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Class-$\mathcal{KL}$ estimates and input-to-state stability analysis of impulsive switched systems☆

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ABSTRACT

In this paper, we investigate input-to-state stability of impulsive switched systems. The goal is to bridge two apparently different, but both useful, stability notions, input-to-state stability and stability in terms of two measures, in the hybrid systems setting. Based on two class-$\mathcal{KL}$ function estimates and a comparison theorem for impulsive differential equations, two sets of sufficient Lyapunov-type conditions for input-to-state stability in terms of two measures are obtained for impulsive switched systems. These conditions exploit some nonlinear integral constraints in terms of generalized dwell-time conditions to balance the continuous dynamics and impulsive dynamics so that input-to-state stability is achieved, despite possible instability of individual continuous subsystems or destabilizing impulsive effects. An illustrative example is presented, together with numerical simulations, to demonstrate the main results.

1. Introduction

Hybrid systems have attracted a lot of attention in recent years due to their numerous applications in various fields of sciences and engineering. Hybrid systems are dynamical systems exhibiting both continuous and discrete dynamic behavior. The interaction of continuous- and discrete-time dynamics in a hybrid system can often lead to rich dynamical behavior and phenomena that are not encountered in purely continuous- or discrete-time systems and hence brings difficulties and challenges to the studies of hybrid systems, such as their stability analysis and control design (see, e.g., [1–4] and references therein). In this paper, we study stability of hybrid systems under the notions of input-to-state stability [5] and stability in terms of two measures [6].

The notion of *input-to-state stability* (ISS), originally introduced in [5], has proved very useful in characterizing the effects of external inputs to a control system. The ISS notion has subsequently been extended to discrete-time systems in [7] and to switched systems in [8,9]. Input-to-state stability properties for hybrid systems are investigated in [10,11], where the hybrid systems are defined on hybrid time domains. More recently, Chen and Zheng [12] and Hespanha et al. [13] have studied Lyapunov conditions for input-to-state stability of impulsive nonlinear systems with and without time-delays, and Liu et al. [14] have investigated input-to-state properties of impulsive and switching hybrid systems with time-delay.

Stability in terms of two measures, on the other hand, provides a unified notion for Lyapunov stability, partial stability, orbital stability, and stability of an invariant set of nonlinear systems [6,15], which would otherwise be treated separately. This notion has recently been adopted in the framework of switched systems [14,16], but not yet exploited for input-to-state stability.

The goal of this paper is to bridge the above two notions of stability, namely input-to-state stability and stability in terms of two measures, in the hybrid systems setting. The clear benefits of doing so include characterization of robustness of hybrid systems affected by noise or disturbances, not only in the Lyapunov stability sense, but also in various other stability notions, such as partial stability, orbital stability, and stability with respect to an invariant set. One additional advantage is that we can choose one of the stability measures to be the output function, the results can also cover input-to-output stability [17].

We focus on two widely studied types of hybrid systems, namely impulsive systems [18] and switched systems [2]. Intuitively, impulsive systems are systems with state jumps and can be used to model real world processes that undergo abrupt changes (impulses) in the state at discrete times [18]. Impulsive dynamical systems can be naturally viewed as a class of hybrid systems

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[19,20]. Switched systems, on the other hand, are systems with dynamic switching and can be used to model real systems whose dynamics are chosen from a family of possible choices according to a switching signal [2]. In order to study these two types of hybrid systems in the same framework, we formulate them as impulsive switched systems by introducing integrated signals called impulsive and switching signals [14]. Input-to-state stability properties in terms of two measures are investigated not only under a particular signal, but rather under various classes of signals as in the stability analysis for switched systems [21].

Besides the conceptual unification of two stability notions mentioned above, the results in this paper also include a key improvement, compared with previous results on input-to-state stability of impulsive or switched hybrid systems. Namely, we do not assume that the continuous dynamics of the system either grow exponentially or decay exponentially, which is in contrast with the results in the literature [9,12,13,22]. Note the presence of the nonlinear function c_i in both of the main theorems in the current paper. These actually give more natural conditions similar to the Lyapunov characterizations of input/output stability notions of continuous nonlinear systems [23].

The rest of this paper is organized as follows. Basic notation and definitions are given in Section 2, where we formulate an impulsive and switching hybrid system with external input. Particularly, the notion of input-to-state stability in terms of two measures is presented. Section 3 provides two lemmas on class-$\mathcal{KL}$ estimates for impulsive switched systems. The main results of this paper, presented in Section 4, give Lyapunov-type sufficient conditions for input-to-state stability of impulsive switched systems in terms of two measures. An example is presented in Section 5 to illustrate the main results. The paper is concluded by Section 6, where the main contributions of this paper are highlighted.

2. Notation and definitions

Let $\mathbb{Z}^+$ denote the set of nonnegative integers, $\mathbb{R}^+$ the set of nonnegative real numbers, and $\mathbb{R}^n$ the n-dimensional real Euclidean space. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x. Let $C[M; N]$ denote the set of all continuous functions from $M \subset \mathbb{R}^m$ to $N \subset \mathbb{R}^n$.

Let $\mathcal{I}_c$ and $\mathcal{I}_d$ be two index sets. Consider the following impulsive switched system

$$\begin{cases} x'(t) = f_{i_k}(t, x(t), u(t)), \quad t \in (t_k, t_{k+1}),\ k \in \mathbb{Z}^+, \quad \text{(a)} \\ \Delta x(t) = I_{j_k}(t, x(t^-), u(t)), \quad t = t_k, k \in \mathbb{Z}^+ \setminus \{0\}, \quad \text{(b)} \\ x(t_0) = x_0. \quad \text{(c)} \end{cases} \tag{2.1}$$

where $i_k \in \mathcal{I}_c, j_k \in \mathcal{I}_d, x(t) \in \mathbb{R}^n$ is the system state, $u : \mathbb{R}^+ \mapsto \mathbb{R}^m$ the system input function (measurable and locally essentially bounded), $\left\{t_k : k \in \mathbb{Z}^+\right\} \subset \mathbb{R}^+$ a strictly increasing sequence such that $t_k \to \infty$ as $k \to \infty$, and $x(t^-) = \lim_{s\to t^-} x(s)$. For each $i \in \mathcal{I}_c$, the function $f_i \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ and is locally Lipschitzian in (x, u). For each $j \in \mathcal{I}_d, I_j : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$. Each solution x is continuous at each $t \neq t_k$ for $t \geq t_0$. We assume that, for each $i \in \mathcal{I}_c$ and $j \in \mathcal{I}_d, f_i(t, 0, 0) \equiv I_j(t, 0, 0) \equiv 0$ so that system (2.1), without input, admits a trivial solution. Note that in (2.1), the sequence of triples $\{(t_k, i_k, j_k)\}$ together imposes the following:

(i) a sequence of indices i_k to *switch* the right-hand side of (2.1)(a) among the family $\{f_i : i \in \mathcal{I}_c\}$;
(ii) a sequence of indices j_k to select the *impulse* functions I_{j_k} from the family $\{I_j : j \in \mathcal{I}_d\}$ to reset the system state according to the difference Eq. (2.1)(b); and
(iii) a sequence of discrete times t_k, called the *impulse and switching times* (except the initial time t_0[1]), to determine when the switching and impulse occur.

Besides the family of functions $\{f_i : i \in \mathcal{I}_c\}$ and $\{I_i : i \in \mathcal{I}_d\}$, which govern the continuous dynamics and the discrete dynamics of system (2.1), respectively, it is expected that properties of solutions to system (2.1) (e.g. input-to-state stability properties to be investigated in this paper) can also be highly affected by the sequence of triples $\{(t_k, i_k, j_k)\}$, which we call an *impulsive and switching signal*. Hence, it is of interest to characterize properties of solutions that are uniform over a certain class of impulsive and switching signals. We may use $\mathcal{S}$ to denote a certain class of such signals.

A function $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to be of *class* $\mathcal{K}$ and we write $\alpha \in \mathcal{K}$, if α is continuous, strictly increasing, and $\alpha(0) = 0$. If $\alpha \in \mathcal{K}$ also satisfies $\alpha(t) \to \infty$ as $t \to \infty$, we say that α is of *class* $\mathcal{K}_\infty$ and write $\alpha \in \mathcal{K}_\infty$. A function $\beta : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to be of *class* $\mathcal{KL}$ and we write $\beta \in \mathcal{KL}$, if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \in \mathbb{R}^+$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \in \mathbb{R}^+$. Let $\Gamma := \{h \in C[\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+] : \inf_{(t,x)} h(t, x) = 0\}$ and fix some $h_0,\ h \in \Gamma$. To unify different notions of stability, the stability analysis is performed in terms of two measures (h_0 and h) as shown in the following two definitions.

