Converse Barrier Functions via Lyapunov Functions

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***Abstract*—We prove a robust converse barrier function theorem via the converse Lyapunov theory. While the use of a Lyapunov function as a barrier function is straightforward, the existence of a converse Lyapunov function as a barrier function for a given safety set is not. We establish this link by a robustness argument. We show that the closure of the forward reachable set of a robustly safe set must be robustly asymptotically stable under mild technical assumptions. As a result, all robustly safe dynamical systems must admit a robust barrier function in the form of a Lyapunov function for set stability. We present the results in both continuous-time and discrete-time settings and remark on connections with various barrier function conditions.**

***Index Terms*—Barrier functions, Lyapunov functions, robustness, safety verification and control, stability.**

I. Introduction

The use of barrier functions to ensure set invariance and safety in control of dynamical systems has gained popularity in recent years in safety-critical control applications [1]–[12]. The readers are referred to Ames *et al.* [9] for a nice introduction on the background of barrier functions.

From the earlier work [1], [2] to recent results [13], [14], converse theorems for barrier functions played an important role in understanding how safety properties can indeed be characterized by barrier functions. The more stringent conditions in [1] and [2] for the existence of converse barrier functions are relaxed in [13] to a class of structurally table dynamical systems (more precisely, Morse–Smale vector fields) and in [14] to a robust safety requirement.

In this article, inspired by the recent work [14] and the connections made in [10] (see also [9]) between a barrier function and a Lyapunov function, we prove that, for all *robustly* safe dynamical systems, barrier functions can be constructed from Lyapunov functions. The use of Lyapunov functions to ensure set invariance is standard [15] (see also [1]). Ames *et al.* [9] and Xu *et al.* [10] also highlighted that if the barrier function conditions are satisfied in a neighborhood of the safety set, then the barrier function can indeed be regarded as a Lyapunov function. What is missing, however, are conditions under which such barrier functions exist assuming safety of the system. We establish this link by proving that the closure of the robust reachable set of a robustly safe set must be robustly asymptotically stable under mild technical assumptions (see Theorem 19). The results of this article could help provide a potentially more unified view of the Lyapunov function and barrier function theories, because how to simultaneously satisfy Lyapunov and barrier function conditions are important in practice but technically challenging [9], [12].

Notation: For $x \in \mathbb{R}^n$ and $r \geq 0$, we denote the ball of radius r centered at x by $B_r(x) = \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$, where $\|\cdot\|$ is the Euclidean norm. For a closed set $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we denote the distance from x to A by $\|x\|_A = \inf_{y \in A} \|x - y\|$ and r-neighborhood of A by $B_r(A) = \cup_{x \in A} B_r(x) = \{x \in \mathbb{R}^n : \|x\|_A \leq r\}$. For convenience, we also write $\mathbb{B} = B_1(0)$ and $r\mathbb{B} = B_r(0)$.

The remainder of this article is organized as follows. We present some preliminaries on barrier and Lyapunov functions for continuous-time systems in Section II. We prove a converse barrier function theorem by a converse Lyapunov function theorem in Section III. The results of Section III are extended to discrete-time systems in Section IV. Finally, Section V concludes this article.

II. Preliminaries

Consider a continuous-time dynamical system

$$x' = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be locally Lipschitz. For each $x_0 \in \mathbb{R}^n$, we denote the unique solution starting from $x(0) = x_0$ and defined on the maximal interval of existence by $x(t; x_0)$ or simply $x(t)$ if x_0 is not emphasized.

Given a scalar $\delta \geq 0$, a *δ-perturbation* of the dynamical system (1) is described by the differential inclusion

$$x' \in F_\delta(x) \tag{2}$$

where $F_\delta(x) = B_\delta(f(x))$. An equivalent description of the δ-perturbation of system (1) can be given by

$$x'(t) = f(x(t)) + d(t) \tag{3}$$

where $d : \mathbb{R} \to \delta\mathbb{B}$ is any measurable signal. We denote system (1) by $\mathcal{S}$ and its δ-perturbation by $\mathcal{S}_\delta$. Note that $\mathcal{S}_\delta$ reduces to $\mathcal{S}$ when $\delta = 0$. A solution of $\mathcal{S}_\delta$ starting from $x(0) = x_0$ can be denoted by $x(t; x_0, d)$ or simply $x(t)$, where d is a given disturbance signal. The set of all solutions for $\mathcal{S}_\delta$ starting from x_0 is denoted by $\mathcal{S}_\delta(x_0)$. We are only interested in forward solutions (i.e., solutions defined in positive time) in this article. We also assume that all solutions are forward complete, i.e., they exist on the time interval $[0, \infty)$. Set invariance, defined below and used in this article, also only concerns forward invariance.

Definition 1 (Invariant set): A set $\Omega \subseteq \mathbb{R}^n$ is said to be an *invariant set* of $\mathcal{S}_\delta$ if all solutions of $\mathcal{S}_\delta$ starting in Ω remain in Ω in positive time.

Definition 2 (Robustly invariant set): A set $\Omega \subseteq \mathbb{R}^n$ is said to be a *δ-robustly invariant set* of $\mathcal{S}$, for some $\delta \geq 0$, if it is an invariant set of $\mathcal{S}_\delta$. It is said to be a *robustly invariant set* of $\mathcal{S}$ if it is a δ-robustly invariant set for some $\delta > 0$.

Definition 3 (Robustly safe set): Given an *unsafe* set $U \subseteq \mathbb{R}^n$, a set $W \subseteq \mathbb{R}^n$ is said to be *δ-robustly safe* w.r.t. to U if all solutions of $\mathcal{S}_\delta$ starting from W will not enter U.

An immediate connection between robustly safe and invariant sets is the following.

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Proposition 4: If there exists a δ-robustly invariant set Ω such that $W \subseteq \Omega$ and $\Omega \cap U = \emptyset$, then W is δ-robustly safe w.r.t. to U.

