Invariance Control Synthesis for Switched Nonlinear Systems: An Interval Analysis Approach

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***Abstract*—This note proposes an interval analysis approach to invariance control synthesis for switched nonlinear systems without assuming that the subsystems are stable or have common equilibrium points. Partition-based controllers are extracted via iterative computation of controlled invariant sets based on an interval branch-and-bound scheme. This method is guaranteed to be finitely determined and complete, provided that the switched system satisfies a robustly controlled invariance condition. Two examples drawn from practical applications are provided to show the effectiveness and efficiency of the proposed method.**

***Index Terms*—Control synthesis, interval analysis, invariance, robustness, switched systems.**

I. Introduction

Control of switched systems can be found in various applications, e.g., electrical power converters [7], robot motion planning [15], and flight management [6]. Invariance control is concerned with seeking a control law such that the solutions of a closed-loop system are restricted to a specified region in the state space for all time. Practical stabilization and safety control, which are two important objectives in various settings, are in the scope of invariance control [7], [8].

We consider switched systems without any stability assumptions. This is in contrast with related work on switched systems in [5] and [8], where subsystems are required to be asymptotically or incrementally stable. For such systems, invariance control relies on a fixed-point algorithm [1], which iteratively approximates the maximal controlled invariant set (the fixed point). Numerical implementation of this fixed-point algorithm is nontrivial even for linear systems because maximal controlled invariant sets are not guaranteed to be finitely determined, i.e., computation may not terminate in a finite number of steps [26]. To circumvent this difficulty, outer and inner approximations of invariant sets are sought. For invariance control purposes, invariant inner approximations are more desirable, because, different from outer approximations, they are subsets of states that can be controlled invariant, for which an invariance controller exists. Finitely determined invariant inner approximations can be obtained by requiring contractivity on the system dynamics around a C-set (i.e., compact and convex set containing the origin) [2] or computing the null-controllable sets (i.e., the set of states that can be controlled to the origin in finite time) [10]. The limitation to C-set is dropped in [26].

For nonlinear systems, an additional challenge comes from the computation of reachable sets under nonlinear dynamics. Lyapunov functions are an important tool for nonlinear invariance control, yet construction of Lyapunov functions is a challenging task. For the purpose of safety control, barrier certificates [29] are proposed, and sum-of-squares techniques [12] are used to search for barrier certificates or Lyapunov functions. Assumptions of polynomial dynamics or particular forms of feedback control functions are usually made. Using deterministic [8] or nondeterministic [18], [19], [21], [23], [30] finite abstractions of the original systems, abstraction-based methods avoid reachable set computation during fixed-point iterations. Because of the gap between nondeterministic abstractions and the original systems, control synthesis results based on such abstractions might be incomplete.

In this note, we propose an invariance control method for discrete-time switched nonlinear systems without assuming that subsystems are asymptotically stable or have common equilibrium points. As an improvement of our preliminary work [17], this note does not assume that all states in the given set can be controlled invariant. Essential to this method is the recursive inner approximations of the maximal controlled invariant sets using interval methods [13]. This is in contrast with [4], [16], since [16] is concerned with outer approximations, and [4] only considers one-step set approximations. Set representation by unions of intervals allows convenient controller extraction after computation terminates. We highlight the main contributions as follows. In this note, we show that a controlled invariant set can be finitely determined if the nonlinear switched system is robustly controlled invariant with respect to a given target set. It implies that robustness is critical for numerical computation of controlled invariant sets. This also extends related results for linear systems [2], [25], [26] to a nonlinear setting. By setting a sufficiently high precision, the algorithm is promised to yield an invariant inner approximation of the maximal controlled invariant set under this robustly controlled invariance condition. Furthermore, incorporating the branch-and-bound scheme, the proposed invariance control method demonstrates a higher efficiency than abstraction-based methods using uniform grids. This is illustrated by two numerical examples.

II. Invariance Control Problem

We consider switched systems of the form

$$x_{k+1} = f_{p_k}(x_k), \quad k \in \mathbb{Z}_{\geq 0} \tag{1}$$

where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers, $x_k \in \mathbb{R}^n$ is the system state, and $p_k \in \mathcal{M}$ is the mode at time k. The set of modes $\mathcal{M}$ is

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assumed to be finite. The function $f_p : \mathbb{R}^n \to \mathbb{R}^n$ is continuous for all $p \in \mathcal{M}$.

Any infinite sequence in $\mathcal{M}$ defines a *switching signal* for system (1), written as $\sigma := \{p_k\}_{k=0}^{\infty}$. Given a switching signal σ and an initial condition $x_0 \in \mathbb{R}^n$, a solution of system (1) is a sequence of states $\{x_k\}_{k=0}^{\infty}$ such that (1) holds for all time.

A set $\Omega \subseteq \mathbb{R}^n$ is said to be *controlled invariant* for system (1) if, for any initial state $x_0 \in \Omega$, there exists a switching signal σ such that the resulting solution $\{x_k\}_{k=0}^{\infty}$ of (1) satisfies $x_k \in \Omega$ for all $k \geq 0$. A controlled invariant set inside Ω is possible to exist, even if Ω is not controlled invariant itself. The one that contains all controlled invariant sets inside Ω is called the *maximal controlled invariant set* of Ω, denoted by $\mathcal{I}^{\infty}(\Omega)$.

