic dependence on perturbation strength in the FGR regime

$$\{F_\infty\} \propto 2\pi/(\Gamma N) = 2\pi/(\delta^2 \overline{\lambda^2} N) \tag{15}$$

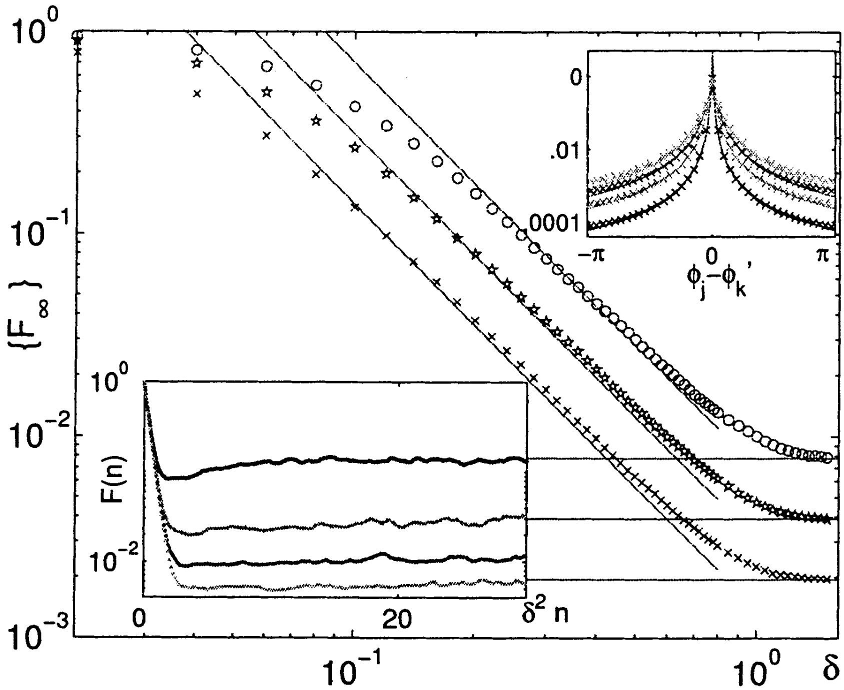

Fig. 1. Saturation level vs. perturbation strength for initial eigenstates of a random CUE map of dimensions 256 (circles), 512 (stars), and 1024 (x). For weak perturbations, below the FGR regime, the fidelity barely decays. In the limit of strong perturbation $\{F_\infty\}$ saturates at $2/N$ (solid line). For intermediate values of δ, $\{F_\infty\}$ is well approximated by Eq. (15) with the proportionality constant $C_{\mathrm{CUE}} = 3.6$. $\{F_\infty\}$ is obtained by averaging over 2000 map iterations starting at iteration $n = 2000$, well after the initial exponential decay. This is averaged over all N initial eigenstates. The lower inset shows the initial exponential fidelity decay of the CUE map with $N = 1024$ averaged over all 1024 system eigenstates. The fidelity decay is plotted vs. $\delta^2 n$ so that the exponential decay rates overlap and the saturation level is easily seen. The perturbation strengths used are $\delta = 0.1, 0.2, 0.3, 0.4$ (top to bottom). The upper inset shows a semi-log plot of the local density of states for a CUE map perturbed by a collective bit z-rotation, $\delta = 0.1, 0.2, 0.3$ and 0.4 (bottom to top). The solid line is a Lorentzian of width $\Gamma = \overline{V_{mn}^2}/\Delta$ with $\overline{V_{mn}^2}$ determined numerically from the CUE map. Note that the LDOS approximates the Lorentzian of Eq. (7) extremely well.

3. NUMERICAL SIMULATIONS

The above predictions were first tested on random circular unitary ensemble (CUE) maps. Random matrix theory predicts the behavior of the fidelity decay in both the Gaussian[11] and FGR[10] perturbation strength