Definition 5 (Robust barrier function): Given sets $W, U \subseteq \mathbb{R}^n$, a continuously differentiable function $B : \mathbb{R}^n \to \mathbb{R}$ is said to be a δ-*robust barrier function* for W and U if the following conditions are satisfied:

1) $B(x) \geq 0$ for all $x \in W$;
2) $B(x) < 0$ for all U; and
3) $\nabla B(x) \cdot (f(x) + d) > 0$ for all x such that $B(x) = 0$ and all $d \in \delta\mathbb{B}$.

Remark 6: The choice of sign for B to indicate a safe set is rather arbitrary because we can always negate it. Here, we use the condition $B(x) \geq 0$ to describe the safe set (the same as [9] and [14]) instead of $B(x) \leq 0$ (as in the original work [3]).

A δ-robust barrier function for (W, U) provides a certificate for δ-robust safety of W w.r.t. U, as summarized in the following result.

Proposition 7 (Sufficiency of barrier functions [2], [3]): If there exists a δ-robust barrier function for (W, U), then W is δ-robustly safe w.r.t. U.

A proof that leads to a slightly different conclusion can be found in [2]. We provide a short proof below for completeness.

Proof: We show that the set $C = \{x \in \mathbb{R}^n : B(x) \geq 0\}$ is an invariant set for $\mathcal{S}_\delta$. Robust safety of W follows immediately in view of Proposition 4 and conditions 1) and 2) of Definition 5. Suppose that C is not invariant for $\mathcal{S}_\delta$. Then, there exists a solution $x(\cdot)$ for $\mathcal{S}_\delta$ such that $x(0) \in C$ and $x(t) \notin C$ for some $t > 0$. Define

$$\bar{t} = \sup\{t \geq 0 : x(t) \in C\}.$$

Then, $\bar{t}$ is well defined and finite. By continuity of $B(x(t))$, we have $B(x(\bar{t})) = 0$. This implies that

$$\frac{dB(x(t))}{dt} = \nabla B(x(t)) \cdot (f(x(t)) + d(t)) > 0$$

at $t = \bar{t}$. Hence, for $\varepsilon > 0$ sufficiently small, we have $B(x(t)) > B(x(\bar{t})) = 0$ for $t \in (\bar{t}, \bar{t} + \varepsilon]$. This contradicts the definition of $\bar{t}$.

Remark 8: Note that the strict inequality $\nabla B(x) \cdot (f(x) + d) > 0$ is needed to guarantee the set $\{x \in \mathbb{R}^n : B(x) \geq 0\}$ is forward invariant. The original paper [3] used the nonstrict inequality condition: $\nabla B(x) \cdot (f(x) + d) \geq 0$ for all x such that $B(x) = 0$. This condition has been known to be unsound (see, e.g., [16, Example 2]; see also [17, Remark after Th. 3]). Safety properties of dynamical systems are intimately related to set invariance, on which there is a rich history of investigation (interested readers can refer to the work in [18] for more information; see also [17, Sec. 3]).

Several converse theorems for barrier functions have been proved in the literature [1], [2], [13], [14]. We quote a most recent result by Ratschan as follows.

Theorem 9 (Necessity of barrier functions [14]): Suppose that the closure of W and U is disjoint and the complement of U is bounded. If W is δ-robustly safe w.r.t. U, then there exists a continuously differentiable function $B : \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions:

1) $B(x) \geq 0$ for all $x \in W$;
2) $B(x) < 0$ for all U; and
3) $\nabla B(x) \cdot f(x) > 0$ for all x such that $B(x) = 0$.

While condition 3) appears to be slightly different from item 3) in Definition 5, we will remark on the connections between them, as well as with other variants of barrier function conditions, in Section III (see Remark 22).

We say a continuous function $\alpha : [0, a) \to \mathbb{R}$ belongs to class $\mathcal{K}$ and write $\alpha \in \mathcal{K}$ if α is strictly increasing and $\alpha(0) = 0$.

Definition 10 (Set stability [19]): A closed set $A \subseteq \mathbb{R}^n$ is said to be δ-*robustly uniformly asymptotically stable* (δ-RUAS) for $\mathcal{S}$ if the following two conditions are met.

1) For every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $\|x(0)\|_A < \delta_\varepsilon$ implies that $x(t)$ is defined and $\|x(t)\|_A < \varepsilon$ for all $t \geq 0$ and for any solution $x(t)$ of $\mathcal{S}_\delta$.
2) There exists some $\rho > 0$ such that, for every $\varepsilon > 0$, there exists some $T > 0$ such that $\|x(t)\|_A < \varepsilon$ for any solution $x(t)$ of $\mathcal{S}_\delta$ whenever $\|x(0)\|_A < \rho$ and $t \geq T$.

It is not difficult to see that a δ-RUAS set A must be δ-robustly invariant.

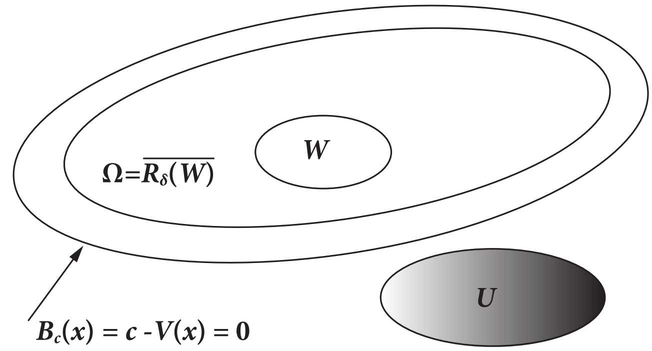

Fig. 1. Illustration of the sets involved in Theorem 16 and Assumption 15. The main idea is to show that the set Ω is δ'-RUAS (see Theorem 19), which guarantees the existence of a robust Lyapunov function V in a neighborhood of Ω (according to Theorem 12). A barrier function $B_c(x) = c - V(x)$ is then constructed to separate the safe and the unsafe sets. This construction of invariant sets as sublevel sets of a Lyapunov function is clearly standard. One of the main messages of this article, however, is that this construction is without loss of generality at least for robustly invariant sets.