Our objective is to design an *invariance controller* for (1) with respect to a set $\Omega \subseteq \mathbb{R}^n$, which is defined as a function $c : \Omega \to 2^{\mathcal{M}}$ such that, for any (state-dependent) switching signal $\sigma = \{p_k\}_{k=0}^{\infty}$, where $p_k \in c(x_k)$ for all $k \geq 0$, the resulting solution $\{x_k\}_{k=0}^{\infty}$ with $x_{k+1} = f_{p_k}(x_k)$ satisfies that $x_k \in \Omega$ for all k.

To design such a controller, one is often interested in approximating the maximal controlled invariant set, since it reveals all the states that can be controlled invariant. Moreover, such approximations are desired to be invariant (we call them invariant approximations) so that invariance controllers can be found for them. Therefore, we address the problem as follows.

Invariance Control Problem: Given a compact set $\Omega \subseteq \mathbb{R}^n$ for system (1), compute an invariant approximation of the maximal controlled invariant set inside Ω and design an invariance controller for system (1) with respect to Ω.

Before presenting the technical details, we recall some essential results on controlled invariance in the literature, which also apply to switched systems based on the following definition.

Definition 1: Given a set $\Omega \subseteq \mathbb{R}^n$, the *one-step backward reachable set* of Ω with respect to system (1) is defined by

$$\mathrm{pre}(\Omega) := \{x \in \mathbb{R}^n : \exists p \in \mathcal{M} \text{ such that } f_p(x) \in \Omega\}.$$

By Definition 1 and continuity of the function f_p for all $p \in \mathcal{M}$, it is straightforward to prove the following results.

Proposition 1 (see [22], [28]): Let $\Omega \subseteq \mathbb{R}^n$, and $A \subseteq B \subseteq \mathbb{R}^n$. Then, 1) if Ω is closed, $\mathrm{Pre}(\Omega)$ is closed; 2) $\mathrm{Pre}(A) \subseteq \mathrm{Pre}(B)$.

Proposition 2 (see [3]): A set $\Omega \subseteq \mathbb{R}^n$ is controlled invariant for system (1) if and only if $\Omega \subseteq \mathrm{Pre}(\Omega)$.

Let I be a mapping between subsets of $\mathbb{R}^n$ defined as

$$I(X) = \mathrm{Pre}(X) \cap X \tag{2}$$

where $X \subseteq \mathbb{R}^n$.

Denote by I^j ($j \in \mathbb{Z}_{\geq 0}$) the jth iterate of the mapping I, and let $I^0(X) = X$. The maximal controlled invariant set inside a given compact set $\Omega \subseteq \mathbb{R}^n$ can be obtained by iterating I infinitely many times as shown in the following proposition.

Proposition 3 (see [1], [16]): Let $\Omega \subseteq \mathbb{R}^n$ be closed. Then

$$\mathcal{I}^{\infty}(\Omega) = \lim_{j \to \infty} I^j(\Omega) = \bigcap_{j=0}^{\infty} I^j(\Omega).$$

Computation of reachable sets and finite termination guarantee are two challenges in implementing iterative computation of (2). We show, in Section III, that computation of controlled invariant sets is finitely determined for general nonlinear systems if a robustly controlled invariant condition is satisfied. This computation can be realized by an interval branch-and-bound technique, and an invariance controller can be easily extracted as presented in Section IV.

Algorithm 1: Approximation of $I(\Omega)$.

1: **procedure** CPRE $([f_p]_{p\in\mathcal{M}}, \Omega, \varepsilon)$
2: $\quad \underline{X} \leftarrow \varnothing, \Delta X \leftarrow \varnothing, X_c \leftarrow \varnothing, List \leftarrow \Omega$
3: $\quad$ **while** $List \neq \varnothing$ **do**
4: $\quad\quad [x] \leftarrow List.first$
5: $\quad\quad$ **if** $[f_p]([x]) \cap \Omega = \varnothing$ for all $p \in \mathcal{M}$ **then**
6: $\quad\quad\quad X_c \leftarrow X_c \cup [x]$
7: $\quad\quad$ **else if** $[f_p]([x]) \subseteq \Omega$ for some $p \in \mathcal{M}$ **then**
8: $\quad\quad\quad \underline{X} \leftarrow \underline{X} \cup [x]$
9: $\quad\quad$ **else if** $w([x]) < \varepsilon$ **then**
10: $\quad\quad\quad \Delta X \leftarrow \Delta X \cup [x]$
11: $\quad\quad$ **else**
12: $\quad\quad\quad \{L[x], R[x]\} = Bisect([x])$
13: $\quad\quad\quad List.add(L[x] \cup R[x])$
14: $\quad\quad$ **end if**
15: $\quad$ **end while**
16: $\quad$ **return** $\underline{X}$
17: **end procedure**