Definition 2.1. System (2.1) is said to be *uniformly* (h_0, h)- *input-to-state stable (ISS)* over a certain class of signals $\mathcal{S}$, if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, independent of the choice of impulsive and switching signals $\{(t_k, i_k, j_k)\}$ in $\mathcal{S}$, such that, for each $x_0 \in \mathbb{R}^n$ and input function u, the solution x of (2.1) exists globally and satisfies

$$h(t, x(t)) \leq \beta(h_0(t_0, x_0), t - t_0) + \gamma(\sup_{t_0\leq s\leq t} |u(s)|). \tag{2.2}$$

Remark 2.1. The above definition generalizes the classical notion of ISS for ordinary system given in [5]. The (h_0, h)-stability notions are considered here in the spirit of the work by Lakshmikantham and Liu [6] to unify different notions of stability, which would otherwise be treated separately. It is worth noting that this notion has been adopted in the framework of switched systems by Chatterjee and Liberzon [16] and Liu et al. [22], but not yet exploited for input-to-state stability.

Remark 2.2. A few choices of the two measures (h_0, h) given below will demonstrate the generality of Definition 2.1. It is easy to see that Definition 2.1 gives

(1) the classical stability of trivial solution $x(t) \equiv 0$, if $h(t, x) = h_0(t, x) = |x|$ and $u(t) \equiv 0$;
(2) the well known input-to-state stability, if $h(t, x) = h_0(t, x) = |x|$;
(3) the input-to-state partial stability, if $h(t, x) = |(x_1, \ldots, x_s)|$, where $s < n$, and $h_0(t, x) = |x|$;
(4) the input-to-state stability of a prescribed motion $y(t)$, if $h(t, x) = h_0(t, x) = |x - y(t)|$;
(5) the input-to-state stability of the invariant set $A \subset \mathbb{R}^n$, if $h(t, x) = h_0(t, x) = d(x, A)$, where $d(x, A)$ is the distance of x from the invariant set A;
(6) the input-to-state orbital stability of a periodic solution, if $h(t, x) = h_0(t, x) = d(x, C)$, where C is the closed orbit in the phase space.

[1] While a switching mode is assigned by i_0 at $t = t_0$, we do not consider a solution to instantly undergo an impulse at the initial time t_0.

To investigate the ISS properties of system (2.1), which has different modes of the continuous dynamics given by $\{f_i : i \in \mathcal{I}_c\}$, we shall choose accordingly a family of multiple Lyapunov functions $\{V_i : i \in \mathcal{I}_c\}$, where each $V_i \in C(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ and is locally Lipschitzian in its second variable. We introduce the upper right-hand derivative of V_i with respect to ith mode of system (2.1), for each $i \in \mathcal{I}_c$, at $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$ by

$$D^+V_i(t,\phi) = \limsup_{h\to 0^+} \frac{1}{h}\left[V_i(t+h, x+hf_i(t,x,u)) - V_i(t,x)\right],$$

where $u \in \mathbb{R}^m$. Moreover, for a function $m : \mathbb{R} \mapsto \mathbb{R}$, $D^+m(t)$, the upper right-hand derivative of $m(t)$, is defined by $D^+m(t) = \limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)]$.

Let $\{(t_k, i_k, j_k)\}$ be an impulsive and switching signal and x be a solution to system (2.1) on $[t_k, t_{k+1})$. Define $m(t) = V_{i_k}(t, x(t))$, for $t \in [t_k, t_{k+1})$. The above definitions for upper right-hand derivatives are connected by the relation

$$D^+m(t) = D^+V_{i_k}(t, x(t)), \quad t \in (t_k, t_{k+1}),$$

because $\{V_i\}$ are locally Lipschitzian in x.

Finally, we introduce two classes of impulsive and switching signals as in [14]. The goal of this paper is to establish uniform input-to-state stability of system (2.1) in terms of two measures over these particular classes of signals. The two classes of signals generalize the well-known dwell-time conditions to *dwell-time conditions with respect to specific switching modes*. We say that an impulsive and switching signal $\{(t_k, i_k, j_k)\}$ belongs to $\mathcal{S}^i_{\inf}(\tau)$, for some $\tau > 0$ and $i \in \mathcal{I}_c$, if it satisfies $\inf\{t_{k+1} - t_k : k \in \mathbb{Z}^+, i_k = i\} \geq \tau$, where t_0 is the initial time. If $\{(t_k, i_k, j_k)\}$ satisfies $\sup\left\{t_{k+1} - t_k : k \in \mathbb{Z}^+, i_k = i\right\} \leq \tau$, it is said to belong to $\mathcal{S}^i_{\sup}(\tau)$. For fixed $\tau > 0$, the well-known dwell-time signals are recovered by $\mathcal{S}_{\inf}(\tau) = \bigcap_{i\in\mathcal{I}_c} \mathcal{S}^i_{\inf}(\tau)$ and $\mathcal{S}_{\sup}(\tau) = \bigcap_{i\in\mathcal{I}_c} \mathcal{S}^i_{\sup}(\tau)$. In other words, an impulsive and switching signal $\{(t_k, i_k, j_k)\}$ belongs to $\mathcal{S}^i_{\inf}(\tau)$ or $\mathcal{S}^i_{\sup}(\tau)$, if it assumes a dwell-time lower bound or upper bound τ, respectively, with respect to the ith mode.

3. Class-$\mathcal{KL}$ estimates

In this section, we establish two general class-$\mathcal{KL}$ estimates for solutions of scalar impulsive switched systems.

Lemma 3.1. *Consider the scalar impulsive switched system*

$$\begin{cases} y'(t) = p_{i_k}(t)\alpha_{i_k}(y(t)), \quad t \in (t_k, t_{k+1}),\ k \in \mathbb{Z}^+, \quad (a) \\ y(t_k) = g(y(t_k^-)), \quad t = t_k,\ k \in \mathbb{Z}^+ \setminus \{0\}, \quad (b) \\ y(t_0) = y_0, \quad (c) \end{cases} \tag{3.1}$$

where $i_k \in \mathcal{I}_c$, $y_0 \geq 0$, $g \in \mathcal{K}_\infty$, $p_i : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is locally integrable, and $\alpha_i \in \mathcal{K}$ is locally Lipschitz for each $i \in \mathcal{I}_c$. If

(i) $\tau_i := \sup\left\{t_k - t_{k-1} : k \in \mathbb{Z}^+,\ i_k = i\right\} < \infty$; *and*
(ii) $N_i := \inf_{q>0} \int_{g(q)}^{q} \frac{ds}{\alpha_i(s)} > M_i := \sup_{t\geq 0} \int_t^{t+\tau_i} p_i(s)ds$,

then the system has a unique solution $y(t)$ defined for all $t \geq t_0$ and there exists a class-$\mathcal{KL}$ function β such that

$$y(t) \leq \beta(y_0, t - t_0).$$

Lemma 3.2. *Consider the scalar impulsive switched system*

$$\begin{cases} y'(t) = -p_{i_k}(t)\alpha_{i_k}(y(t)), \quad t \in (t_k, t_{k+1}),\ k \in \mathbb{Z}^+, \quad (a) \\ y(t_k) = g(y(t_k^-)), \quad t = t_k,\ k \in \mathbb{Z}^+ \setminus \{0\}, \quad (b) \\ y(t_0) = y_0, \quad (c) \end{cases} \tag{3.2}$$

where $i_k \in \mathcal{I}_c$, $y_0 \geq 0$, $g \in \mathcal{K}_\infty$ with $g(s) > s$ for all $s > 0$, $p_i : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is locally integrable, and $\alpha_i \in \mathcal{K}$ is locally Lipschitz for each $i \in \mathcal{I}_c$. If

(i) $\tau_i := \inf\left\{t_k - t_{k-1} : k \in \mathbb{Z}^+,\ i_k = i\right\}$;
(ii) $N_i := \inf_{t\geq 0} \int_t^{t+\tau_i} p_i(s)ds > M_i := \sup_{q>0} \int_q^{g(q)} \frac{ds}{\alpha_i(s)}$,

then the system has a unique solution $y(t)$ defined for all $t \geq t_0$ and there exists a class-$\mathcal{KL}$ function β such that

$$y(t) \leq \beta(y_0, t - t_0).$$

Remark 3.1. Before proving these two lemmas, a few comments are in order.

(1) Lemmas 3.1 and 3.2, together with a comparison theorem for impulsive differential equation [18], will be used to derive input-to-state stability for impulsive switched systems in the next section.
(2) A key feature here is that the continuous dynamics of the system can be fully nonlinear. Note that the functions α_i are general nonlinear class $\mathcal{K}$ functions. This feature allows us to derive more general Lyapunov characterizations without restricting the continuous dynamics of a hybrid system either to be exponentially growing or exponentially decaying.
(3) Conditions (i) in the above lemmas give a dwell-time upper bound and a dwell-time lower bound on the impulsive and switching signals, respectively. Conditions (ii) specify balancing conditions on the continuous dynamics and impulsive dynamics to guarantee the existence of a $\mathcal{KL}$-estimate. Particularly, in Lemma 3.1, the continuous dynamics can be unstable, while the impulsive dynamics are stabilizing. Lemma 3.2 covers the opposite situation, i.e., the impulsive effects are destabilizing, while continuous dynamics are stabilizing. The balancing conditions are given in general integral forms.