Definition 11 (Robust Lyapunov function): Let $D \subseteq \mathbb{R}^n$ be an open set containing a closed set $A \subseteq \mathbb{R}^n$. A continuously differentiable function $V : D \to \mathbb{R}$ is said to be a δ-*robust Lyapunov function* for $\mathcal{S}$ w.r.t. A if the following two conditions are satisfied.

1) There exist class $\mathcal{K}$ functions α_1 and α_2 such that

$$\alpha_1(\|x\|_A) \leq V(x) \leq \alpha_2(\|x\|_A)$$

for all $x \in D$.

2) There exists a class $\mathcal{K}$ functions α_3 such that

$$\nabla V(x) \cdot ((f(x) + d) \leq -\alpha_3(\|x\|_A)$$

for all $x \in D$ and $d \in \delta\mathbb{B}$.

There are well-known Lyapunov characterizations of set stability.

Theorem 12 (Lyapunov characterization of set stability [19]–[22]): A closed set $A \subseteq \mathbb{R}^n$ is δ-RUAS for $\mathcal{S}$ if and only if there exists a δ-robust Lyapunov function for $\mathcal{S}$ w.r.t. A.

Remark 13: The Lyapunov function V can be made smooth on its domain D. There is a rich literature on converse Lyapunov functions. The readers are referred to the introduction of Teel and Praly [22] for an excellent historical account of the topic. Among the works in [19]–[22], Wilson [20] established a smooth converse Lyapunov function for uniform asymptotic stability w.r.t. a closed (and not necessarily compact) set. Lin *et al.* [19] proved a smooth converse Lyapunov function theorem for global uniform asymptotic stability w.r.t. a closed set A under the additional assumption that either A is compact or all solutions exist globally in backward time. It is later shown by Teel and Praly [22, Corollary 1], in a more general framework of converse Lyapunov function theorems for stability in terms of two measures [21, Th. 3.4.1], that the backward completeness assumption is not necessary.

III. Robust Converse Barrier Functions via Lyapunov Functions

In this section, we prove a version of converse barrier function theorem by resorting to converse Lyapunov theory.

We first introduce some notation. Let $R_\delta^t(x_0)$ denote the set reached by solutions of $\mathcal{S}_\delta$ at time t starting from x_0, i.e.,

$$R_\delta^t(x_0) = \{x(t) : x(\cdot) \in S_\delta(x_0))\}.$$

We further define

$$R_\delta(x_0) = \bigcup_{t\geq 0} R_\delta^t(x_0)$$

and, for a set $W \subseteq \mathbb{R}^n$

$$R_\delta^t(W) = \bigcup_{x_0 \in W} R_\delta^t(x_0), \quad R_\delta(W) = \bigcup_{x_0\in W} R_\delta(x_0).$$

Clearly, $R_\delta(W)$ is a δ-robustly invariant set of $\mathcal{S}$. If W is δ-robustly safe, then $R_\delta(W) \cap U \neq \emptyset$. If the complement of U is bounded (as assumed in Theorem 9), then $R_\delta(W)$ is bounded. Let $\Omega = \overline{R_\delta(W)}$. Then, Ω is compact. Without further assumption, Ω may intersect with U, as shown in the following example.

Example 14: Consider $\mathcal{S}$ defined by $x' = -x$. Let $W = [-0.1, 0.1]$ and $\delta = 0.2$. Then, $R_\delta(W) = (-0.2, 0.2)$ and $\Omega = [-0.2, 0.2]$. If $U = (-\infty, -2] \cup [2, \infty)$, then W is δ-robustly safe w.r.t. U because $R_\delta(W) \cap U = \emptyset$. Yet $\Omega \cap U \neq \emptyset$.

Note that the assumptions of Theorem 9 are indeed satisfied by the aforementioned example. While additionally assuming U to be open will lead to $\Omega \cap U = \emptyset$, we need a slightly stronger assumption for the purpose of this section, that is, $\Omega \cap \overline{U} = \emptyset$. This is summarized in the following assumption.

Assumption 15: The set W is δ-robustly safe w.r.t. U and $\Omega \cap \overline{U} = \emptyset$, where $\Omega = \overline{R_\delta(W)}$. Furthermore, we assume either of the following:

1) Ω is compact;
2) f is globally Lipschitz and there exists some $\rho > 0$ such that $\|x\|_\Omega \leq \rho$ implies $x \notin U$.

A few remarks on the aforementioned assumption are in order. The main result from the closely related work [14] (quoted as Theorem 9 above) did assume that Ω is compact. We also consider the case when Ω is not bounded, under the additional assumption that f is globally Lipschitz (as required by Lemma 17 below) and a mild technical assumption that essentially says at least some small neighborhood of Ω does not intersect U (a condition that clearly holds when Ω is compact).

With this assumption, we prove the following result on converse barrier functions.

Theorem 16 (Robustly safe sets admit robust barrier functions): Suppose that Assumption 15 holds. Then, for any $\delta' \in (0, \delta)$, there exists a δ'-robust barrier function for (W, U).

The conclusion of the aforementioned result is slightly stronger than the main result in [14] (quoted as Theorem 9 in Section II above) in two aspects: we show the existence of a δ'-robust barrier function for any $\delta' \in (0, \delta)$; we do not assume Ω to be compact, when f is globally Lipschitz.[1] Assumption 15 appears to be stronger than that of Theorem 9 in that it requires $\Omega \cap \overline{U} = \emptyset$. Nonetheless, the proof of Theorem 9 (see, e.g., [14, Lemma 5]) seems to be using this fact without explicitly mentioning or proving it. Example 14 shows that this does not readily follow from the assumptions of Theorem 9. Despite these subtle technical differences, the main message of this section, however, is that converse barrier functions can be constructed from Lyapunov functions.