III. Computation of Controlled Invariant Sets

We use the following notation: the Minkowski sum and the Pontryagin difference of sets $A, B \subseteq \mathbb{R}^n$ is defined as $A \oplus B := \{a + b \mid a \in A, b \in B\}$, and $A \ominus B := \{c \in \mathbb{R}^n \mid c + b \in A, \forall b \in B\}$, respectively. $B \setminus A := \{x \in B \mid x \notin A\}$. The boundary, interior and closure of A are ∂A, $\mathrm{int}(A)$ and $\mathrm{cl}(A)$, respectively. Define $\mathcal{B}_r := \{y \in \mathbb{R}^n \mid |y| \leq r\}$, where $|\cdot|$ is the infinity norm in $\mathbb{R}^n$. Denote by $[x] := [x_1] \times \cdots \times [x_n] \subseteq \mathbb{R}^n$ the interval vector in $\mathbb{R}^n$ where $[x_i] = [\underline{x}_i, \overline{x}_i] \subseteq \mathbb{R}$ ($\underline{x}_i$ is the infimum and $\overline{x}_i$ is the supremum of $[x_i]$) for $i = 1, \ldots, n$. The set of all interval vectors in $\mathbb{R}^n$ is denoted by $\mathbb{IR}^n$, and $w([x]) := \max_{1 \leq i \leq n}\{\overline{x_i} - \underline{x_i}\}$ is the width of $[x]$.

A. Interval Approximation of Backward Reachable Sets

We use interval methods for approximating backward reachable sets under nonlinear dynamics, not only for its simplicity, but also for its convergence guarantee under mild assumptions. Fundamental to computing interval images is the concept of convergent inclusion functions.

Definition 2 (see [13]): Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$. The interval function $[f] : \mathbb{IR}^n \to \mathbb{IR}^m$ is called a *convergent inclusion function* of f if 1) $f([x]) \subseteq [f]([x])$ and 2) $\lim_{w([x]) \to 0} w([f]([x])) = 0$ for all $[x] \in \mathbb{IR}^n$.

For a vector-valued function f, its convergent inclusion function is not unique. One straightforward inclusion function is called the natural inclusion function, which is obtained by replacing the variables and operations of a function by their interval counterparts. Natural inclusion functions are known to have at least a linear convergence rate. For higher precision, centered-form and mean-value form can be used [13].

Given Ω as a list of intervals and a set of vector field $\{f_p\}_{p\in\mathcal{M}}$, Algorithm 1 approximates $I(\Omega)$ by applying a branch-and-bound scheme [13]. The assumption that Ω is a list of intervals is without loss of generality, because any compact set can be arbitrarily approximated by a union of intervals. In line 12, the interval vectors $L([x])$ and $R([x])$ are given by $L[x] = [\underline{x}_1, \overline{x}_1] \times \cdots [\underline{x}_j, (\underline{x}_j + \times\overline{x}_j)/2] \times \cdots \times [\underline{x}_n, \overline{x}_n]$, and $R[x] = [\underline{x}_1, \overline{x}_1] \times \cdots \times [(\underline{x}_j + \overline{x}_j)/2, \overline{x}_j] \times \cdots \times [\underline{x}_n, \overline{x}_n]$, respectively, where j is the bisected dimension.

Algorithm 1 refines Ω into three lists of intervals, denoted by $\underline{X}$, ΔX, and X_c. The intervals entirely inside $I(\Omega)$ are included in $\underline{X}$, while the ones outside of $I(\Omega)$ are collected in X_c. The list ΔX consists of the intervals that are partially inside $I(\Omega)$, i.e., undetermined intervals. The parameter ε controls the size of the undetermined intervals.

We show, in Lemma 1, the relation between ε and the approximation precision of $\text{Pre}(\Omega)$ based on the following assumption.

Assumption 1: Let $\Omega \subseteq \mathbb{R}^n$. For system (1), there exists a $\rho_1 > 0$ such that $|f_p(x) - f_p(y)| \le \rho_1 |x - y|$ for all $x, y \in \Omega$ and $p \in \mathcal{M}$.

This is essentially a local Lipschitz assumption on f_p for all $p \in \mathcal{M}$. If f_p is continuously differentiable in a neighborhood of Ω for all $p \in \mathcal{M}$ and Ω is compact, then $\rho_1 = \max_{x \in \overline{\text{co}}(\Omega), p \in \mathcal{M}} \|J_x f_p\|$, where $J_x f_p$ is the Jacobian matrix of f_p at x, and $\overline{\text{co}}(\Omega)$ is the closed convex hull of Ω. With Assumption 1, it is always possible to choose an inclusion function $[f_p]$ for each f_p (e.g., mean-value form [13]) such that

$$w([f_p]([x])) \le \rho_1 w([x]) \qquad \forall [x] \subseteq \Omega. \tag{3}$$

Lemma 1: Suppose that Ω is compact. Let $\overline{X} := \underline{X} \cup \Delta X$, where $\underline{X}$ and ΔX are outputs of Algorithm 1 with precision control parameter ε. If Assumption 1 holds on Ω, and the interval functions $[f_p]$ ($p \in \mathcal{M}$) are chosen to satisfy (3), then

$$I(\Omega \oplus \mathcal{B}_{\rho_1 \varepsilon}) \supseteq \text{Pre}(\Omega \oplus \mathcal{B}_{\rho_1 \varepsilon}) \cap \Omega \supseteq \overline{X} \supseteq I(\Omega)$$
$$I(\Omega \ominus \mathcal{B}_{\rho_1 \varepsilon}) \subseteq \text{Pre}(\Omega \ominus \mathcal{B}_{\rho_1 \varepsilon}) \cap \Omega \subseteq \underline{X} \subseteq I(\Omega).$$

Proof: It follows straightforwardly from Algorithm 1 that $\underline{X} \subseteq I(\Omega) \subseteq \overline{X} \subseteq \Omega$, and $w([x]) < \varepsilon$ for all $[x] \in \Delta X$.