Proof of Lemma 3.1. Note that $y(t) \geq 0$ for all $t \geq t_0$. We will show that

(i) Given any $\varepsilon > 0$, there is $\delta = \delta(\varepsilon)$ such that $y_0 \leq \delta$ implies $y(t) \leq \varepsilon$ for all $t \geq t_0$.
(ii) Given any $r > 0$ and $\eta > 0$, there exists $T = T(\eta, r) \geq 0$ (dependent on η and r, but independent of t_0) such that $y(t) < \eta$ for all $t \geq t_0 + T$.

Following a standard argument [24], a class-$\mathcal{KL}$ function β can be constructed based on parts (i) and (ii) above.

To show part (i), pick $\delta = \delta(\varepsilon) = g(\varepsilon)$ for any given $\varepsilon > 0$. The conditions on N_i shows that $0 < g(q) < q$ for all $q > 0$. We claim that if $y_0 \leq \delta$, then $y(t)$ is defined for all $t \geq t_0$ and $y(t) \leq \varepsilon$. Suppose $y(t; t_0, y_0)$ is continued to its maximal interval of existence $[t_0, \beta)$. If $\beta < \infty$, there must exist some $t \in (t_0, \beta)$ such that $y(t) > \varepsilon$. We shall show that $y(t) \leq \varepsilon$ for all $t \in [t_0, \beta)$, which in turn will imply that $\beta = \infty$ and therefore part (i) is proved.

Suppose for the sake of contradiction that $y(t) > \varepsilon$ for some $t \in [t_0, \beta)$. Let $t^* = \inf\{t \in [t_0, \beta) : y(t) > \varepsilon\}$. It is clear that $y(t) \leq \varepsilon$ for all $t \in [t_0, t^*)$. Moreover, either $y(t^*) = \varepsilon$ or $y(t^*) > \varepsilon$ and $t^* = t_k$ for some k. In either case, $y(t)$ is defined for all $t \in [t_0, t^*]$. Since $y_0 \leq \delta = g(\varepsilon) < \varepsilon$ and $y(t^*) \geq \varepsilon$, then $t^* > t_0$. Moreover, $y(t) < \varepsilon$ for $t \in [t_0, t^*)$. We claim that $t^* \neq t_k$ for any $k \geq 1$ and hence $y(t^*) = \varepsilon$. Actually if $t^* \neq t_k$ for some $k \geq 1$, then $0 < \varepsilon \leq y(t^*) = g(y(t^{*-})) < y(t^{*-}) \leq \varepsilon$, which is impossible.

Now suppose that $t^* \in (t_k, t_{k+1})$ for some k. Since $y(t) < \varepsilon$ for $t \in [t_0, t^*)$, we have $y(t_k) = g(y(t_k^-)) < g(\varepsilon)$. Integrating (3.1)(a) on $[t_k, t^*]$ gives

$$N_{i_k} \leq \int_{g(\varepsilon)}^{\varepsilon} \frac{ds}{c(s)} \leq \int_{y(t_k)}^{y(t^*)} \frac{ds}{\alpha_{i_k}(s)} \leq \int_{t_k}^{t^*} p_{i_k}(s)ds \leq \int_{t_k}^{t_k+\tau_{i_k}} p_{i_k}(s)ds \leq M_{i_k},$$

which clearly contradicts with condition (iv). This completes the proof for part (i).

We proceed to show part (ii). Let $r > 0$ and $\eta > 0$ be arbitrarily given. Following the proof of part (i), we can show that there exists some $\rho = \rho(r)$ such that $y(t) \le \rho$ for all $t \ge t_0$ if $y_0 \le r$. Actually we can choose $\rho = g^{-1}(r)$, which is always possible since $g \in \mathcal{K}_\infty$. Then repeating the proof of uniform stability shows $y(t) \le \rho$ for all $t \ge t_0$.

Let $\delta = \delta(\eta) = g(\eta)$. From part (i), $y_0 \le \delta$ implies that $y(t) \le \eta$ for all $t \ge t_0$, where t_0 can be arbitrary. Without loss of generality, we can assume $\eta < \rho$. Define

$$L_i = L_i(\gamma) = \sup\left\{\frac{1}{\alpha_i(s)} : g(\delta) \le s \le \rho\right\}.$$

It is clear that $0 < L_i < \infty$. For $\delta \le q \le \rho$, we have $g(\delta) \le g(q) < q \le \rho$ and so

$$N_i \le \int_{g(q)}^{q} \frac{ds}{\alpha_i(s)} \le L_i[q - g(q)],$$

from which we can obtain $g(q) \le q - N_i/L_i < q - d$, where $d = d(\eta, r) > 0$ is chosen so that $d < \inf_{i\in\mathcal{I}_c}(N_i - M_i)/L_i$. Let $\tau = \sup_{i\in\mathcal{I}_c} \tau_i$. Choose $T = \tau(N + 1)$, where N is the first integer such that $N > \rho/d$. Then T depends on τ, r, and η, but is independent of t_0. The proof is complete by proving the following claim.

Claim. *There exists some $t \in [t_0, t_0 + T]$ such that $|x(t)| \le \delta$.*

Note that the conclusion of part (ii) *would follow from this claim and part* (i) *.* □

Proof of the Claim. Suppose it is not true. Then $y(t) > \delta$ for all $t \in [t_0, t_0 + T]$. We prove by induction that, for $[t_k, t_{k+1}) \subset [t_0, t_0 + T]$, where $k \in \mathbb{Z}^+$, we have $y(t) \le A_k := \rho - kd$. By the choice of T, there are at least $N + 1$ such intervals, where $N > \rho/d$, which would eventually lead to $y(t) \le \rho - Nd < 0$ on $[t_N, t_{N+1})$, a clear contradiction.

On $[t_0, t_1)$, we have $m(t) \le A_0 = \rho$. Suppose that $0 < \delta < y(t) \le A_k \le \rho$ on $[t_k, t_{k+1})$, $k < N$. Assume for the sake of contradiction that there exists some $t \in [t_{k+1}, t_{k+2})$ such that $y(t) > A_k - d = A_{k+1}$. Let $t^* = \inf\{t \in [t_{k+1}, t_{k+2}) : y(t) > A_k - d\}$. Since $y(t_{k+1}) = g(y(t_{k+1}^-)) \le g(A_k) < A_k - d$, we have $t^* \in (t_{k+1}, t_{k+2})$ and $y(t^*) = A_k - d$. Integrating (3.1)(a) on $[t_{k+1}, t^*]$ gives

$$\int_{y(t_{k+1})}^{y(t^*)} \frac{ds}{\alpha_{i_{k+1}}(s)} \le \int_{t_{k+1}}^{t^*} p_{i_{k+1}}(s)ds \le \int_{t_{k+1}}^{t_{k+1}+\tau} p_{i_{k+1}}(s)ds \le M_{i_{k+1}},$$

and

$$\begin{aligned}\int_{y(t_{k+1})}^{y(t^*)} \frac{ds}{\alpha_{i_{k+1}}(s)} &\ge \int_{g(A_k)}^{A_k - d} \frac{ds}{\alpha_{i_{k+1}}(s)} \\ &= \int_{g(A_k)}^{A_k} \frac{ds}{\alpha_{i_{k+1}}(s)} - \int_{A_k-d}^{A_k} \frac{ds}{\alpha_{i_{k+1}}(s)} \\ &\ge N_{i_{k+1}} - dL_{i_{k+1}} > M_{i_{k+1}}.\end{aligned}$$

Again, we reach a contradiction. Therefore, the claim is proved and so are both parts (i) and (ii). The proof is complete. □

Proof of Lemma 3.2. We prove the same statements as parts (i) and (ii) in the proof of Lemma 3.1 and following a standard argument [24], a class-$\mathcal{KL}$ function β can be constructed based on parts (i) and (ii).

Note that $0 < M_i < N_i < \infty$ for all $i \in \mathcal{I}_c$. For part (i), we choose $\delta = \delta(\varepsilon) = g^{-1}(\varepsilon) < \varepsilon$. We show that if $y_0 \le \delta$, then $y(t)$ is defined for all $t \ge t_0$ and $y(t) \le \varepsilon$. Suppose $y(t; t_0, y_0)$ is continued to its maximal interval of existence $[t_0, \beta)$. If $\beta < \infty$, there must exist some $t \in (t_0, \beta)$ such that $y(t) > \varepsilon$. We shall show that $y(t) \le \varepsilon$ for all $t \in [t_0, \beta)$, which in turn will imply that $\beta = \infty$ and therefore part (i) is proved.