[1]In fact, f being Lipschitz in a neighborhood of Ω suffices.

The construction relies on showing that the closure of the reachable set of the robustly safe set, i.e., the set $\Omega = \overline{R_\delta(W)}$, is robustly asymptotically stable (see Theorem 19). The following technical lemma on reachable sets of a perturbed system plays an important role in proving Theorem 19.

Lemma 17: Fix any $\delta' \in [0, \delta)$ and $\tau > 0$. Let $K \subseteq \mathbb{R}^n$ be a compact set. Then, there exists some $r = r(K, \tau, \delta,' \delta) > 0$ such that the following holds: if there is a solution x of $\mathcal{S}_{\delta'}$ such that $x(s) \in K$ for all $s \in [0, T]$, where $T \geq \tau$, then for any $y_0 \in B_r(x(0))$ and any $y_1 \in B_r(x(T))$, we have $y_1 \in R_\delta^T(y_0)$, i.e., y_1 is reachable at T from y_0 by a solution of $\mathcal{S}_\delta$. Furthermore, if f is globally Lipschitz, r can be chosen to be independent of K.

Proof: Let $x_0 = x(0)$ and $x_1 = x(T)$. Consider

$$y(s) = x(s) + \frac{s}{T}[y_1 - x_1 + x_0 - y_0] - x_0 + y_0, \quad s \in [0, T].$$

Then, $y(0) = y_0$ and $y(T) = y_1$. Furthermore

$$\begin{aligned}\|y(s) - x(s)\| &\leq \|y_1 - x_1\| \frac{s}{T} + \|y_0 - x_0\| \left(1 - \frac{s}{T}\right)\\ &\leq r\left(\frac{s}{T} + 1 - \frac{s}{T}\right) = r\end{aligned}$$

and

$$\begin{aligned}\|y'(s) - x'(s)\| &\leq \left\|\frac{1}{T}[y_1 - x_1 + (x_0 - y_0)]\right\|\\ &\leq \frac{1}{T}[\|y_1 - x_1\| + \|x_0 - y_0\|] \leq \frac{2r}{T}\end{aligned}$$

for all $s \in [0, T]$. Hence

$$\begin{aligned}&\|y'(s) - f(y(s))\|\\ &= \|y'(s) - x'(s) + x'(s) - f(x(s)) + f(x(s)) - f(y(s))\|\\ &\leq \frac{2r}{T} + \delta' + Lr\end{aligned}$$

where we used the triangle inequality, the fact that x is a solution of $\mathcal{S}_{\delta'}$, and Lipschitz continuity of f on the set $B_r(K)$. By picking r sufficiently small such that $\frac{2r}{\tau} + \delta' + Lr < \delta$, then we have $\|y'(s) - f(y(s))\| < \delta$ for all $s \in [0, T]$. Thus, y is a solution of $\mathcal{S}_\delta$ and the conclusion follows. Note that the choice of r only depends on K, τ, δ', and δ. The dependence on K is removed if f is globally Lipschitz.

Remark 18: Lemma 17 extends the statement of [14, Lemma 1], where the proof was omitted. Lemma 17 is slightly stronger because it says that we can steer any point in a small neighborhood of $x(0)$ (as opposed to only $x(0)$) to a small neighborhood of $x(T)$. This fact is needed in the proof of Theorem 19 below. Lemma 17 also allows T to vary as long as it is lower bounded by τ. The proof given here is elementary and constructive. Similar argument (of a simpler version) appeared in the proof of [23, Th. 1].

Theorem 19 (Robustly invariant sets are robustly asymptotically stable): If Assumption 15 holds, then for any $\delta' \in [0, \delta)$, the set $\Omega = \overline{R_\delta(W)}$ is δ'-RUAS for $\mathcal{S}$.

Proof: We verify conditions 1) and 2) of Definition 10.

1) For any $\varepsilon > 0$, let $\tau > 0$ be the minimal time that is required for solutions of $\mathcal{S}_{\delta'}$ to escape from $B_{\frac{\varepsilon}{2}}(\Omega)$ to $B_\varepsilon(\Omega)$. The existence of such τ follows from that f is locally Lipschitz and an argument using Gronwall's inequality. Pick $\delta_\varepsilon < \min(r, \frac{\varepsilon}{2})$, where r is from Lemma 17, applied to the set $B_\varepsilon(\Omega)$ and scalars τ, δ', and δ. Let x be any solution of $\mathcal{S}_{\delta'}$ such that $\|x(0)\|_\Omega < \delta_\varepsilon$. We show that $\|x(t)\|_\Omega < \varepsilon$ for all $t \geq 0$. Suppose that this is not the case. Then, $\|x(t_1)\|_\Omega \geq \varepsilon$ for some $t_1 \geq \tau > 0$. Since $\delta_\varepsilon < r$, we can always pick $y_0 \in R_\delta(W)$ such that $y_0 \in B_r(x(0))$ by the triangle inequality. By Lemma 17, there exists a solution of $\mathcal{S}_\delta$ from $y_0 \in R_\delta(W)$ to $x(t_1) \notin \Omega$. This contradicts that $R_\delta(W)$ is δ-robustly invariant.