By (3), we have $w([f_p]([x])) \le \rho_1 w([x]) < \rho_1 \varepsilon$. For any $[x] \in \Delta X$, there exists a $p \in \mathcal{M}$ such that $[f_p]([x]) \cap \Omega \neq \varnothing$. By the definition of the Minkowski sum, it follows that $[f_p]([x]) \subseteq \Omega \oplus \mathcal{B}_{\rho_1 \varepsilon}$. Also, $\underline{X} \subseteq (\Omega \cap \text{Pre}(\Omega))$. Hence, $\overline{X} = (\underline{X} \cup \Delta X) \subseteq (\text{Pre}(\Omega \oplus \mathcal{B}_{\rho_1 \varepsilon}) \cap \Omega) \subseteq I(\Omega \oplus \mathcal{B}_{\rho_1 \varepsilon})$.

Since $I(\Omega \ominus \mathcal{B}_{\rho_1 \varepsilon}) \subseteq (\text{Pre}(\Omega \ominus \mathcal{B}_{\rho_1 \varepsilon}) \cap \Omega)$ always holds, we only show that $(\Omega \cap \text{Pre}(\Omega \ominus \mathcal{B}_{\rho_1 \varepsilon})) \subseteq \underline{X}$. If not, there exists an $x \in (\Omega \cap \text{Pre}(\Omega \ominus \mathcal{B}_{\rho_1 \varepsilon}))$, but $x \notin \underline{X}$. Then, x has to be in ΔX, since $x \in X_c$ implies that $x \notin \Omega \cap \text{Pre}(\Omega)$, which is contradictory to the fact that $x \in (\Omega \cap \text{Pre}(\Omega \ominus \mathcal{B}_{\rho_1 \varepsilon}))$. Let $x \in [x] \subseteq \Delta X$. By [14, Th. 2.1 (ii)], $f_p(x) \in [f_p]([x]) \subseteq \Omega \ominus \mathcal{B}_{\rho_1 \varepsilon} \oplus \mathcal{B}_{\rho_1 \varepsilon} \subseteq \Omega$. It implies that $[x] \subseteq \underline{X}$, which is a contradiction. Hence, $I(\Omega \ominus \mathcal{B}_{\rho_1 \varepsilon}) \subseteq (\text{Pre}(\Omega \ominus \mathcal{B}_{\rho_1 \varepsilon}) \cap \Omega) \subseteq \underline{X}$, which completes the proof. ■

Furthermore, the sets $\underline{X}$ and $\overline{X}$ converge to $I(\Omega)$ under some continuity condition of I [13, Th. 3.1]. This implies that $I(\Omega)$ can be approximated to an arbitrary precision.

B. Robustly Controlled Invariant Sets

Based on the preceding section, now we show a close connection between the computation of controlled invariant sets and a robustness property of the systems.

The following definition generalizes the concept of contractive invariant sets [2]. We do not assume that the given set is of particular shapes or contains the origin.

Definition 3: A set Ω is said to be a *r-robustly controlled invariant set* ($r \ge 0$) for system (1) if

$$\Omega \subseteq \text{Pre}(\Omega \ominus \mathcal{B}_r). \tag{4}$$

We call Ω robustly controlled invariant if $r > 0$. The supremum of r satisfying (4) is called the *robust invariance margin* of Ω.

Intuitively, a set Ω with a positive robust invariance margin is able to be controlled invariant even under a certain degree of uncertainties, including computational errors.

It is interesting to note that by definition the maximal invariant set itself is not robustly controlled invariant. This also indicates that the determination of the maximal invariant set is numerically nontrivial because of approximation errors.

Algorithm 2: Inner Approximation of $\mathcal{I}^\infty(\Omega)$.

Require: $\{[f_p]\}_{p \in \mathcal{M}}, \Omega, \varepsilon$

1: $\widetilde{Y} \leftarrow \Omega, Y \leftarrow \varnothing$
2: **while** $Y \neq \widetilde{Y}$ **do**
3: $\quad Y \leftarrow \widetilde{Y}$
4: $\quad \underline{X} = \text{CPRE}(\{[f_p]\}_{p \in \mathcal{M}}, Y, \varepsilon)$
5: $\quad \widetilde{Y} \leftarrow \cup_{[x] \in \underline{X}} [x]$
6: **end while**
7: **return** Y

Proposition 4: Let $\Omega \subseteq \mathbb{R}^n$ be compact and $\mathcal{I}^\infty(\Omega)$ be the maximal invariant set within Ω. Suppose that $\mathcal{I}^\infty(\Omega) \neq \Omega$. Then, $\mathcal{I}^\infty(\Omega)$ is not robustly controlled invariant.