Suppose for the sake of contradiction that $y(t) > \varepsilon$ for some $t \in [t_0, \beta)$. Let $t^* = \inf\{t \in [t_0, \beta) : y(t) > \varepsilon\}$. It is clear that $y(t) \le \varepsilon$ for all $t \in [t_0, t^*)$. Moreover, either $y(t^*) = \varepsilon$ or $y(t^*) > \varepsilon$ and $t^* = t_k$ for some k. In either case, $y(t)$ is defined for all $t \in [t_0, t^*]$. Since $y_0 \le \delta = g^{-1}(\varepsilon) < \varepsilon$ and $y(t^*) \ge \varepsilon$, then $t^* > t_0$. Moreover, $y(t) < \varepsilon$ for $t \in [t_0, t^*)$. Suppose that $t^* \in [t_k, t_{k+1})$. Eq. (3.2)(a) implies that $y(t)$ is decreasing on $[t_k, t^*]$. Therefore $t^* = t_k$. We then show that $y(t^{*-}) \le g^{-1}(\varepsilon)$. Suppose this is not true. Then we have $g^{-1}(\varepsilon) < y(t) \le \varepsilon$ on $[t_{k-1}, t_k)$. Integrating (3.2)(a) on $[t_{k-1}, t_k)$ gives

$$\begin{aligned}\int_{y(t_k^-)}^{y(t_{k-1})} \frac{ds}{\alpha_{i_{k-1}}(s)} &\ge \int_{t_{k-1}}^{t_k} p_{i_{k-1}}(s)ds \\ &\ge \int_{t_{k-1}}^{t_{k-1}+\tau_{i_{k-1}}} p_{i_{k-1}}(s)ds \ge N_{i_{k-1}},\end{aligned}$$

and

$$\int_{y(t_k^-)}^{y(t_{k-1})} \frac{ds}{\alpha_{i_{k-1}}(s)} \le \int_{g^{-1}(\varepsilon)}^{g(g^{-1}(\varepsilon))} \frac{ds}{\alpha_{i_{k-1}}(s)} \le M_{i_{k-1}} < N_{i_{k-1}},$$

which gives a contradiction in view of condition (ii). Therefore, $y(t^{*-}) = y(t_k^-) \le g^{-1}(\varepsilon)$ and $y(t^*) = g(y(t^{*-})) \le \varepsilon$. But $y(t^*) \ge \varepsilon$. We must have $y^* = \varepsilon$. However, $y(t)$ is decreasing on $[t^*, t_{k+1})$ and therefore $y(t) \le \varepsilon$ on $[t^*, t_{k+1})$, which contradicts the definition of t^*. This completes the proof for part (i).

We proceed to show part (ii). Let $r > 0$ and $\eta > 0$ be arbitrarily given. Following the proof of part (i), we can show that there exists some $\rho = \rho(r)$ such that $y(t) \le \rho$ for all $t \ge t_0$ if $y_0 \le r$. Actually we can choose $\rho = g(r)$, which is always possible since $g \in \mathcal{K}_\infty$. Then repeating the proof of uniform stability shows $y(t) \le \rho$ for all $t \ge t_0$.

Let $\delta = \delta(\eta) = g^{-1}(\eta)$. From part (i), $y_0 \le \delta$ implies that $y(t) \le \eta$ for all $t \ge t_0$, where t_0 can be arbitrary. Without loss of generality, we can assume $\eta < \rho$. Define

$$L_i = L_i(\eta) = \sup\left\{\frac{1}{\alpha_i(s)} : \delta \le s \le \rho\right\}.$$

It is clear that $0 < L_i < \infty$. Let $d = \inf_{i\in\mathcal{I}_c}(N_i - M_i)/L_i$, $\tau = \sup_{i\in\mathcal{I}_c} \tau_i$, and N be an integer such that $N > \rho/d$. Choose $T = (N + 1)^2\tau$. The proof is again complete by proving the following claim.

Claim. *There exists some $t \in [t_0, t_0 + T]$ such that $|x(t)| \le \delta$.*

Note that the conclusion of part (ii) *would follow from this claim and part* (i) *.* □

Proof of the Claim. Suppose it is not true. Then $\delta < y(t) \le \rho$ for all $t \in [t_0, t_0 + T]$. By the choice of T, there are two possible cases: (I) there are at least N impulse times t_k, $k \ge 1$, in $[t_0, t_0 + T]$; (II) there are at least one subinterval of $[t_0, t_0 + T]$ with length greater than $N\tau$ which does not contain any impulse time. In either case, we will derive a contradiction.

Consider case (I) first. We show that by induction that $y(t_k^-) \le g^{-1}(\rho) - kd$, where $1 \le k \le N$, Eventually this would lead to $y(t_N^-) \le g^{-1}(\rho) - Nd < 0$, which is not impossible. We first show that $y(t_1^-) \le g^{-1}(\rho) - d$. Integrating (3.2)(a) on $[t_0, t_1)$ gives

$$\begin{aligned}\int_{y(t_1^-)}^{y_0} \frac{ds}{\alpha_{i_0}(s)} &= \int_{g^{-1}(\rho)}^{\rho} \frac{ds}{\alpha_{i_0}(s)} + \int_{y(t_1^-)}^{g^{-1}(\rho)} \frac{ds}{\alpha_{i_0}(s)} \\ &\le M_{i_0} + \int_{y(t_1^-)}^{g^{-1}(\rho)} \frac{ds}{\alpha_{i_0}(s)},\end{aligned}$$

and

$$\int_{y(t_1^-)}^{y_0} \frac{ds}{\alpha_{i_0}(s)} \ge \int_{t_0}^{t_1} p_{i_0}(s)ds \ge \int_{t_0}^{t_0+\tau_{i_0}} p_{i_0}(s)ds \ge N_{i_0},$$

which gives

$$M_{i_0} + \int_{y(t_1^-)}^{g^{-1}(\rho)} \frac{ds}{\alpha_{i_0}(s)} \geq N_{i_0}. \tag{3.3}$$

Condition (ii) implies that

$$\int_{y(t_1^-)}^{g^{-1}(\rho)} \frac{ds}{\alpha_{i_0}(s)} > 0,$$

which in turn implies $\rho > g^{-1}(\rho) > y(t_1^-) \geq \delta$. Therefore, by the definition of L_{i_0},

$$\int_{y(t_1^-)}^{g^{-1}(\rho)} \frac{ds}{\alpha_{i_0}(s)} \leq [g^{-1}(\rho) - y(t_1^-)]L_{i_0}.$$

Combining this and Eq. (3.3) gives

$$[g^{-1}(\rho) - y(t_1^-)]L_{i_0} + M_{i_0} \geq N_{i_0},$$

which eventually gives

$$y(t_1^-) \leq g^{-1}(\rho) - \frac{N_{i_0} - M_{i_0}}{L_{i_0}} \leq g^{-1}(\rho) - d.$$

Now we have $y(t_1^-) \leq g^{-1}(\rho) - d$ and $y(t_1) = g(y(t_1^-)) \leq g(g^{-1}(\rho) - d)$. Repeating the same argument on $[t_1, t_2)$, we can show $y(t_2^-) \leq g^{-1}(y(t_1)) - d \leq g^{-1}(\rho) - 2d$. By induction, this shows $y(t_k^-) \leq g^{-1}(\rho) - kd$, where $1 \leq k \leq N$, and $y(t_N^-) \leq g^{-1}(\rho) - Nd < 0$, a contradiction.

Now consider case (II). Suppose $[t_k, t^*)$ is an interval in $[t_0, t_0 + T]$ without any impulse, where t^* can be $t_0 + T$. Since $t^* - t_k \geq N\tau$, we can break this interval into N intervals, each with a length greater than τ. Label these intervals as $[s_0, s_1]$, $[s_1, s_2]$, ..., $[s_{N-1}, s_N]$, where $s_0 = t_k$ and $s_N = t^*$. Repeating the same argument as in case (I), we can show $y(s_1^-) \leq g^{-1}(y(s_0)) - d$. Since $y(t)$ is continuous at s_1, we have $y(s_1) = y(s_1^-) \leq g^{-1}(y(s_0)) - d < y(s_0) - d$. By induction, we can show $y(s_N^-) \leq y(s_0) - Nd \leq \rho - Nd < 0$, again a contradiction. The proof is complete. □

4. Input-to-state stability

In this section, we establish some sufficient conditions for input-to-state stability of impulsive switched systems, as applications of the class-$\mathcal{KL}$ estimates we have obtained in the previous section. As mentioned in the introduction, to unify different notions of stability, the input-to-state stability analysis is performed in terms of two measures (h_0 and h).

Our first result is concerned with (h_0, h)-ISS properties of system (2.1), in the case when all the subsystems governing the continuous dynamics of (2.1) can be unstable and the impulses, on the other hand, are stabilizing. Intuitively, the conditions in the following theorem consist of four aspects (corresponding to each of the conditions): (i) the Lyapunov–Krasovskii functionals satisfy certain positive definite and decrescent conditions; (ii) there exist some positive estimates of the upper right-hand derivatives of the functionals with respect to each unstable mode of (2.1); (iii) the jumps induced by the stabilizing impulses satisfy certain diminishing conditions; (iv) the estimates on the derivatives and the growth by jumps satisfy certain integral form balancing conditions in terms of the dwell-time upper bounds.