2) Fix any $\varepsilon_0 > 0$. Following part 1), choose δ_{ε_0} such that $\|x(0)\|_\Omega < \delta_{\varepsilon_0}$ implies $\|x(t)\|_\Omega < \varepsilon_0$ for any solution $x(t)$ of $\mathcal{S}_{\delta'}$. Let r be chosen according to Lemma 17 with the set $B_{\varepsilon_0}(\Omega)$ and scalars $\tau = 1, \delta'$, and δ. Choose $\rho \in (0, r)$. Let x be any solution of $\mathcal{S}_{\delta'}$. We show that $\|x(0)\|_\Omega < \rho$ implies that $x(t) \in R_\delta(W)$ for all $t \ge 1$. Suppose that this is not the case. Then, there exists some $t_1 \ge 1$ such that $x(t_1) \in \partial\Omega$ or $x(t_1) \notin \Omega$. In either case, we can pick $y_1 \in B_r(x(t_1))$ such that $y_1 \notin \Omega$ and $y_0 \in B_r(x(0))$ such that $y_0 \in R_\delta(W)$. By Lemma 17, there exists a solution of $\mathcal{S}_\delta$ from $y_0 \in R_\delta(W)$ to $y_1 \notin \Omega$. This contradicts that $R_\delta(W)$ is δ-robustly invariant. Hence, $x(t) \in R_\delta(W) \subseteq \Omega$ for all $t \ge 1$. This clearly implies (2).

The conclusion of Theorem 19 cannot be strengthened in the sense that the set $\Omega = \overline{R_\delta(W)}$ may not be δ-RUAS for $\mathcal{S}$, as shown in the simple example below.

Example 20: Consider $\mathcal{S}$ defined by $x' = -x + x^2$. Let $W = [-0.1, 0.1]$ and $\delta = 0.25$. Then, $R_\delta(W) = (\frac{1}{2} - \frac{\sqrt{2}}{2}, 0.5)$ and $\Omega = [\frac{1}{2} - \frac{\sqrt{2}}{2}, 0.5]$. Solutions of $\mathcal{S}_\delta$ starting from $x_0 = 0.5 + \varepsilon$, where $\varepsilon > 0$, with $d(t) = \delta$ will tend to infinity. Hence, Ω cannot be δ-RUAS.

Theorem 16 can be obtained as a corollary of Theorems 19 and 12.

Proof of Theorem 16: By Theorem 19, Ω is δ'-RUAS for any $\delta' \in [0, \delta)$. By Theorem 12, there exists a neighborhood D of Ω and a smooth $V : D \to \mathbb{R}$ such that V satisfies conditions 1) and 2) in Definition 11.

Let

$$B_c(x) = c - V(x) \tag{4}$$

where $c > 0$ is a scalar chosen sufficiently small such that $B_c(x) \ge 0$ implies $x \notin U$, which is always possible under Assumption 15. Then, $B_c(x)$ also verifies all the conditions of a δ'-robust barrier function. In particular, we have

$$\begin{aligned} \nabla B_c(x) \cdot (f(x) + d) &= -\nabla V(x) \cdot (f(x) + d) \\ &\ge \alpha_3(\alpha_2^{-1}(V(x))) \\ &= \alpha_3(\alpha_2^{-1}(c - B_c(x))) \\ &= \alpha_3(\alpha_2^{-1}(c)) > 0 \end{aligned} \tag{5}$$

for all x such that $B_c(x) = 0$ and all $d \in \delta'\mathbb{B}$.

Remark 21: The construction of a Barrier function via a Lyapunov function is inspired by the works in [10] (see also [9]) and [14]. In [10], Xu *et al.* showed that if there exists a barrier function $B : \mathbb{R}^n \to \mathbb{R}$ satisfying the condition

$$\nabla B(x) \cdot f(x) \ge -\alpha(B(x)) \quad \forall x \in D \tag{6}$$

for some open set D containing $C = \{x \in \mathbb{R}^n : B(x) \ge 0\}$ and extended class $\mathcal{K}$ function[2] α, then C is asymptotically stable. This is straightforward to see because one can construct a Lyapunov function based on B by $V(x) = 0$ if $x \in C$ and $V(x) = -B(x)$ if $x \in D \backslash C$. Xu *et al.* [10] also discussed robustness implied by condition (6). The results of this section can be seen as a converse fact. We start with the assumption that a set W is robustly safe and show that the closure of the robustly invariant reachable set $\Omega = \overline{R_\delta(W)}$ is robustly asymptotically stable. Our proof of the latter fact is inspired by the work in [14]. A converse Lyapunov function is then used to construct a robust barrier function. Fig. 1 illustrates the sets involved in Theorem 16 and Assumption 15.

The following remark aims to connect the choice of barrier function condition in this article, i.e., condition 3) in Definition 5, with several different variants of barrier function conditions present in the literature.

[2]A function $\alpha : (-b, a) \to \mathbb{R}$, $a, b > 0$, is said to belong to extended class $\mathcal{K}$ if α is strictly increasing and $\alpha(0) = 0$.

Remark 22: Condition 3) in Definition 5 for a barrier function has different variants. The original work [3] had a condition like 3) and the following variant:

$$\nabla B(x) \cdot f(x, d) \ge 0 \quad \forall (x, d) \in \mathcal{X} \times \mathcal{W} \tag{7}$$

where $\mathcal{X} \times \mathcal{W}$ is the set on which f is defined and $\mathcal{W}$ is an arbitrary disturbance set. According to the work in [3], this variant makes the set of functions satisfying the barrier function conditions convex and amenable to computation by convex optimization. Condition (7) appears to be restrictive (from a computational perspective) because it needs to be satisfied for all $(x, d) \in \mathcal{X} \times \mathcal{W}$. Ames *et al.* [9] proposed (6) as a variant. Following the construction $B(x) = -V(x)$ in the proof of Theorem 16, we have

$$\begin{aligned} \nabla B(x) \cdot (f(x) + d) &= -\nabla V(x) \cdot (f(x) + d) \\ &\ge \alpha_3(\|x\|_A) \\ &\ge \alpha_3(\alpha_2^{-1}(V(x))) \\ &= \alpha_3(\alpha_2^{-1}(-B(x))) \end{aligned} \tag{8}$$

for all $x \in D$ and $d \in \delta\mathbb{B}$. Defining $\alpha_0(s) = -\alpha_3(\alpha_2^{-1}(-s))$, we obtain

$$\nabla B(x) \cdot (f(x) + d) \ge -\alpha_0(B(x)) \quad \forall (x, d) \in D \times \delta'\mathbb{B}. \tag{9}$$