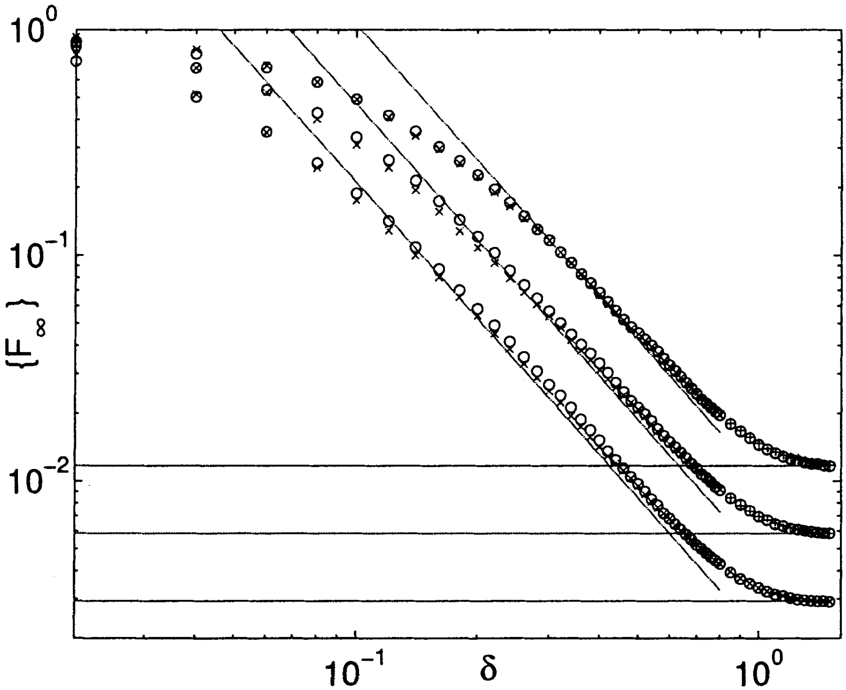

Fig. 2. $\{F_\infty\}$ vs. perturbation strength for initial eigenstates of a random COE map (x) and the QKT with $k = 12$ (circles) of dimensions 256, 512, and 1024 (from top to bottom). For weak perturbations, below the FGR regime, the fidelity barely decays. In the limit of strong perturbation $\{F_\infty\}$ saturates at $3/N$ (solid line). For intermediate values of δ, $\{F_\infty\}$ is well approximated by the estimate of Eq. (15) with the proportionality constant $C_{\mathrm{COE}} = 5.4$. The numerical value of $\{F_\infty\}$ is determined in the same manner as for the CUE maps.

regimes. The use of a random matrix as the evolution operator to study dynamical aspects of quantum chaos has been suggested in Ref. 9.

We assume that our system is composed of a collection of two-level subsystems or qubits. The perturbation used is a z-rotation of all of these qubits through an angle δ

$$U_p = \prod_{j=1}^{n_q} e^{-i\delta\sigma_z^j/2} \tag{16}$$

where $n_q = \log_2 N$ is the number of qubits in the system. In the context of quantum information processing, this perturbation corresponds to an error in the phase of all the quantum bits in a quantum information processor. We note that this perturbation also arises in quantum control studies as a

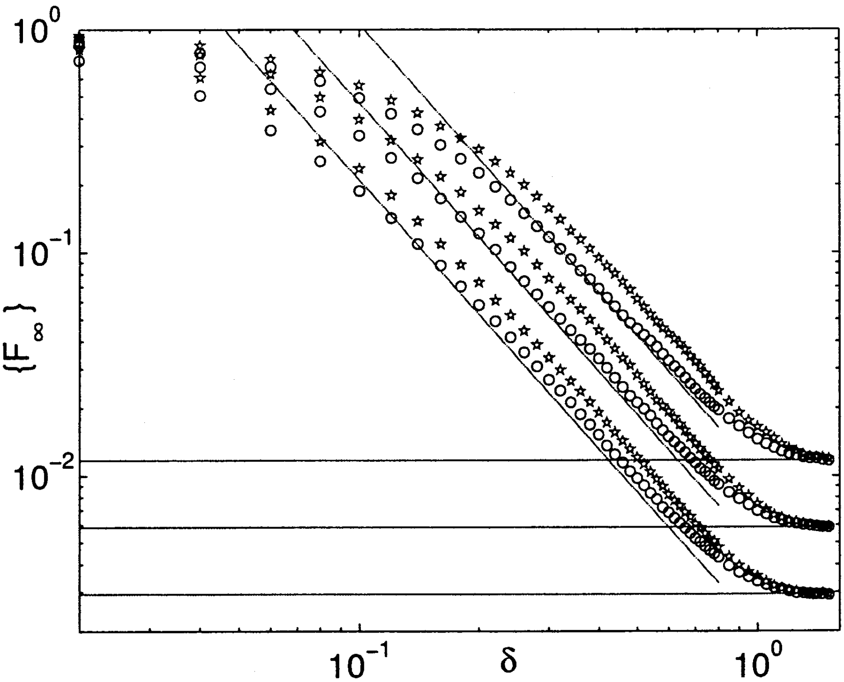

Fig. 3. $\{F_\infty\}$ vs. perturbation strength for initial eigenstates of the oe subspace of the QKT with $k = 12$ (stars) and the full QKT with $k = 12$ (circles) for Hilbert space dimensions 256, 512, and 1024 (top to bottom). For weak perturbations, below the FGR regime, the fidelity barely decays. In the limit of strong perturbation $\{F_\infty\}$ saturates at $3/N$ (solid line). For intermediate values of δ of the full kicked top, $\{F_\infty\}$ is well approximated by the estimate of Eq. (15) with the proportionality constant $C_{\mathrm{COE}} = 5.4$. However, $\{F_\infty\}$ for the oe subspace does not match F_∞ of the full kicked top for these perturbation strengths.

model of coherent far-field errors.[19] For this perturbation, CUE maps exhibit exponential fidelity decay and a Lorentzian shaped LDOS[9] as shown in the insets of Fig. 1.