Proof: We prove this by showing that some boundary points of $\mathcal{I}^\infty(\Omega)$ will be mapped into the boundary of $\mathcal{I}^\infty(\Omega)$. We only consider the case $\text{int}(\Omega) \neq \varnothing$; otherwise the conclusion trivially holds by Definition 3 because $\text{int}(\mathcal{I}^\infty(\Omega)) = \varnothing$. For the purpose of contradiction, we assume that $x \in (\partial \mathcal{I}^\infty(\Omega) \cap \text{int}(\Omega))$, and there exists a $p \in \mathcal{M}$ such that $f_p(x) \in \text{int}(\mathcal{I}^\infty(\Omega))$. That implies there exists a $r > 0$ such that $B_r(f_p(x)) \subseteq \mathcal{I}^\infty(\Omega)$. By continuity of f_p, we can find a $\delta(r) > 0$ such that any $x' \in B_{\delta(r)}(x)$ satisfies $f_p(x') \in B_r(f_p(x))$ and, thus, $f_p(B_{\delta(r)}(x)) \subseteq \mathcal{I}^\infty(\Omega)$, which means x is an interior point of $\mathcal{I}^\infty(\Omega)$. This is a contradiction. ■

Likewise, the proposition below characterizes the maximal r-robustly controlled invariant sets for system (1) in a given closed set. We use the set limit defined in [24, Definition 4.1].

Proposition 5: Let $\Omega \subseteq \mathbb{R}^n$ be closed and $\mathcal{I}_r^\infty(\Omega)$ be the maximal r-robustly invariant set inside Ω for system (1). Define a mapping between subsets of $\mathbb{R}^n$:

$$I_r(X) = \text{Pre}(X \ominus \mathcal{B}_r) \cap X \tag{5}$$

where $X \subseteq \mathbb{R}^n$. Let I_r^j ($j \in \mathbb{Z}_{>0}$) be the jth iterate of I_r. Then, $\mathcal{I}_r^\infty(\Omega) = \lim_{j \to \infty} I_r^j(\Omega) = \cap_{j=1}^\infty I_r^j(\Omega)$.

Proof: Given that Ω is closed, $\Omega \ominus \mathcal{B}_r$ is closed [14, Th. 2.1]. By Proposition 1, $\text{Pre}(\Omega \ominus \mathcal{B}_r)$ and, hence, $I_r^j(\Omega)$ ($\forall j \ge 1$), is closed. Since $\{I_r^j\}$ is nonincreasing, by Painlevé–Kuratowski convergence [24, p. 111], $\lim_{i \to \infty} I^j(\Omega) = \bigcap_{j=1}^\infty I_r^j(\Omega)$ is closed and nonempty if $I_r^j(\Omega) \neq \varnothing$.

First, we claim that $\cap_{j=1}^\infty I_r^j(\Omega) \subseteq \mathcal{I}_r^\infty(\Omega)$. We only consider $I_r^j(\Omega) \neq \varnothing$ for all $j \ge 1$, otherwise the claim trivially holds. For any $x \in \cap_{j=1}^\infty I_r^j(\Omega)$, $x \in I_r^j(\Omega)$ for all $j \ge 1$. Thus, there exists $p_j \in \mathcal{M}$ such that $f_{p_j}(x) \in (I_r^{j-1}(\Omega) \ominus \mathcal{B}_r)$ for all $j \ge 1$. Since $\mathcal{M}$ is finite, the sequence $\{p_j\}_{j=1}^\infty \subseteq \mathcal{M}$ must admit a constant subsequence, i.e., there exists $p \in \mathcal{M}$ such that $f_p(x) \in I_r^{j-1}(\Omega)$ for infinitely many $j \ge 1$. This implies $f_p(x) \in \bigcap_{j=1}^\infty I_r^j(\Omega)$, which proves the claim.

Next, we show that $\mathcal{I}_r^\infty(\Omega) \subseteq \cap_{j=1}^\infty I_r^j(\Omega)$. We assume that $\mathcal{I}_r^\infty(\Omega) \ominus \mathcal{B}_r \neq \varnothing$, otherwise $\mathcal{I}_r^\infty(\Omega) = \varnothing$, which means the conclusion trivially holds. For $j = 0$, we have $\mathcal{I}_r^\infty(\Omega) \subseteq I_r^0(\Omega) = \Omega$. Suppose that $\mathcal{I}_r^\infty(\Omega) \subseteq I_r^j(\Omega)$. By (4), for any $x \in (I_r^j(\Omega) \setminus I_r^{j+1}(\Omega))$, $f_p(x) \notin (I_r^j(\Omega) \ominus \mathcal{B}_r)$ for all $p \in \mathcal{M}$, which also means $f_p(x) \notin (\mathcal{I}_r^\infty(\Omega) \ominus \mathcal{B}_r)$. By definition of $\mathcal{I}_r^\infty(\Omega)$, $x \notin \mathcal{I}_r^\infty(\Omega)$. It follows that $\mathcal{I}_r^\infty(\Omega) \subseteq I_r^{j+1}(\Omega)$. Hence, $\mathcal{I}_r^\infty(\Omega) \subseteq \bigcap_{j=1}^\infty I_r^j(\Omega)$. ■

Now we present Algorithm 2 for approximating the maximal controlled invariant set.