While in the absence of disturbance, (9) appears in the same form as (6), it has a subtle difference because $\alpha_0(s)$ in (9) is not defined for $s > 0$. Note that, since $B(x) = -V(x)$, $B(x)$ is never positive by this construction. Nonetheless, (9) does match (6) when $B(x) \le 0$ in the absence of disturbance. With the barrier function $B_c(x)$ defined in (4) in the proof of Theorem 16, we have

$$\nabla B_c(x) \cdot (f(x) + d) \ge -\alpha_0(B_c(x)) \tag{10}$$

with $\alpha_0(s) = -\alpha_3(\alpha_2^{-1}(c - s))$. Note that, compared with (9), $B_c(x)$ now can take positive value and $\alpha_0(s)$ is defined for $s \in (0, c]$ as well. Nonetheless, while (10) agrees with (6) for $B_c(x) \le 0$ in the absence of disturbance, it is in fact stronger than (6) when $B_c(x) > 0$ because $\alpha_0(s) < 0$ for $s \in (0, c)$. This is not surprising because $B_c(x)$ is constructed using a Lyapunov function. Furthermore, in the absence of disturbance d, the strictly inequality (5) established in the proof of Theorem 16 recovers condition 3) for the barrier function in Theorem 9. Ratschan [14] seems to be using this strict positiveness, as well as strict positiveness of B on W, to indicate a robust barrier certificate. Here, we formally define a robust barrier function by requiring condition 3) in Definition 5 to hold under disturbance. We also remark that, when the set $B(x) = 0$ is compact, condition 3) in Theorem 9 also holds under sufficiently small disturbance. The construction given by Theorem 16, however, allows any disturbance of size $\delta' \in [0, \delta)$.

Remark 23: Another commonly used class of barrier functions is called reciprocal barrier functions [9], inspired by barrier methods from optimization [24]. Given a set C defined by

$$C = \{x \in \mathbb{R}^n : h(x) \ge 0\}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, a reciprocal barrier function $B : C^\circ \to \mathbb{R}$, where $C^\circ = \{x \in \mathbb{R}^n : h(x) > 0\}$ is the interior of C, such that

$$\frac{1}{\alpha_1(h(x))} \le B(x) \le \frac{1}{\alpha_2(h(x))} \tag{11}$$

$$\nabla B(x) \cdot f(x) \le \alpha_3(h(x)) \tag{12}$$

for all $x \in C^\circ$, where α_i $(i = 1, 2, 3)$ are class $\mathcal{K}$ functions. The reciprocal of the construction of barrier functions based on Lyapunov function directly gives a reciprocal barrier function. Let $h(x) = c - V(x)$ as in (4) and $B(x) = \frac{1}{h}$. Then, it is straightforward to verify that (11) is satisfied and (12) is robustly satisfied.

IV. Discrete-Time Converse Barrier Functions

Having built a link between Lyapunov functions and barrier functions, we extend the results in the previous section to the discrete-time setting and provide a converse theorem for discrete-time barrier function. The presentation parallels that of Section III, but formulated for discrete-time systems. We first present the preliminaries for discrete-time systems.

A. Preliminaries on Discrete-Time Systems

Consider a discrete-time dynamical system

$$x(t+1) = f(x(t)) \tag{13}$$

where $x(t) \in \mathbb{R}^n$ for $t \in \mathbb{Z}^+\{0, 1, 2, \cdots\}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be locally Lipschitz.

Given a scalar $\delta \geq 0$, a *δ-perturbation* of the dynamical system (13) is described by the difference inclusion

$$x(t+1) \in F_\delta(x(t)) \tag{14}$$

where $F_\delta(x) = B_r(f(x))$, or equivalently

$$x(t+1) = f(x(t)) + d(t) \tag{15}$$

where $d(t) \in \delta\mathbb{B}$ for each t. We denote system (13) by $\mathcal{DTS}$ and its δ-perturbation by $\mathcal{DTS}_\delta$. Note that $\mathcal{DTS}_\delta$ reduces to $\mathcal{DTS}$ when $\delta = 0$. A solution of $\mathcal{DTS}_\delta$ is a sequence denoted by $x(t; x_0, d)$ or $x(t)$, where $t = 0, 1, 2, \cdots$ and $d(t)$ is a disturbance sequence.

Since robustly safe sets, robustly invariant sets, and robust stability w.r.t. a closed set for $\mathcal{DTS}$ can be defined almost verbatim as for continuous-time systems, by replacing solutions of $\mathcal{S}_\delta$ with that of $\mathcal{DTS}_\delta$, they are omitted. We define discrete-time barrier and Lyapunov functions as follows.

Definition 24 (Discrete-time robust barrier function): Given sets $W, U \subseteq \mathbb{R}^n$, a continuously differentiable function $B : \mathbb{R}^n \to \mathbb{R}$ is said to be a *δ-robust barrier function* for W and U if the following conditions are satisfied:

1) $B(x) \geq 0$ for all $x \in W$;
2) $B(x) < 0$ for all U;
3) $B(f(x) + d) \geq 0$ for all x such that $B(x) \geq 0$ and all $d \in \delta\mathbb{B}$.

Proposition 25 (Sufficiency of discrete-time barrier functions): If there exists a δ-robust barrier function for (W, U), then W is δ-robustly safe w.r.t. U.

Proof: The conclusion follows from the fact that the set $C = \{x \in \mathbb{R}^n : B(x) \geq 0\}$ is δ-robustly invariant and $C \cap U \neq \emptyset$.