Figure 1 shows $\{F_\infty\}$ for initial eigenstates of the CUE matrix vs. perturbation strength. We see that below the FGR regime there is very little decay while in the limit of strong perturbation $\{F_\infty\} = 2/N$ as expected for CUE maps. Between these we see a power law decrease of $\{F_\infty\}$ with increased perturbation strength. Since the LDOS is Lorentzian the discrepancy seen in Fig. 1 may be due to the approximation made by replacing the Lorentzian LDOS with a rectangle of width Γ. The actual slope of the data is between 1.8 and 1.9. The data is compared to $\{F_\infty\} = C_{\mathrm{CUE}}/(\delta^2\overline{\lambda^2}N)$, where the proportionality constant, $C_{\mathrm{CUE}} = 3.6$ is chosen to best fit the data.

A similar analysis was carried out for random circular orthogonal ensemble (COE) maps. Random COE matrices can be created from CUE matrices, $COE = CUE * transpose(CUE)$.[3] Like the CUE maps, the COE maps have no classical analog and we introduce them here as models for the behavior of quantum chaotic maps with COE eigenvector statistics and energy level spacings. Figure 2 shows $\{F_\infty\}$ vs. perturbation strength for COE maps. Again, an approximate quadratic relationship emerges but with a different proportionality coefficient, $C_{\mathrm{COE}} = 5.4$.

The calculated numerical average of $\{F_\infty^{\mathrm{COE}}\}/\{F_\infty^{\mathrm{CUE}}\}$ for the three Hilbert space dimensions explored with perturbations in the FGR regime is 1.48. In the limit of strong perturbation the ratio $\{F_\infty^{\mathrm{COE}}\}/\{F_\infty^{\mathrm{CUE}}\}$ is equal to 3/2[7] as follows from the discussion leading up to Eq. (12). However, numerically we find that the proportionality remains fixed also for perturbations in the FGR regime. This can be explained by assuming random statistics within a top-hat distribution of width Γ, as done in the LDOS approximation above. The same argument as above will give $\{F_\infty^{\mathrm{COE}}\}/\{F_\infty^{\mathrm{CUE}}\} = 3/2 \simeq 1.48$ for perturbation strengths in the FGR regime.

We next study $\{F_\infty\}$ for a quantum system with a well defined classical analog, the quantum kicked top (QKT).[3,20] The QKT is an exemplary model of quantum chaos and has been used in previous studies of fidelity decay.[4,7,8] The QKT is a unitary map, $U_{QKT} = \exp(-i\pi J_y/2)\exp(-ikJ_z^2/j)$, acting on a Hilbert space of dimension $N = 2j + 1$. $\boldsymbol{J}$ is the angular momentum operator in the irreducible representation and k is the kick strength. A kick strength of $k = 12$ is used which is in the chaotic region of the QKT. Since the QKT shows anti-unitary symmetry, it is part of the COE class. The QKT has COE-like nearest neighbor level spacings[3] and eigenvector statistics.[17] The same perturbation, the collective z-rotation, is used.

It should be noted that the data for the QKT and COE maps are very similar. This is expected since, as has been conjectured and demonstrated in a number of works, quantum chaotic systems have statistical[2,17] and dynamic features[9] similar to those of the canonical random matrix theory ensembles.

The QKT is a system with a classical analog and has symmetries not found in random matrices. It is interesting to see whether $\{F_\infty\}$ for just one of these subspaces behaves differently from that of the full QKT. To do this, $\{F_\infty\}$ is calculated for the oe subspace (odd under 180° rotations around the y-axis[4]) of the QKT. We note that while for the kicked top $N = 2j + 1$, the oe subspace has Hilbert space dimension which has dimension $N = j$. The results of $\{F_\infty\}$ vs. perturbation strength are shown in Fig. 3 and again we can approximate quadratic decrease of $\{F_\infty\}$ with increased perturbation strength. However, while the saturation level at the limit of strong perturbation does reach the expected $3/N$ at the same perturbation strength as for the full QKT, the intermediate perturbation strengths lead to a saturation level that is higher than for the full QKT. The coefficient C_{oe} is significantly higher than that of the CUE or COE maps.

In conclusion, we have given a theoretical argument estimating the dependence of the fidelity decay saturation level, $\{F_\infty\}$, on perturbation strength for initial states that are eigenstates of the system and intermediate perturbation strengths. Numerical simulations for systems with and without classical analogs support these predictions. Interestingly, the full QKT with its invariant subspaces behaves as predicted for a map with COE-like statistics, while the behavior of the map consisting of only one of these subspaces deviates from the predicted saturation level of the fidelity decay.

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