Theorem 1: Let $\Omega \subseteq \mathbb{R}^n$ be compact, and Assumption 1 holds. Denote by $\underline{Y}^\varepsilon$ the output of Algorithm 2 for a given precision ε. Then, Algorithm 2 terminates in a finite number of steps. Furthermore, if $\rho_1 \varepsilon \le r$, then

1) If $\underline{Y}^\varepsilon = \varnothing$, then no r-robustly controlled invariant subset of Ω exists for system (1).

2) If $\underline{Y}^{\varepsilon} \neq \varnothing$, then $\underline{Y}^{\varepsilon}$ and

$$\mathcal{I}_r^{\infty}(\Omega) \subseteq \underline{Y}^{\varepsilon} \subseteq \mathcal{I}^{\infty}(\Omega).$$

Proof: Let $X_{c,j}$, ΔX_j, and Y_j be X_c, ΔX, and Y in the jth iteration ($j \in \mathbb{Z}_{\geq 0}$), respectively. Let $Y_0 = \Omega$, and $Y_{j+1} = Y_j \setminus (\Delta X_j \cup X_{c,j})$. If $Y_{j+1} \neq Y_j$ for any j, then $(\Delta X_j \cup X_{c,j}) \neq \varnothing$, and $\{Y_j\}$ is strictly decreasing. Under a given precision ε and the compactness of Ω, Y_j contains a finite number of intervals. Then, there must exists an $N \in \mathbb{Z}_{>0}$ such that $Y_N = \varnothing$, which results in $Y_{N+1} = \varnothing$. It means that $Y_j = \widetilde{Y_j}$ for all $j \geq N$. Hence, this algorithm will terminate in finite iterations.

To prove 1) and 2), suppose $\{X_j\}$ and $\{X_j^r\}$ is the sequence generated by (2) and (5), respectively. Then, $X_0 = X_0^r = \Omega$, $X_{j+1} = \mathrm{Pre}(X_j) \cap X_j$, and $X_{j+1}^r = (\mathrm{Pre}(X_j \ominus \mathcal{B}_r) \cap X_j)$. By Lemma 1, $Y_0 = X_0 = \Omega$, $(\mathrm{Pre}(Y_j \ominus \mathcal{B}_{\rho_1 \varepsilon}) \cap Y_j) \subseteq Y_{j+1} \subseteq (\mathrm{Pre}(Y_j) \cap Y_j)$. Hence, $\{X_j\}$, $\{X_j^r\}$, and $\{Y_j\}$ are nonincreasing, and $X_j^r \subseteq Y_j \subseteq X_j$ for all $j \in \mathbb{Z}_{\geq 0}$.

If $\underline{Y}^{\varepsilon} = \varnothing$, then there exists some integer $N > 0$ such that $Y_N = \varnothing$. It follows that $\mathcal{I}_r^{\infty}(\Omega) = \bigcap_{j=1}^{N} I_r^j(\Omega) = \varnothing$. Hence, 1 is proved.

If $\underline{Y}^{\varepsilon} \neq \varnothing$, then there exists an integer $J > 0$ such that $\underline{Y}^{\varepsilon} = Y_J = Y_{J+1} \supseteq X_J \supseteq (\bigcap_{j=1}^{\infty} I_r(\Omega)) = \mathcal{I}_r^{\infty}(\Omega)$. Since $Y_J = Y_{J+1} \subseteq (\mathrm{Pre}(Y_J) \cap Y_J)$, we have $Y_J \subseteq \mathrm{Pre}(Y_J)$, i.e., Y_J is controlled invariant by definition. Hence, $Y_J \subseteq \mathcal{I}^{\infty}(\Omega)$. ■

Theorem 1 provides a criterion for choosing the precision control parameter ε if the robust invariance margin is known *a priori*. For switched linear systems, one can refer to quadratic Lyapunov functions for estimations of their robust invariance margins [2]. In practice, it is not required to know this margin to use Algorithm 2. One can choose a sufficiently small ε according to the available computational resources. Algorithm 2 can also be used to estimate the robust invariance margin by starting with a large ε and iteratively reducing it until the algorithm achieves a nonempty result.

Furthermore, Theorem 1 suggests that robustly controlled invariance is a sufficient condition for invariant inner approximation of the maximal controlled invariant set to be finitely determined. Consider a discrete-time system $x(t+1) = A_\theta x(t)$, where

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Every state moves on a circle centered at the origin. Algorithm 2 returns an empty set, since the system is not robustly invariant. In fact, any interval-based approximation of invariant sets for this example will fail to be invariant.

Considering Proposition 4, if Ω is not controlled invariant itself, the exact $\mathcal{I}^{\infty}(\Omega)$ is nontrivial to obtain, no matter how small the precision control parameter ε is chosen. But if Ω contains a robustly controlled invariant set, Algorithm 2 is guaranteed to return a controlled invariant set by choosing ε sufficiently small. Theorem 1 also justifies the use of interval approximation of Ω in both Algorithms 1 and 2, since Ω and its interval approximation can be close enough such that their maximal robustly controlled invariant sets are the same.

IV. Extraction of Invariance Controller

In this section, we show that, if admissible switching modes are recorded while performing Algorithm 2, an invariance controller can be extracted once the algorithm stops.