Theorem 4.1. *Suppose that there exist a family of functions $\{V_i : i \in \mathcal{I}_c\}$ in $C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$ that are locally Lipschitzian in the second variable, functions a, b, g, ρ, and χ of class $\mathcal{K}_\infty$, $p_i : \mathbb{R}^+ \mapsto \mathbb{R}^+$ locally integrable and $\alpha_i \in \mathcal{K}$ locally Lipschitz for each $i \in \mathcal{I}_c$, and positive constants τ_i ($i \in \mathcal{I}_c$) such that, for all $i, \hat{\imath} \in \mathcal{I}_c, j \in \mathcal{I}_d$, and $(t, x, u) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m$,*

(i) $a(h(t, x)) \leq V_i(t, x) \leq b(h_0(t, x))$;
(ii) $D^+V_i(t, x) \leq p_i(t)c_i(V_i(t, x))$ if $V_i(t, x) \geq \chi(|u|)$;
(iii) $V_i(t, x + I_j(t, x, u)) \leq g(V_{\hat{\imath}}(t^-, x))$ if $V_{\hat{\imath}}(t^-, x) \geq \chi(|u|)$, and $V_i(t, x + I_j(t, x, u)) < \rho(|u|)$ if $V_{\hat{\imath}}(t^-, x) < \chi(|u|)$; and
(iv) $M_i := \sup_{t\geq 0} \int_t^{t+\tau_i} p_i(s)ds < N_i := \inf_{q>0} \int_{g(q)}^q \frac{ds}{c_i(s)}$.

Then system (2.1) is uniformly (h_0, h)-ISS on $\bigcap_{i\in\mathcal{I}_c} \mathcal{S}^i_{\sup}(\tau_i)$. In particular, system (2.1) is uniformly (h_0, h)-ISS on $\mathcal{S}_{\sup}(\tau)$, where $\tau = \inf_{i\in\mathcal{I}_c} \tau_i$.

Proof. Let $\{(t_k, i_k, j_k) : k \in \mathbb{Z}^+\}$ be a given impulsive and switching signal. Define

$$m(t) := V_{i_k}(t, x(t)), \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}^+.$$

Let $\|u\|_\infty$ denote the sup norm of the input on $\mathbb{R}^+$. Let $\mu = \max(g^{-1} \circ \chi, \rho)$ and $C = \mu(\|u\|_\infty)$.

Claim. *If $m(T) \leq g(C)$ for some $T \geq t_0$, then $m(t) \leq C$ for all $t \geq T$.*

Proof of the Claim. Suppose this is not true. Note that $g(C) < C$. Let

$$t^* := \inf\{t \geq T : m(t) > C\}.$$

Then $t^* \in [t_k, t_{k+1})$ for some k. Define

$$t_* := \sup\left\{t \in [t_k, t^*] : m(t) \leq g(C)\right\}.$$

Then $t^* \geq t_* \geq t_k$. We show that $t^* > t_*$. If $t^* = t_*$, then we must have $t^* = t_* = t_k$ by continuity of $m(t)$ on (t_*, t^*). If $m(t_k^-) \geq \chi(\|u\|_\infty)$, condition (iii) implies that $m(t_k) \leq g(m(t_k^-)) \leq g(C) < C$, which contradicts the definition of t^*. If $m(t_k^-) < \chi(\|u\|_\infty)$, condition (iii) implies that $m(t_k) < \rho(\|u\|_\infty)$, which also contradicts the definition of t^*. Therefore, $t^* > t_*$. By continuity, we have $m(t^*) = C$ and $m(t_*) = g(c)$. Moreover, we have $g(C) \leq m(t) \leq C$ on $[t_*, t^*] \subset [t_k, t_{k+1})$, which implies that $m(t) \geq \chi(\|u\|_\infty)$. Condition (ii) implies that

$$D^+m(t) \leq p_{i_k}(t)m(t), \quad t \in [t_*, t^*].$$

By integration, this differential inequality gives

$$N_{i_k} \leq \int_{g(C)}^{C} \frac{ds}{c_{i_k}(s)} = \int_{m(t_*)}^{m(t^*)} \frac{ds}{c_{i_k}(s)} \leq \int_{t_*}^{t^*} p_{i_k}(s)ds \leq \int_{t_*}^{t_*+\tau} p_{i_k}(s)ds \leq M_{i_k},$$

which contradicts condition (iv). The claim is proved. □

Let $\bar{t} = \inf\{t \geq t_0 : m(t) \leq g(C)\}$. It follows from the above claim that $m(t) \leq C$ for all $t \geq \bar{t}$, which implies

$$|h(t, x(t))| \leq a^{-1} \circ \mu(\|u\|_\infty), \quad t \geq \bar{t}. \tag{4.1}$$

For $t < \bar{t}$, we have $m(t) > g(C)$, which implies $m(t) \geq \chi(\|u\|_\infty)$ for all $t \leq \bar{t}$. Consequently, conditions (ii) and (iii) imply that

$$D^+m(t) \leq p_{i_k}(t)m(t), \quad t \in (t_k, t_{k+1}),\ t \leq \bar{t},$$

and

$$m(t) \leq g(m(t^-)), \quad t = t_k,\ t \leq \bar{t}.$$

Using Lemma 3.1 and a comparison principle for impulsive differential equation (see, e.g., Theorem 2.6.1 of [18]), there exists a function $\hat{\beta} \in \mathcal{KL}$ such that

$$m(t) \leq \hat{\beta}(m(t_0), t - t_0), \quad t_0 \leq t \leq \bar{t},$$

which implies

$$h(t, x(t)) \leq a^{-1} \circ \hat{\beta}(b(h_0(t, x(t))), t - t_0), \quad t_0 \leq t \leq \bar{t}. \tag{4.2}$$

Let $\gamma(r) = a^{-1}\max(g^{-1}(\chi(r)), \rho(r))$ and $\beta(r,s) = a^{-1}(\hat{\beta}(b(r), s))$ for $(r,s) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$. Combining (4.1) and (4.2) gives

$$h(t, x(t)) \leq \beta(h_0(t, x(t)), t - t_0) + \gamma(\|u\|_\infty), \quad t \geq t_0.$$

This guarantees that $x(t)$ is defined for all $t \geq t_0$. Since both β and γ are independent of a particular choice of signal $\{(t_k, i_k, j_k)\}$ in $\mathcal{S}^i_{\sup}(\tau)$, we have established (h_0, h)-uniform ISS of (2.1) on $\mathcal{S}^i_{\sup}(\tau)$. The proof is complete. □

Remark 4.1. We remark that both the continuous dynamics and the impulsive dynamics can be fully nonlinear due to the nonlinearity of the functions c_i and g. Hence the results are more general than most of the current results on input-to-state stability of impulsive or switched systems [9,12,13,22].

The following corollary is a special case of Theorem 4.1, which still generalizes the result in [13] to systems with switching dynamics with stability analysis in terms of two measures.

Corollary 4.1. *Suppose that conditions* (i)–(iii) *of Theorem* 4.1 *hold with the functions* p_i, c_i, *and* g *defined by* $p_i(s) = \mu_i$, $c_i(s) = s$, *and* $g(s) = \rho s$, $s \in \mathbb{R}^+$, *where* μ_i *and* ρ *are some positive constants. Then the theorem conclusions hold, if condition* (iv) *is replaced by*

(iv′) $\ln \rho < -\mu_i \tau_i$ *for all* $i \in \mathcal{I}_c$.

Our second result is concerned with (h_0, h)-ISS properties of system (2.1), in the case when all the subsystems governing the continuous dynamics of (2.1) are stable and the impulses, on the other hand, are destabilizing. Intuitively, the conditions in the following theorem consist of four aspects (corresponding to each of the conditions): (i) the Lyapunov functions satisfy certain positive definite and (h_0, h)-decrescent conditions; (ii) there exist some negative estimates of the upper right-hand derivatives of the functionals with respect to each stable mode of (2.1); (iii) the jumps induced by the destabilizing impulses satisfy certain growth conditions; (iv) the estimates on the derivatives and the growth by jumps satisfy certain integral form balancing conditions in terms of the dwell-time lower bounds.