Remark 26: Condition 3) in Definition 24 for a discrete-time barrier function appears to be weaker than the ones used in practice. For instance, the following condition was proposed in [8]:

$$B(f(x)) - B(x) \geq -\alpha(B(x)), \quad x \in D \tag{16}$$

where $D \supseteq C = \{x \in \mathbb{R}^n : B(x) \geq 0\}$ and α is class $\mathcal{K}$ function satisfying $\alpha(r) < r$ when $r > 0$. Note that one needs to extend the definition of α to $(-b, 0)$ for some $b > 0$ if the set D contains x such that $B(x) < 0$. A special case of (16) is given by $\alpha(r) = \eta r$ for $\eta \in (0, 1)$. When $\eta = 1$ and $D = C$, we obtain condition 3) of Definition 24. When $\eta = 0$, we obtain a condition that is stronger than (16) on C

$$B(f(x)) - B(x) \geq 0, \quad x \in C \tag{17}$$

which clearly implies (16) for any $\alpha \in \mathcal{K}$ and $D = C$ because $\alpha(B(x)) \geq 0$ for $B(x) \geq 0$. Similar to Remark 22 on continuous-time barrier functions, the construction of discrete-time converse barrier functions by Lyapunov functions below in fact satisfy an even stronger form

$$B(f(x)) - B(x) \geq -\alpha_0(B(x)), \quad x \in D \tag{18}$$

where D is an open neighborhood of C and $\alpha_0(s) \leq 0$ for all s in its domain. Clearly, (18) implies both (17) and (16).

Definition 27: Let $D \subseteq \mathbb{R}^n$ be an open set containing a closed set $A \subseteq \mathbb{R}^n$. A continuously differentiable function $V : D \to \mathbb{R}$ is said to be a *δ-robust Lyapunov function* for $\mathcal{DTS}$ w.r.t. A if the following two conditions are satisfied.

1) There exist class $\mathcal{K}$ functions α_1 and α_2 such that
$$\alpha_1(\|x\|_A) \leq V(x) \leq \alpha_2(\|x\|_A)$$
for all $x \in D$.
2) There exists a class $\mathcal{K}$ functions α_3 such that
$$V(f(x) + d) - V(x) \leq -\alpha_3(\|x\|_A)$$
for all $x \in D$ and $d \in \delta\mathbb{B}$.

There are also Lyapunov characterizations of set stability for discrete-time systems.

Theorem 28 (Lyapunov characterization of set stability for $\mathcal{DTS}$ [25]): A closed set $A \subseteq \mathbb{R}^n$ is δ-RUAS for $\mathcal{DTS}$ if and only if there exists a δ-robust Lyapunov function for $\mathcal{DTS}$ w.r.t. A.

B. Converse Barrier Functions via Lyapunov Functions for Discrete-Time Systems

The notation and definitions for reachable sets remain the same, with continuous-time solutions replaced with discrete-time ones. We define $R^t_\delta(x_0)$, $R_\delta(x_0)$, $R_\delta(W)$, and $\Omega = \overline{R_\delta(W)}$ as in Section III, replacing continuous-time solutions with discrete-time ones. The following is a discrete-time version of Lemma 17.

Lemma 29: Fix any $\delta' \in [0, \delta)$. Let $K \subseteq \mathbb{R}^n$ be a compact set. Then, there exists some $r = r(K, \delta,' \delta) > 0$ such that the following holds: if there is a solution x of $\mathcal{S}_{\delta'}$ such that $x(s) \in K$ for all $s \in [0, T]$, where $T \geq 1$, then for any $y_0 \in B_r(x(0))$ and any $y_1 \in B_r(x(T))$, we have $y_1 \in R^T_\delta(y_0)$, i.e., y_1 is reachable at T from y_0 by a solution of $\mathcal{S}_\delta$. Furthermore, if f is globally Lipschitz, r can be chosen to be independent of K.

Proof: Let $x_0 = x(0)$ and $x_1 = x(T)$. Consider

$$y(s) = x(s) + \frac{s}{T}[y_1 - x_1 + x_0 - y_0] - x_0 + y_0$$

for $s \in \{0, 1, \ldots, T\}$. Then, $y(0) = y_0$ and $y(T) = y_1$. Furthermore

$$\begin{aligned}\|y(s) - x(s)\| &\leq \|y_1 - x_1\| \frac{s}{T} + \|y_0 - x_0\| \left(1 - \frac{s}{T}\right)\\ &\leq r\left(\frac{s}{T} + 1 - \frac{s}{T}\right) = r\end{aligned}$$

for all $s \in \{0, 1, \ldots, T\}$. Hence

$$\begin{aligned}&\|y(s+1) - f(y(s))\|\\ &= \|y(s+1) - x(s+1)\| + \|x(s+1) - f(x(s))\|\\ &\quad + \|f(x(s)) - f(y(s))\|\\ &\leq \frac{r}{T} + \delta' + Lr, \quad s \in \{0, 1, \ldots, T-1\}\end{aligned}$$

where we used the triangle inequality, the fact that x is a solution of $\mathcal{S}_{\delta'}$, and Lipschitz continuity of f on the set $B_r(K)$. By picking r sufficiently small such that $r + \delta' + Lr < \delta$, then we have $\|y(s+1) - f(y(s))\| < \delta$ for all $s \in \{0, 1, \ldots, T-1\}$. Thus, y is a solution of $\mathcal{DTS}_\delta$ and the conclusion follows. Note that the choice of r only depends on K, δ', and δ. If f is globally Lipschitz, the dependence on K can be removed.

The following is a discrete-time version of Theorem 19.

Theorem 30 (Robustly invariant sets are robustly asymptotically stable): If Assumption 15 holds, then Ω is δ'-RUAS for $\mathcal{DTS}$ for any $\delta' \in [0, \delta)$.