Definition 4: Given a set $\Omega \subseteq \mathbb{R}^n$, a finite collection of sets $\mathcal{P} = \{P_1, P_2, \ldots, P_N\}$ is said to be a *partition* of Ω if 1) $P_i \subseteq \Omega$; 2) $\mathrm{int}(P_i) \cap \mathrm{int}(P_j) = \varnothing$; 3) $\Omega \subseteq \bigcup_{i=1}^{N} P_i$, for all $i \in \{1, \ldots, N\}$.

The following conclusion follows immediately from Theorem 1, which constructs a partition-based invariance controller.

Corollary 1: Let the assumptions in Theorem 1 hold. If Ω has a r-robustly ($r > 0$) controlled invariant set, then there exists a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_N\}$ of Ω and an invariance controller $c : \Omega \to 2^{\mathcal{M}}$ with

$$c(x) = \bigcup_{i \in N} \psi_{P_i}(x), \quad x \in \Omega. \tag{6}$$

The map ψ_{P_i} is given by

$$\psi_{P_i}(x) = \begin{cases} \varnothing & \text{if } x \notin P_i \\ \{p_{i_k}\} & \text{if } x \in P_i \end{cases}$$

where $p_{i_k} \in \mathcal{M}$ for $i \in \{1, \ldots, N\}$, $k \in \{1, \ldots, |\mathcal{M}|\}$.

Proof: By Assumption 1 and Theorem 1, for any $r > 0$ there exists $0 < \varepsilon \leq r/\rho_1$ such that Algorithm 2 returns a controlled invariant set that are represented by union of intervals, denoted by $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_N\}$ with admissible switching modes stored in a corresponding list $\mathcal{C} = \{C_1, C_2, \ldots, C_N\}$. Then, the controller in the form of (6) with $\mathcal{P} = \mathcal{Y}$ and

$$\psi_{Y_i} = \begin{cases} \varnothing & \text{if } x \notin Y_i \\ C_i & \text{if } x \in Y_i \end{cases}$$

renders the closed-loop system invariant with respect to Ω. ■

This partition-based controller resembles the ones generated by abstraction-based methods. The proposed method adaptively partitions the target area with respect to system dynamics. With the same minimum grid size, our method induces fewer grid points than abstraction-based methods with uniform grids, which demonstrates a lower complexity in iterative computation. Besides, construction of abstractions and control synthesis are separated in abstraction-based methods. Whether an abstraction needs to be refined is determined by the control synthesis results. A refinement scheme on top of these two stages usually incurs repeated computation in each stage [8], [21] without termination guarantee. In this aspect, the proposed approach provides an integrated direct control synthesis procedure, which is guaranteed to terminate under a robust invariance condition. To apply our control synthesis algorithm, only the parameter ε of Algorithm 2 has to be set to control the size of partitions, which is relatively simple.

V. Examples

In this section, we present two examples and compare the performance of the proposed method with abstraction-based methods in terms of computational time and abstraction size.

A. Boost DC-DC Converter

Consider a typical boost dc–dc converter [9] with two switching modes and linear affine dynamics $\dot{x} = A_p x + b$, where $p = 1, 2$, and

$$b = \begin{bmatrix} \frac{v_s}{v_l} & 0 \end{bmatrix}, A_1 = \begin{bmatrix} -\frac{r_l}{x_l} & 0 \\ 0 & -\frac{1}{x_c(r_c+r_0)} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -\frac{1}{x_l}\left(r_l + \frac{r_0 r_c}{r_0+r_c}\right) & -\frac{r_0}{x_l(r_0+r_c)} \\ \frac{r_0}{x_c(r_0+r_c)} & -\frac{1}{x_c(r_0+r_c)} \end{bmatrix}.$$

In our simulation, $x_c = 70$ p.u.(per unit), $r_c = 0.005$ p.u., $x_l = 3$ p.u., $r_l = 0.05$ p.u., $r_0 = 1$ p.u., and $v_s = 1$ p.u. With a sampling time $\tau_s = 0.5$ s, we obtain the discrete-time model $x(t+1) = e^{A_p \tau_s} x(t) + \int_0^{\tau_s} e^{A_p(\tau_s - s)} b \, \mathrm{d}s$.

The invariance specification is given by $\Omega = [1.15, 1.55] \times [1.09, 1.17]$. We use the natural inclusion function with $\varepsilon = 0.001$ in Algorithm 2 and extract the corresponding invariance controller.

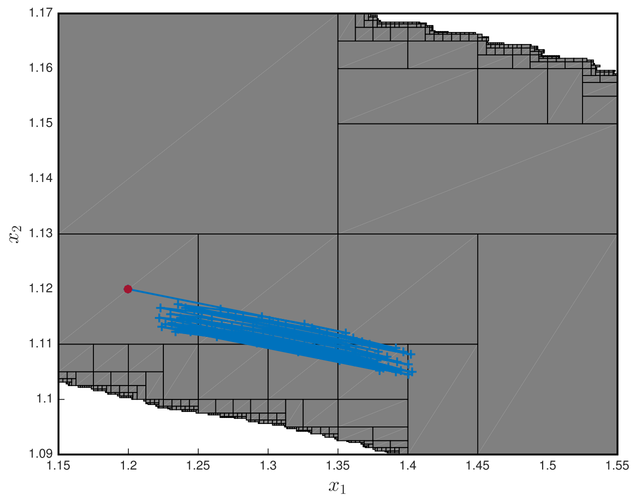

Fig. 1. Closed-loop phase portrait.