Theorem 4.2. *Suppose that there exist a family of functions* $\{V_i : i \in \mathcal{I}_c\}$ *in* $C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$ *that are locally Lipschitzian in the second variable, functions* a, b, g, ρ, *and* χ *of class* $\mathcal{K}_\infty$, $p_i : \mathbb{R}^+ \mapsto \mathbb{R}^+$ *locally integrable and* $\alpha_i \in \mathcal{K}$ *locally Lipschitz for each* $i \in \mathcal{I}_c$, *and positive constants* τ_i $(i \in \mathcal{I}_c)$ *such that, for all* $i, \hat{\imath} \in \mathcal{I}_c, j \in \mathcal{I}_d$, *and* $(t, x, u) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m$,

(i) $a(h(t,x)) \leq V_i(t,x) \leq b(h_0(t,x))$;
(ii) $D^+V_i(t,x) \leq -p_i(t)c_i(V_i(t,x))$ *if* $V_i(t,x) \geq \chi(|u|)$;
(iii) $V_i(t, x + I_j(t,x,u)) \leq g(V_{\hat{\imath}}(t^-, x))$ *if* $V_{\hat{\imath}}(t^-, x) \geq \chi(|u|)$, *and* $V_i(t, x + I_j(t,x,u)) < \rho(|u|)$ *if* $V_{\hat{\imath}}(t^-, x) < \chi(|u|)$; *and*
(iv) $N_i := \inf_{t\geq 0} \int_t^{t+\tau_i} p_i(s)ds > M_i := \sup_{q>0} \int_q^{g(q)} \frac{ds}{c_i(s)}$.

Then system (2.1) *is uniformly* (h_0, h)*-ISS on* $\bigcap_{i\in\mathcal{I}_c} \mathcal{S}^i_{\inf}(\tau_i)$. *In particular, system* (2.1) *is uniformly* (h_0, h)*-ISS on* $\mathcal{S}_{\inf}(\tau)$, *where* $\tau = \sup_{i\in\mathcal{I}_c} \tau_i$.

Proof. Let $\left\{(t_k, i_k, j_k) : k \in \mathbb{Z}^+\right\}$ be a given impulsive and switching signal. Let $m(t)$ and $\|u\|_\infty$ be the same as in the proof of Theorem 4.1. Let $\mu = \max(g \circ \chi, \rho)$ and $C = \mu(\|u\|_\infty)$.

Claim. *If* $m(T) \leq g^{-1}(C)$ *for some* $T \geq t_0$, *then* $m(t) \leq C$ *for all* $t \geq T$.

Proof of the Claim. Suppose this is not true. Note that $g^{-1}(C) < C$. Let

$$t^* := \inf\{t \geq T : m(t) > C\}.$$

Then $t^* \in [t_k, t_{k+1})$ for some k. We show that $t^* = t_k$. Suppose this is not the case. Then $m(t^*) = C > g^{-1}(C) \geq \chi(\|u\|_\infty)$. Moreover, by continuity of $m(t)$ on (t_k, t_{k+1}), we have $m(t) > \chi(\|u\|_\infty)$ for $t \in [t^*, t^* + h]$, provided that h is sufficiently small. By condition (ii), $D^+m(t) \leq 0$ and $m(t)$ is decreasing on $[t^*, t^* + h]$, which is a contradiction with the definition of t^*. Therefore, we must have $t^* = t_k$ and $m(t^*) > C$. Moreover, t^* cannot be t_0. Otherwise, we must have $t^* = T = t_0$, which is impossible.

Now consider the interval $[t_{k-1}, t_k)$. It follows from the definition of t^* that $m(t) \leq C$ on $[t_{k-1}, t_k)$. If $m(t) > g^{-1}(C) \geq \chi(\|u\|_\infty)$ for all $t \in [t_{k-1}, t_k)$, then condition (ii) implies that

$$D^+m(t) \leq p_{i_{k-1}}(t)c_{i_{k-1}}(m(t)), \quad t \in [t_{k-1}, t_k).$$

Integrating this on $[t_{k-1}, t_k)$ gives

$$\int_{m(t_k^-)}^{m(t_{k-1})} \frac{ds}{c_{i_{k-1}}(s)} \geq \int_{t_{k-1}}^{t_k} p_{i_{k-1}}(s)ds \geq \int_{t_{k-1}}^{t_{k-1}+\tau_{i_{k-1}}} p_{i_{k-1}}(s)ds \geq N_{i_{k-1}},$$

and

$$\int_{m(t_k^-)}^{m(t_{k-1})} \frac{ds}{c_{i_{k-1}}(s)} \leq \int_{g^{-1}(C)}^{g(g^{-1}(C))} \frac{ds}{c_{i_{k-1}}(s)} \leq M_{i_{k-1}} < N_{i_{k-1}}.$$

We have a contradiction in view of condition (iv). If $m(t) < g^{-1}(C)$ for some $t \in [t_{k-1}, t_k)$, let

$$t_* := \sup\left\{t \in [t_{k-1}, t_k) : m(t) \leq g^{-1}(C)\right\}.$$

We show that $t_* = t_k$. Suppose this is not the case, then $m(t_*) = g^{-1}(C)$ and $m(t) \geq g^{-1}(C) \geq \chi(\|u\|_\infty)$ on $[t_*, t_k)$. Condition (ii) then implies that $m(t)$ is decreasing on $[t_*, t_k)$, which violates the definition of t_*. Therefore, $t^* = t_* = t_k$ and $m(t^{*-}) = m(t_*^-) \leq g^{-1}(C)$. Now if $m(t^{*-}) \geq \chi(\|u\|_\infty)$, condition (iii) implies that $m(t^*) \leq g(m(t^{*-})) \leq g(g^{-1}(C)) \leq C$, which contradicts that $m(t^*) > C$. If $m(t^{*-}) < \chi(\|u\|_\infty)$, condition (iii) implies that $m(t^*) < \rho(\|u\|_\infty) \leq C$, which also contradicts that $m(t^*) > C$. In either case, we have a contradiction. Therefore, the claim must be true. The rest of the proof is similar to that of Theorem 4.1. □

Similarly, the following corollary is a special case of Theorem 4.2, which also generalizes the result in [13] to systems with switching dynamics with stability analysis in terms of two measures.

Corollary 4.2. *Suppose that conditions* (i)–(iii) *of Theorem* 4.2 *hold with the functions* p_i, c_i, *and* g *defined by* $p_i(s) = \mu_i$, $c_i(s) = s$, *and* $g(s) = \rho s$, $s \in \mathbb{R}^+$, *where* μ_i *and* ρ *are some positive constants. Then the theorem conclusions hold, if condition* (iv) *is replaced by*

(iv′) $\ln \rho > \mu_i \tau_i$ *for all* $i \in \mathcal{I}_c$.

5. Example

In this section, we present an example to illustrate our main results.

Example 5.1. Consider the following networked hybrid control system

$$\begin{cases} x'(t) = A_{i_k}x(t) + f_{i_k}(x(t)) + B_{i_k}w(t), & t \in (t_k, t_{k+1}), \quad \text{(a)} \\ y(t) = Cx(t) + v(t), & t \geq t_0, \quad \text{(b)} \end{cases} \tag{5.1}$$

and

$$\begin{cases} \hat{x}'(t) = A_{i_k}\hat{x}(t) + \hat{f}_{i_k}(\hat{x}(t)), \quad t \in (t_k, t_{k+1}),\ k \in \mathbb{Z}^+, \quad \text{(a)} \\ \hat{y}(t) = C\hat{x}(t), \quad t \in (t_k, t_{k+1}),\ k \in \mathbb{Z}^+, \quad \text{(b)} \\ \hat{y}_l(t) = \begin{cases} y_{j_k}(t^-), & l = j_k, \\ \hat{y}_l(t^-), & l \neq j_k, \end{cases}, \quad t = t_k, \quad \text{(c)} \\ \hat{x}(t) = C^T(CC^T)^{-1}\hat{y}(t), \quad t = t_k, \quad \text{(d)} \end{cases} \tag{5.2}$$

where $k \in \mathbb{Z}^+, l \in \{1, 2, \ldots, n\}$, $x(t) \in \mathbb{R}^n$ is the system state, $w(t) \in \mathbb{R}^m$ is the disturbance input, $C \in \mathbb{R}^{p\times n}$ is the observation matrix ($p \leq n$), $y(t) \in \mathbb{R}^p$ is the state measurement, $v(t) \in \mathbb{R}^n$ is the measurement noise, $\hat{x}(t) \in \mathbb{R}^n$ is the remote estimate of $x(t), f_i(x(t))$ and $\hat{f}_i(\hat{x}(t))$ are the nonlinear perturbations of the state and their estimations, respectively. Here, without loss of generality, we assume that the observation matrix C is of full rank (i.e., rank$(C) = p$). Therefore, $C^T(CC^T)^{-1}$ gives the right-side inverse of the matrix C.