Proof:

1) For any $\varepsilon > 0$, let r be from Lemma 29, applied to the set $B_\varepsilon(\Omega)$ and scalars δ' and δ. Pick $\delta_\varepsilon = r$. Let x be any solution of $\mathcal{DTS}_{\delta'}$

such that $\|x(0)\|_\Omega < \delta_\varepsilon$. We show that $\|x(t)\|_\Omega < \varepsilon$ for all $t \geq 0$. Suppose that this is not the case. Then, $\|x(k)\|_\Omega \geq \varepsilon$ for some $k \geq 1$. Since $\|x(0)\|_\Omega < r$, we can always pick $y_0 \in R_\delta(W)$ such that $y_0 \in B_r(x(0))$ by the triangle inequality. By Lemma 29, there exists a solution of $\mathcal{DTS}_\delta$ from $y_0 \in R_\delta(W)$ to $x(k) \notin \Omega$. This contradicts that $R_\delta(W)$ is δ-robustly invariant.

2) Fix any $\varepsilon_0 > 0$. Following part 1), choose δ_{ε_0} such that $\|x(0)\|_\Omega < \delta_{\varepsilon_0}$ implies $\|x(t)\|_\Omega < \varepsilon_0$ for any solution $x(t)$ of $\mathcal{DTS}_{\delta'}$. Let r be chosen according to Lemma 29 with the set $B_{\varepsilon_0}(\Omega)$ and scalars δ' and δ. Choose $\rho \in (0, r)$. Let x be any solution of $\mathcal{DTS}_{\delta'}$. We show that $\|x(0)\|_A < \rho$ implies that $x(t) \in R_\delta(W)$ for all $t \geq 1$. Suppose that this is not the case. Then, there exists some $k \geq 1$ such that $x(k) \in \partial\Omega$ or $x(k) \notin \Omega$. In either case, we can pick $y_1 \in B_r(x(k))$ such that $y_1 \notin \Omega$ and $y_0 \in B_r(x(0))$ such that $y_0 \in R_\delta(W)$. By Lemma 29, there exists a solution of $\mathcal{DTS}_\delta$ from $y_0 \in R_\delta(W)$ to $y_1 \notin \Omega$. This contradicts that $R_\delta(W)$ is δ-robustly invariant. Hence, $x(t) \in R_\delta(W)$ for all $t \geq 1$. This completes part 2) of the definition of δ'-RUAS.

Theorem 31 (Robustly safe sets admit robust discrete-time barrier functions): Suppose that Assumption 15 holds. If either Ω is compact or f is globally Lipschitz, then for any $\delta' \in (0, \delta)$, there exists a δ'-robust barrier function for (W, U).

Proof: By Theorem 30, Ω is δ'-RUAS for $\mathcal{DTS}$ with any $\delta' \in [0, \delta)$. By Theorem 28, there exists a neighborhood D of Ω and a smooth $V : D \to \mathbb{R}$ such that

$$\alpha_1(\|x\|_\Omega) \leq V(x) \leq \alpha_2(\|x\|_\Omega)$$

and

$$V(f(x) + d) - V(x) \leq -\alpha_3(\|x\|_\Omega)$$

for all $x \in D$ and $d \in \delta'\mathbb{B}$, where α_i $(i = 1, 2, 3)$ are class $\mathcal{K}$ functions. Define

$$B(x) = -V(x), \quad x \in D.$$

It is straightforward to verify that B satisfies conditions 1)–3) of Definition 24 for a δ'-robust discrete-time barrier function.

Remark 32: By the construction of the barrier function $B(x)$ in the proof of Theorem 31, we in fact have a stronger condition than condition 3) in Definition 24

$$B(f(x) + d) - B(x) \geq -\alpha(B(x)) \tag{19}$$

for all $x \in D$ and $d \in \delta'\mathbb{B}$, where α is defined and increasing on $(-a, 0]$ for some $a > 0$ and $\alpha(0) = 0$.

Remark 33: Similar to Remark 23, we can construct discrete-time reciprocal barrier functions via Lyapunov functions. A discrete-time reciprocal barrier function [6] $B : C^\circ \to \mathbb{R}$ satisfies

$$\frac{1}{\alpha_1(h(x))} \leq B(x) \leq \frac{1}{\alpha_2(h(x))} \tag{20}$$

$$B(f(x)) - B(x) \leq \alpha_3(h(x)) \tag{21}$$

for all $x \in C^\circ$, where C° is the interior of the set $C = \{x \in \mathbb{R}^n : h(x) \geq 0\}$ for some continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$. Clearly, $h(x) = c - V(x)$ for some sufficiently small $c > 0$ and $B(x) = \frac{1}{h}$ satisfy the aforementioned conditions robustly.

V. Conclusion

In this article, we established a connection between Lyapunov functions and barrier functions. We proved that for all robustly safe dynamical systems, the closure of the robust reachable set of the robustly safe set must be robustly asymptotically stable. The converse Lyapunov function theory can then be brought to bear to yield a robust barrier function. We made remarks on several variants of the barrier function conditions and showed that they can all be satisfied by the construction of barrier functions using Lyapunov functions. We also formulated the results for discrete-time in a similar fashion.

For future work, it would be interesting to investigate how the viewpoint of robust barrier functions via Lyapunov functions can be utilized in practice. One immediate connection seems to be using this idea to unify converse Lyapunov functions for set stability and converse barrier functions for safety (see [12] for sufficient Lyapunov-barrier conditions). An initial step is taken in [26]. It remains a question to what extent the theoretical results can be leveraged for the actual computation of barrier functions via computational techniques for Lyapunov functions. A key technical challenge, however, seems to be that, while safety requirements can be specified rather arbitrarily by a designer (e.g., by defining the unsafe region U and safe initial region W in this article), the barrier function conditions are only met at the boundary of reachable set from the safe initial region W, if this set W can indeed be certified to be safe. While the computing of reachable sets can be highly nontrivial, it would be interesting to investigate whether the adaptive refinement techniques for computing maximal controlled invariant sets (see, e.g., [27]), combined with computational techniques for constructing barrier functions (see, e.g., [28]), can be used to determine a smaller set on which (control) barrier functions can be algorithmically constructed. A related theoretical question is that whether such procedures can be approximately complete in the sense that any δ-robustly safe sets admit a computable δ'-robust barrier certifications for any $\delta' \in [0, \delta)$.

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