TABLE I
COMPARISON OF RUN TIMES

	CPU [GHz]	t_{abs}[s]	t_{syn}[s]
Pessoa	i7 3.5	478.7	65.2
SCOTS	i7 3.5	18.1	75.4
CoSyMA	N/A	N/A	8.32
intvl	i5 2.4	0	0.84

Closed-loop simulation is performed by using the control policy that keeps the switching mode unchanged unless the state is going to leave Ω. As shown in Fig. 1, the state evolution of the controlled system with the initial condition $x_0 = [1.2, 1.12]$ is confined to the controlled invariant set (shaded) of Ω as required.

We compare the run-time of our algorithm, coded in c++, with abstraction-based methods reported in [27] in Table I, where "t_{abs}" and "t_{syn}" denote the time spent on computing abstractions and control synthesis, respectively. In terms of efficiency, our algorithm outperforms other existing methods.

B. Inverted Pendulum on Cart

We aim to control the angle of an inverted pendulum on a cart modeled by the continuous-time ordinary differential equations (ODEs)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{mgl}{J_t}\sin x_1 - \frac{b}{J_t}x_2 + \frac{l}{J_t}\cos x_1 u$$

where x_1 is the angle of the pendulum to the upper vertical line φ (rad), x_2 is the angle change rate $\dot{\varphi}$ (rad/s), $m = 0.2$ kg, $g = 9.8\,\mathrm{m/s^2}$, $l = 0.3$ m, $J = 0.006$ kg $\cdot\mathrm{m}^2$, and $b = 0.1$ N/m/s.

We apply our method and abstraction-based methods to the corresponding sample-and-hold system with the sampling time $\tau_s = 0.01$ s. This system is neither globally asymptotically stable nor incrementally asymptotically stable. Hence, bisimilar symbolic models do not apply. For comparison with abstraction-based methods, we choose a local growth bound[1]

$$\beta(\eta, u) = e^{L(u)\tau_s}\eta, L(u) = \begin{bmatrix} 0 & 1 \\ \sqrt{24.5^2 + 12.5^2 u^2} & -4.17 \end{bmatrix}$$

[1] See [23] for the definition and the construction.

TABLE II
TWO INVARIANCE CONTROL SPECIFICATIONS

Case	Ω (Target Area)	U (Control Inputs)
# 1	$[-0.05, 0.05] \times [-0.01, 0.01]$	$[-0.1, 0.1]$, $\mu_1 = 0.02$
# 2	$[0.10, 0.17] \times [-0.01, 0.01]$	$[-0.4, -0.1]$, $\mu_2 = 0.05$

TABLE III
COMPARISON FOR CASE #1

	N_q	N_{trans}	W/Ω	Time(s)
abst ($\eta = 0.001$)	1881	93 597	87.45%	1959.24
intvl ($\varepsilon = 0.001$)	78	1774	94.90%	58.61
abst ($\eta = 0.004$)	100	5424	65.60%	31.87
intvl ($\varepsilon = 0.004$)	26	696	86.91%	15.47

TABLE IV
COMPARISON FOR CASE #2

	N_q	N_{trans}	W/Ω	Time(s)
abst ($\eta = 0.001$)	1131	41 457	92.86%	1543.80
intvl ($\varepsilon = 0.001$)	126	1626	99.60%	10.62
abst ($\eta = 0.004$)	2338	–	–	–
intvl ($\varepsilon = 0.004$)	44	657	98.24%	4.07

where $\eta = [\eta_1, \eta_2]$ is the grid width. This growth bound can also serve as the inclusion function in our scheme. Other methods to obtain the inclusion function for time-flow maps include the interval-solution of ODEs [20].

We consider sampled control inputs (μ denotes the sampling grid size) in two settings as shown in Table II. Comparison results obtained using interval toolbox [11] are given in Tables III and IV. We refer to the abstraction-based method by "abst", and our method by "intvl". Denote by N_q and N_{trans} the number of abstract states and transitions, respectively. The ratio between the volumes of the controlled invariant set obtained and Ω is denoted by W/Ω.

As observed in Table IV, abstraction-based methods produce more transitions and are easily affected by the choice of the grid size (e.g., no controller can be found for $\eta = 0.004$). This is because the growth bound gives a conservative estimation of state trajectories, and finer grids are needed to generate an invariance controller. Such effect of the conservative growth bound is mitigated by the proposed adaptive partition algorithm, since the grid points are automatically refined.

VI. CONCLUSION

We have presented an interval analysis approach to the invariance control problem for discrete-time switched nonlinear systems. We have introduced a robustly controlled invariance condition for the finite determination of invariant inner approximations. This condition also implies the existence of a partition-based invariance controller, which can be extracted from invariant inner approximations of the maximal controlled invariant sets. Experimental studies showed that this approach is effective and efficient for invariance control of switched nonlinear systems.

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