Random Quantum Circuits and Pseudo-Random Operators: Theory and Applications

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Abstract. Pseudo-random operators consist of sets of operators that exhibit many of the important statistical features of uniformly distributed random operators. Such pseudo-random sets of operators are most useful whey they may be parameterized and generated on a quantum processor in a way that requires exponentially fewer resources than direct implementation of the uniformly random set. Efficient pseudo-random operators can overcome the exponential cost of random operators required for quantum communication tasks such as super-dense coding of quantum states and approximately secure quantum data-hiding, and enable efficient stochastic methods for noise estimation on prototype quantum processors. This paper summarizes some recently published work demonstrating a random circuit method for the implementation of *pseudo-random* unitary operators on a quantum processor [Emerson et al., Science 302:2098 (Dec. 19, 2003)], and further elaborates the theory and applications of pseudo-random states and operators.

INTRODUCTION

Random numbers play a fundamental role in classical information theory, with practical applications including system identification, stochastic simulation, and a variety of cryptographic protocols. Similarly, random quantum states and operators have both fundamental and practical importance to quantum information theory. The operator ensemble with greatest relevance to quantum information is the circular unitary ensemble (CUE), defined as the distribution of unitary operators drawn uniformly with respect to the unique, unitarily invariant (Haar) measure on $U(D)$, where $D = 2^{n_q}$ is the dimension of Hilbert space and n_q is the number of qubits. The Haar measure induces a uniform measure over pure states and thus a random unitary matrix may be applied to randomize an arbitrary pure state. This leads to a number of practical applications for quantum communication: sets of randomizing unitaries enable the super-dense coding of arbitrary quantum states [1], lead to a decrease in the classical communication cost for remote state preparation and allow the construction of more efficient data hiding schemes [2], and provide a means to reduce the shared key length for the (approximate) encryption of quantum states [3]. The usefulness of these protocols is limited by the fact that generating *uniformly random* operators and states requires quantum resources that grow exponentially in the number of quantum bits and hence these protocols become impractical for large quantum systems. Below I describe the random circuit method for generating *efficient* sets of *pseudo-randomly* distributed unitary operators that was originally introduced in Ref. [4], and then describe how random and pseudo-random operators enable efficient methods for noise-estimation on quantum processors.

RANDOM CIRCUITS

A unitary operator drawn from the Haar measure on $U(D)$ is conveniently parameterized via the Hurwitz decomposition [5]. Using standard techniques this decomposition can be re-expressed as a quantum circuit requiring $O(\log(D)^2 D^2)$ elementary (one and two qubit) gates and D^2 independent random 'input' parameters, and therefore, as expected, requires resources growing exponentially in the number of qubits.

To overcome this impracticality, we consider instead a random circuit comprised of a sequence of m iterations of a constant-depth gate parameterized by independent random input parameters. The constant-depth gate has two steps, as illustrated in Figure 1. The first step of the gate consists of rotating each qubit by independent, random unitary operators drawn from the Haar measure on $U(2)$. This first step requires $3n_q$ independent input variables.