Moreover, $i_k \in \mathcal{I}_c$ and $\mathcal{I}_c$ is a finite index set; $\{t_k\}$ is a monotonically increasing transmission time sequence satisfying $t_k \to \infty$ as $k \to \infty$; at each transmission time $t = t_k$, a try-once-discard (TOD)-like protocol [25] to determine the index $j_k \in \{1, 2, \ldots, n\}$, i.e., j_k is the index j corresponding to the largest $|\hat{y}_j(t_k^-) - y_j(t_k^-)|$, where $j \in \{1, 2, \ldots, n\}$. When $t \in (t_k, t_{k+1})$, we can estimate $x(t)$ by letting $\hat{x}(t)$ evolve according to (5.2)(a) and forming $\hat{y}(t)$ according to $\hat{y}(t) = C\hat{x}(t)$; at $t = t_k$, a measurement $y(t_k)$ is sent to the remote estimator and update the output estimate $\hat{y}_{j_k}$ (based on a TOD-like protocol). Note that the state estimate $\hat{x}(t)$ is also updated at $t = t_k$ by (5.2)(d) such that $\hat{y}(t) = C\hat{x}(t)$ is satisfied for all $t \geq t_0$. It is assumed that, for all $x,\ \hat{x} \in \mathbb{R}^n$ and $i \in \mathcal{I}_c$,

(i) there exist constants λ_i such that

$$C^TCA_i + A_i^TC^TC \leq \lambda_i C^TC, \tag{5.3}$$

(ii) there exist constants L_i such that

$$[C(\hat{x} - x)]^T C[\hat{f}_i(\hat{x}) - f_i(x)] \leq L_i|C(\hat{x} - x)|^2. \tag{5.4}$$

The objective is to achieve ISS properties of the estimation error. To this end, we introduce $h(t, \hat{x}) = |\hat{y}(t) - Cx(t)| = |C(\hat{x} - x(t))|$ and $h_0(t, \hat{x}) = |\hat{x} - x(t)|$, for $(t, \hat{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$, where $x(t)$ is regarded as a prescribed trajectory of (5.1)(a). Intuitively, h is associated with the output estimation error $E_y(t) = \hat{y}(t) - Cx(t)$ and h_0 is associated with the state estimation error $E_x(t) = \hat{x}(t) - x(t)$. The ISS properties of the estimation error are equivalent to the (h_0, h)-ISS of the estimating system, i.e., the impulsive switched system given by (5.2). To investigate the (h_0, h)-ISS properties of (5.2), we choose a (common) Lyapunov function $V_i(t, \hat{x}) = V(t, \hat{x}) = |C(\hat{x} - x(t))|^2$ for $(t, \hat{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$. Condition (i) of Theorem 4.1 is clearly satisfied, since $h^2(t, \hat{x}) = V(t, \hat{x}) \leq \|C\|^2 h_0^2(t, \hat{x})$. Similar to Hespanha et al. [13], for each ρ in $((n-1)/n, 1)$, one can show that there exists a function $\chi \in \mathcal{K}_\infty$ such that

$$V(t, \hat{x}(t)) \leq \rho V(t^-, \hat{x}(t^-)), \tag{5.5}$$

provided that $V(t^-, \hat{x}(t^-)) \geq \chi(v(t^-))$. Therefore, condition (iii) of Theorem 4.1 is satisfied with $g(s) = \rho s$. Computing the upper right-hand derivative of $V(t, \hat{x})$ along the ith mode of (5.2) gives

$$\begin{aligned} D^+V_i(t, \hat{x}) &\leq 2[C(\hat{x} - x(t))]^T C\Big[A_i(\hat{x} - x(t)) \\ &\quad + \hat{f}_i(\hat{x}) - f_i(x(t)) - B_iw(t)\Big] \\ &\leq [\lambda_i + 2L_i + \varepsilon]\, V(t, \hat{x}), \end{aligned} \tag{5.6}$$

provided that $V(t, \hat{x}) \geq \chi_\varepsilon(|w(t)|)$, where $\varepsilon > 0$ is an arbitrary positive constant and χ_ε is a function in $\mathcal{K}_\infty$ which depends on ε. Indeed, we can choose $\chi_\varepsilon(s) = 4s \max_{i\in\mathcal{I}_c} \|B_i\|/\varepsilon^2$ for (5.6) to hold. The results are summarized in the following proposition.

Proposition 5.1. *If*

$$\ln\left(\frac{n-1}{n}\right) < -[\lambda_i + 2L_i]\tau_i < 0, \tag{5.7}$$

where λ_i and L_i are given by (5.3) and (5.4),[2] then system (5.2) is uniformly (h_0, h)-ISS over $\bigcap_{i\in\mathcal{I}_c} \mathcal{S}^i_{\sup}(\tau_i)$. In particular, for $\tau = \min_{i\in\mathcal{I}_c} \tau_i > 0$, system (5.2) is uniformly (h_0, h)-ISS over $\mathcal{S}_{\sup}(\tau)$.

Actually, if (5.7) holds, one can choose $\rho \in ((n-1)/n, 1)$ and $\mu_i > \lambda_{\max}(A_i^T + A_i) + L_i^2 > 0$ such that $\ln\rho < -\mu_i\tau_i$. In view of (5.6), we can choose $\varepsilon > 0$ sufficiently small such that condition (ii) of Theorem 4.1 is satisfied with $p_i(s) \equiv \mu_i$ and $c_i(s) = s$. Condition (iv) of Theorem 4.1 now reads

$$\tau_i\mu_i = N_i > M_i = \sup_{q>0} \int_{\rho q}^{q} \frac{1}{s} ds = -\ln\rho.$$

The conclusion of Proposition 5.1 follows from that of Theorem 4.1. Now we consider two numerical cases:

(i) We first take $B_i = [1 \quad 0.1 \quad 0]$,

$$f_i(x) = \hat{f}_i(x) = [\alpha_i(a - b) \quad 0 \quad 0]^T \operatorname{sat}(x_1),$$

and

$$A_i = \begin{bmatrix} -\alpha_i(1-b) & \alpha_i & 0 \\ 1 & -1 & 1 \\ 0 & -\beta_i & 0 \end{bmatrix},$$

where $i \in \mathcal{I}_c = \{1, 2\}$, and $\alpha_1 = 9, \beta_1 = 100/7, \alpha_2 = 10, \beta_2 = 16, a = 8/7, b = 5/7$. Therefore, the state system can be regarded as a hybrid system switching between two Chua's circuits with slightly different parameters, both of which exhibit chaotic behaviors under the given parameters. We consider $Cx = (x_1, x_2, x_3)^T$, i.e., the plant state x is fully observable. It is easy to verify that (5.4) is satisfied with $L_1 = 27/7$ and $L_2 = 30/7$. Moreover, (5.3) is satisfied with $\lambda_1 = \lambda_{\max}(A_1 + A_1^T) = 14.8685$ and $\lambda_2 = \lambda_{\max}(A_2 + A_2^T) = 16.7839$. With $n = 3$, (5.7) reduces to $\tau_1 < 0.0180$ and $\tau_2 < 0.0160$. Therefore, with $\tau_1 = 0.017$ and $\tau_2 = 0.015$, Theorem 4.1 guarantees that system (5.2) is uniformly (h_0, h)-ISS over $\bigcap_{i=1,2} \mathcal{S}^i_{\sup}(\tau_i)$. In particular, for $\tau = 0.015$, system (5.2) is uniformly (h_0, h)-ISS over $\mathcal{S}_{\sup}(\tau)$. By the definitions of h_0 and h, (h_0, h)-ISS of system (5.2) corresponds to ISS properties of the state estimation error. Simulation results for the state estimation error, under the given parameters, are shown in Fig. 1.

(ii) We then take

$$A_i = \begin{bmatrix} \gamma_i & 1 & -2 \\ 4 & \eta_i & 0 \\ 2 & 1 & \theta_i \end{bmatrix},$$

where $i \in \mathcal{I}_c = \{1, 2\}, \gamma_1 = 1, \eta_1 = 3, \theta_1 = 1, \gamma_2 = 1.1, \eta_2 = 3.1, \theta_2 = 1.1$, and we keep other parameters the same. We consider $Cx = (x_1, x_2 - 2x_3)$, which means that the plant state x is only partially observable. We can verify

[2] Note that the conditions given by (5.3) and (5.4) do not require either λ_i or L_i to be positive, which means that we can have more relaxed conditions by allowing either the linear part or the nonlinear part of the plant in (5.1)(a) to contribute to the convergence of the estimation error if they do so. If, however, this renders $\lambda_i + 2L_i$ negative in (5.7), we may replace $\lambda_i + 2L_i$ by an arbitrarily small positive number such that both (5.6) and (5.7) hold.

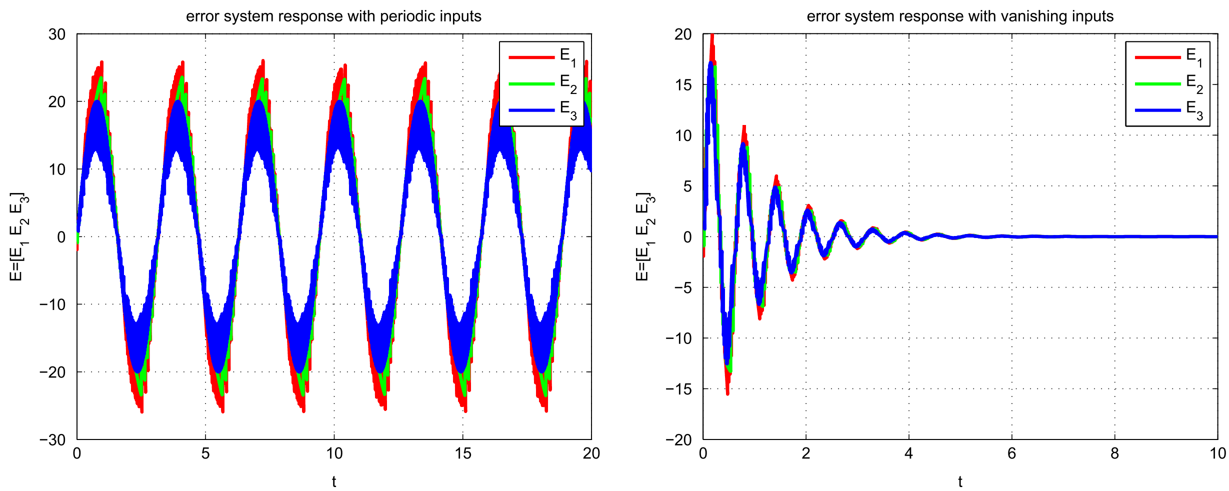

Fig. 1. Simulation results for Example 5.1 showing ISS properties of the state estimation error.

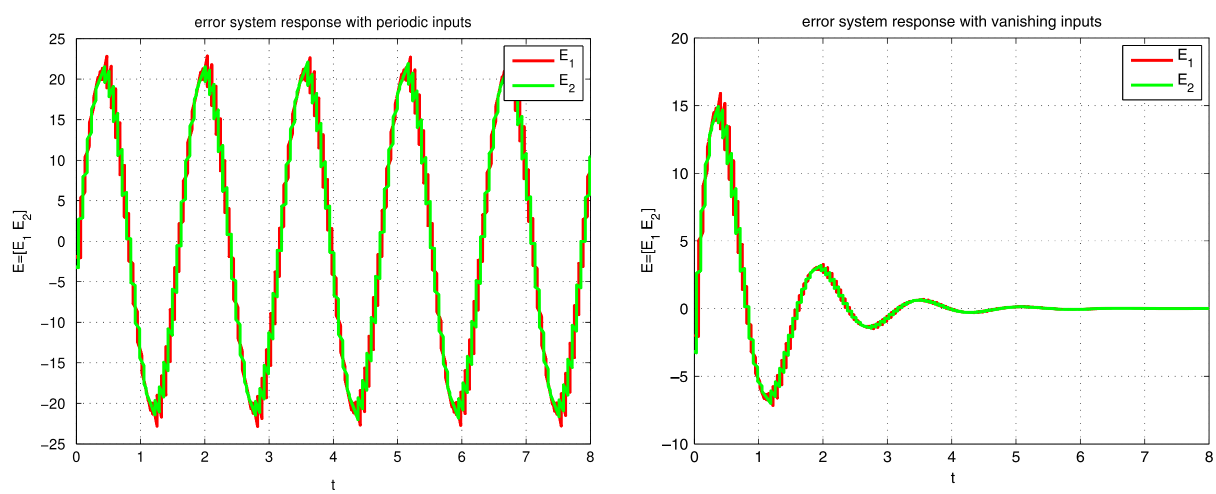

Fig. 2. Simulation results for Example 5.1 showing ISS properties of the output estimation error.

that (5.4) is still satisfied with $L_1 = 27/7$ and $L_2 = 30/7$. Moreover, we can verify that (5.3) is satisfied with $\lambda_1 = 3$ and $\lambda_2 = 3.1$. Similarly, with $n = 3$, (5.7) reduces to $\tau_1 < 0.0378$ and $\tau_2 < 0.0347$. Therefore, with $\tau_1 = 0.037$ and $\tau_2 = 0.034$, Theorem 4.1 guarantees that system (5.2) is uniformly (h_0, h)-ISS over $\bigcap_{i=1,2} \mathcal{S}^i_{\text{sup}}(\tau_i)$. In particular, for $\tau = 0.034$, system (5.2) is uniformly (h_0, h)-ISS over $\mathcal{S}_{\text{sup}}(\tau)$. Since h is associated with the output estimation error, we have actually shown that the output estimation error is stable with respect to disturbance inputs and measurement noise. Simulation results for the output estimation error, under the given parameters, are shown in Fig. 2.

6. Conclusions

In this paper, we have investigated the input-to-state stability properties of impulsive switched systems. Sufficient conditions have been established for input-to-state stability in terms of two measures for hybrid systems with both switching and impulse effects. The formulation of hybrid systems is quite general in that it allows both the continuous dynamics and the discrete dynamics to be chosen from a certain family, according to a general impulsive and switching signal. The stability results unify the notions of input/output stability and stability in terms of two measures in the hybrid systems setting. By exploiting some nonlinear integral constraints in terms of generalized dwell-time conditions, we do not assume the existence of a linear estimate for the continuous dynamics. An illustrative example has been provided to demonstrate the main theoretical results.

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References

[1] R. Goebel, R.G. Sanfelice, A.R. Teel, Hybrid dynamical systems: robust stability and control for systems that combine continuous-time and discrete-time dynamics, IEEE Control Syst. Mag. 29 (2009) 28–93.
[2] D. Liberzon, Switching in Systems and Control, Birkhäuser, Boston, 2003.
[3] A. van der Schaft, H. Schumacher, An Introduction to Hybrid Dynamical Systems, in: Lecture Notes in Control and Information Sciences, vol. 251, Springer-Verlag, London, 2000.
[4] R. Shorten, F. Wirth, O. Mason, K. Wulff, C. King, Stability criteria for switched and hybrid systems, SIAM Rev. 49 (2007) 543–732.
[5] E.D. Sontag, Smooth stabilization implies coprime factorization, IEEE Trans. Automat. Control 34 (1989) 435–443.
[6] V. Lakshmikantham, X.Z. Liu, Stability Analysis in Terms of Two Measures, World Scientific, 1993.
[7] Z.P. Jiang, Y. Wang, Input-to-state stability for discrete-time nonlinear systems, Automatica 37 (2001) 857–869.
[8] J.L. Mancilla-Aguilar, R.A. Garcia, On converse Lyapunov theorems for ISS and iISS switched nonlinear systems, Systems Control Lett. 42 (2001) 47–53.

[9] L. Vu, D. Chatterjee, D. Liberzon, Input-to-state stability of switched systems and switching adaptive control, Automatica 42 (2007) 639–646.
[10] C. Cai, A.R. Teel, Results on input-to-state stability for hybrid systems, in: Proc. of the 44th Conf. on Decision and Control, 2005, pp. 5403–5408.
[11] C. Cai, A.R. Teel, Characterizations of input-to-state stability for hybrid systems, Systems Control Lett. 58 (2009) 47–53.
[12] W.-H. Chen, W.X. Zheng, Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays, Automatica 45 (2009) 1481–1488.
[13] J.P. Hespanha, D. Liberzon, A.R. Teel, Lyapunov conditions for input-to-state stability of impulsive systems, Automatica 44 (2008) 2735–2744.
[14] J. Liu, X. Liu, W.-C. Xie, Input-to-state stability of impulsive and switching hybrid systems with time-delay, Automatica 47 (2011) 899–908.
[15] X.Z. Liu, On (h_0, h)-stability of autonomous systems, J. Appl. Math. Stoch. Anal. 4 (1992) 331–338.
[16] D. Chatterjee, D. Liberzon, Stability analysis of deterministic and stochastic switched systems via a comparison principle and multiple Lyapunov functions, SIAM J. Control Optim. 45 (2006) 174–206.
[17] E.D. Sontag, Y. Wang, Notions of input to output stability, Systems Control Lett. 38 (1999) 235–248.
[18] V. Lakshmikantham, D. Baĭnov, P. Simeonov, Theory of Impulsive Differential Equations, World Scientific Publishing, Teaneck, NJ, 1989.
[19] B. Liu, X. Liu, X. Liu, Stability and robustness of quasi-linear impulsive hybrid systems, J. Math. Anal. Appl. 283 (2003) 416–430.
[20] W.M. Haddad, V. Chellaboina, S.G. Nersesov, Impulsive and Hybrid Dynamical Systems, Princeton Univ. Press, New Jersey, 2006.
[21] J.P. Hespanha, Uniform stability of switched linear systems: extension of LaSalle's invariance principle, IEEE Trans. Automat. Control 49 (2004) 470–482.
[22] J. Liu, X. Liu, W.-C. Xie, On the (h_0, h)-stabilization of switched nonlinear systems via state-dependent switching rule, Appl. Math. Comput. 217 (2010) 2067–2083.
[23] E.D. Sontag, Input to state stability: basic concepts and results, in: Nonlinear and Optimal Control Theory, in: Lecture Notes in Mathematics, vol. 1932, Springer, Berlin, 2008, pp. 163–220.
[24] H.K. Khalil, Nonlinear Systems, third ed., Prentice Hall, 2002.
[25] G.C. Walsh, O. Beldiman, L.G. Bushnell, Asymptotic behavior of nonlinear networked control systems, IEEE Trans. Automat. Control 46 (2001) 1093–1097.