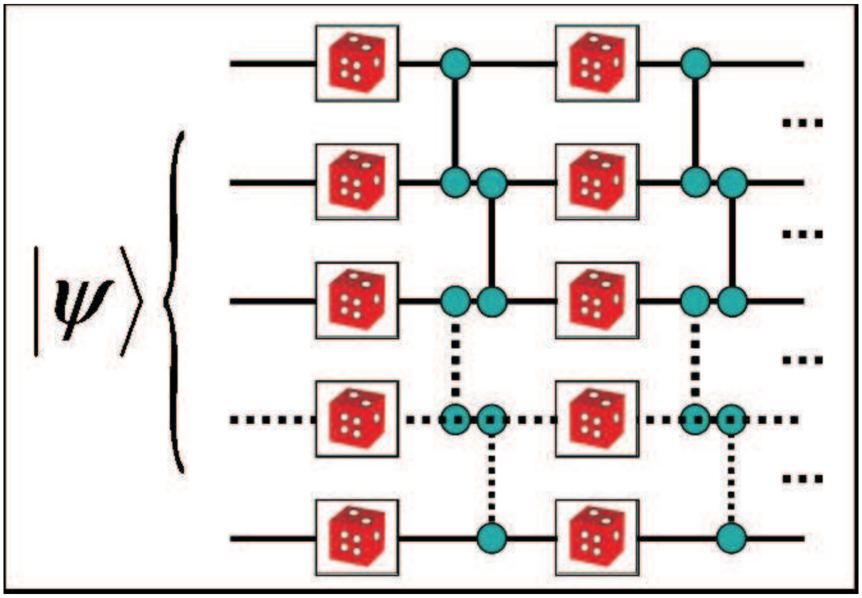

FIGURE 1. A random circuit comprised of a sequence of iterations of a constant-depth gate parameterized by independent random input parameters. The constant-depth gate has two steps. The first step of the gate consists of rotating each qubit by random unitary operators drawn independently from the Haar measure on $U(2)$. These random local rotations are indicated by the boxes containing dice. The second step of the constant-depth gate consists of any convenient nearest-neighbor coupling between the qubits. The set of pseudo-random unitary operators generated by this method converges asymptotically (in the strong sense of uniform convergence) to the Haar measure on $U(D)$ [6] and can efficiently reproduce practical signatures of uniformly (Haar) distributed random unitary operators [4, 13].

The second step of the constant-depth gate consists of the following set of simultaneous two-body interactions, $U = \exp(i(\pi/4)\sum_{j=1}^{n_q-1} \sigma_z^j \otimes \sigma_z^{j+1})$, where σ_z^j is the usual Pauli operator of the j'th qubit and the coupling angle is fixed at $\pi/4$ to maximize the entanglement produced by the two-body interactions. Since we imagine a simple 1-D array of qubits and a spatially local coupling interaction, we have limited our basic circuit to include only nearest-neighbor couplings.

Our construction induces a measure over the set of circuits of length m. For finite m the distribution of unitary operators generated by the random circuit is of course biased with respect to the uniform (Haar) measure on $U(D)$. In this sense the random circuits generate “pseudo-random” unitary operators. For $m > m_c = O(n_q^3 D^2)$, this distribution, though biased, does enjoy non-vanishing support over all elements of the group $U(D)$. Moreover, in the limit of increasing numbers m of iterations, the measure over the composed circuits converges exponentially (in the uniform sense) to the uniform measure on the group [6]. This result follows from properties of the Fourier transform on compact groups and requires that the initial distribution has non-vanishing support over a universal gate set. It is worth stressing that this argument indicates that a random circuit based on *any* universal gate set generates the uniform measure asymptotically, no matter how biased the method with which the gates are drawn from the universal set.

We now turn to the question of whether the measure over random circuits of length $m <$ polylog(D), i.e., efficient random circuits, can mimic some practical subset of the statistical features of the uniform measure. We are in particular interested in properties of unitary operators and states that are relevant to quantum information tasks, and so we first consider the distribution of entanglement produced by the circuit. Specifically, we consider the entanglement of the states obtained by applying the random circuit to the computational basis states. As a practical indicator of entanglement for multi-partite systems we consider the indicator proposed by Meyer and Wallach [7], which can be expressed as an average of the bi-partite entanglement between each qubit and the rest of the system [8],

$$Q = 2 - \frac{2}{n_q}\sum_{i=1}^{n_q} \mathrm{Tr}[\rho_i^2], \tag{1}$$

where ρ_i is the reduced density operator of the i'th qubit. One has $0 \leq Q \leq 1$, with $Q = 0$ for completely factorable states and $Q = 1$ for the cat states. In Ref. [4] we report the distribution of Q produced by the random circuits, as a function of increasing m, for an 8 qubit system. The distribution becomes indistinguishable (on the scale of the figure) from the distribution produced by CUE maps. It is useful to note that the exact CUE average,

$$\langle Q \rangle = \frac{D-2}{D+1} \simeq 1 - \frac{3}{D} + O(D^{-2}), \tag{2}$$

approaches the maximum value of Q exponentially as a function of n_q. Moreover, the standard deviation decreases exponentially with increasing n_q [9]. Thus in the limit of large n_q almost all random (and pseudo-random) states have nearly maximal entanglement $Q \simeq \langle Q \rangle \simeq Q_{max}$. This "concentration of measure" effect is a typical feature of sufficiently smoothly varying functions of states in Hilbert spaces of large dimension [10, 11, 12]. Broadly speaking, a single complex system may be represented by a generic pure state in a large Hilbert space. Generic pure states are "typical" or "universal" in the sense that many of their properties are well approximated by the value of that property obtained by averaging with respect to the uniform measure over all states. Indeed it is this universality feature that has allowed a wide variety of complex quantum systems to be characterized by the universal and analytically tractable averages calculated with respect to the relevant ensemble.

While the above results and arguments indicate that pseudo-random unitaries can mimic the uniformly distributed random unitaries for the smoothly varying properties of interest, it is crucial to verify that they can do so efficiently. As shown in the inset of Fig. 2 in Ref. [4], the rate at which the average Q for the random circuits approaches the CUE average is indeed exponential. However, it should be noted that as n_q increases the exponential convergence rate decreases. The rate of decrease is itself decreasing, strongly suggesting the possibility that at worst only polylog(D) resources are required to converge to the CUE average. Indeed preliminary results from subsequent work suggest that the rate actually saturates at a finite asymptotic value [13], indicating that the random circuit method is indeed efficient, at least for the entanglement indicator we have considered.

Another important feature of random unitary operators is the marginal distribution of their matrix elements. This distribution plays a key practical role in the approximate randomization of quantum states [1, 3, 2]. Numerical results in Ref. [4] indicate that the distribution of matrix elements of the set of random circuits converges to the distribution expected for the uniform measure at an exponential rate. Similar results were obtained for the distribution of eigenvector components. This latter feature has particular relevance to quantum processing since it is the randomness of the eigenvectors that enables methods for the unbiased estimation of unknown noise sources [14] via the algorithm described below.

STOCHASTIC METHODS FOR NOISE ESTIMATION

Circuit implementations of random and pseudo-random unitaries enable novel methods for efficiently characterizing the strength and type of noise sources in imperfect quantum processors. As is well-known, identification and correction of the dominant coherent, incoherent and decoherent noise sources for a given quantum device is an essential step toward the eventual realization of fault-tolerant computation. The distribution and strength of these noise operators will differ for different devices and, moreover, the effect of the noise generators will generally depend on the applied control fields [15, 16]. The direct approach to this problem is quantum process tomography, which scales inefficiently and becomes impossible to implement in practice for devices with more than just a few qubits [17]. However, when the target transformation is sufficiently random, some of the measurable signatures of noise and decoherence, such as the rate of the average fidelity decay [18, 14] and the average rate of purity loss [19], become independent of any system properties and depend only on the intrinsic properties of the noise [14]. Moreover, by implementing a motion-reversal algorithm, where a pseudo-random unitary operator is applied n times followed by its inverse n times, the rate of fidelity decay and purity loss arising from any unknown noise sources may be measured via efficient algorithms [14] and subsequently compared to the analytic (ensemble average) predictions (which depend only on the characteristics of the noise model). It should be noted that an exponentially large (complete set) of initial states are not required (as in process tomography) since almost all initial states lead to fidelity decay rates that are exponentially close to the ensemble average [20]. In this way the implementation of pseudo-random unitary operators on a quantum processor enables *efficient stochastic* algorithms for the unbiased measurement of intrinsic properties of the unknown noise sources.

DISCUSSION

While numerical evidence indicates that random circuits can efficiently reproduce many of the statistical features of the uniform measure that are relevant to quantum information tasks, rigorous proofs of efficiency and error bounds are still needed. This is especially important in the case of the quantum cryptography applications, where a proper security analysis is required. In the context of noise estimation, new techniques are needed that can determine a wider set of features of the noise, and in particular those features that are most relevant to assessing the scalability of a device as a platform for fault-tolerant quantum computation, and to optimizing the appropriate choice of active and passive error-correction protocols.

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