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Brief paper

On asymptotic convergence and boundedness of stochastic systems with time-delay[☆]

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ABSTRACT

Analyzing the long-term behavior of a nonlinear dynamical system is an important but challenging task, especially in the presence of noise and delays. This paper investigates asymptotic convergence and boundedness properties of stochastic systems with time-delay. We are particularly interested in asymptotic convergence that may not be exponential, as observed in many practical applications. General criteria for checking both moment and almost sure asymptotic convergence and boundedness are established. Such criteria are mainly motivated by stochastic systems with time-varying coefficients and multiple delays and/or different orders of nonlinearities. As shown by several examples, the existing results cannot be applied to analyze such systems. Thus, we generalize the existing theory by allowing the diffusion operator associated with a Lyapunov function to satisfy a weaker assumption, which involves time-varying coefficients and several auxiliary functions to cope with possible multiple delays and/or different orders of nonlinearities. The results presented in this paper provide an effective tool for checking asymptotic convergence and boundedness properties of general stochastic nonlinear systems with time-delay.

1. Introduction

Stochastic functional or delay differential equations (SFDEs or SDDEs) are widely used across science and engineering to model stochastic systems whose evolution depends on the past history of the state (see, e.g., Kolmanovskii and Myshkis (1992); Kolmanovskii and Nosov (1981); Kushner (2010); Mao (1997); Mao and Yuan (2006); Mohammed (1986)). Stability, convergence, and boundedness properties of such systems have been studied extensively in the literature (see, e.g., Basin and Rodkina (2008); Berezansky and Braverman (2006); Gao, Lam, and Wang (2006); Huang and Deng (2008); Jankovic, Randjelovic, and Jovanovic (2009); Kadiev and Ponosov (2004); Liu, Liu, and Xie (2009, 2011); Luo, Mao, and Shen (2011); Mao (2000); Mao, Lam, and Huang (2008); Rodkina and Basin (2007); Shen, Luo, and Mao (2006); Wang, Liu, and Liu (2010); Xie and Xie (2000); Yuan and Lygeros (2006); Yuan and Mao (2004)).

In particular, the recent work by Luo et al. (2011) presents a generalized theory for checking asymptotic convergence and boundedness for SFDEs based on a Lyapunov-type condition on the diffusion operator. More specifically, the diffusion operator $\mathcal{L}V$ of an Lyapunov function (to be defined in Section 2) is required to satisfy the following condition:

$$\mathcal{L}V(t,\varphi) \le \alpha_1 - \alpha_2 V(t,\varphi(0)) + \alpha_3 \int_{-r}^{0} V(t+\theta,\varphi(\theta))d\bar{\mu}(\theta) - U(t,\varphi(0)) + \alpha \int_{-r}^{0} U(t+\theta,\varphi(\theta))d\mu(\theta), \quad (1.1)$$

where V is a Lyapunov function candidate, φ is a function in $\mathcal{C}([-r,0];\mathbb{R}^n)$ for some $r>0$, $\bar{\mu}$ and μ are probability measures on $[-r,0]$ for describing the time delays, $\alpha_1 \ge 0$, $\alpha_2 > \alpha_3 \ge 0$, and $\alpha \in (0,1)$ are some constants. Using (1.1) as a standing assumption, the results by Luo et al. (2011) improve earlier results in the literature (e.g., Mao (2000)) by introducing the auxiliary function U, which can be potentially used to cope with nonlinearities and delays encountered in a stochastic system.

However, there are many SFDEs or SDDEs to which the results based on (1.1) or other existing criteria for convergence and boundedness cannot be applied. The main aim of this paper is to extend the existing theory (e.g., Luo et al. (2011)) so that it can be applied to more general stochastic systems with time-delay. Let us look at two motivating examples.

The first example is motivated by stochastic systems with time-varying coefficients. Systems of this type can arise from applications involving stochastic approximation type algorithms

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(e.g., Liu, Liu, Xie, and Zhang (2011)). Consider the scalar stochastic functional differential equation

$$dx(t) = c(t)\big[-x(t) + \alpha\mathcal{D}(x_t)\big]dt + c(t)\sigma dW(t), \tag{1.2}$$

where $W(t)$ is a one-dimensional standard Wiener process (or Brownian motion), $c : \mathbb{R}^+ \to \mathbb{R}^+$ is a time-varying function that satisfies $\int_0^\infty c(s) = \infty$ and $\int_0^\infty c^2(s) < \infty$. In addition, $\alpha \in (0, 1)$ and σ are some constant, and $\mathcal{D}$ is a bounded linear operator from $\mathcal{C}([-r, 0]; \mathbb{R}^n)$ to $\mathbb{R}$ satisfying $|\mathcal{D}(\varphi)| \le \int_{-r}^0 |\varphi(\theta)|\,\mu(\theta)$, where μ is a probability measure on $[-r, 0]$. Consider $V(t, x) = x^2$. Then

$$\begin{aligned}\mathcal{L}V(t, \varphi) &= 2c(t)\varphi(0)\big[-\varphi(0) + \alpha\mathcal{D}(\varphi)\big] + \sigma^2c^2(t)\\ &\le -(2-\alpha)c(t)\,|\varphi(0)|^2 + \alpha c(t)\int_{-r}^0 |\varphi(\theta)|^2\,d\mu(\theta)\\ &\quad + \sigma^2c^2(t).\end{aligned} \tag{1.3}$$

It can be seen that the existing theory in Luo et al. (2011) cannot be applied here due to the time-varying coefficient $c(t)$. In Section 4, we will show that results of the current paper can be applied to conclude both moment and almost sure asymptotic convergence and boundedness results for the above system.

The second example is motivated by stochastic systems with multiple high-order nonlinearities and multiple delays. Consider the scalar stochastic functional differential equation

$$\begin{aligned}dx(t) &= \big[-2x(t) - x^3(t) - x^5(t) + \mathcal{D}_1^2(x_t) + \mathcal{D}_2^3(x_t)\big]dt\\ &\quad + \mathcal{D}_3(x_t)dW(t),\end{aligned} \tag{1.4}$$

$\mathcal{D}_i$ are bounded linear operators from $\mathcal{C}([-r, 0]; \mathbb{R}^n)$ to $\mathbb{R}$ satisfying $|\mathcal{D}_i(\varphi)| \le \int_{-r}^0 |\varphi(\theta)|\,\mu_i(\theta)$, where μ_i, $i = 1, 2, 3$, are probability measures on $[-r, 0]$. Consider $V(t, x) = x^2$. We can verify

$$\begin{aligned}\mathcal{L}V(t, \varphi) &\le -\frac{5}{2}\,|\varphi(0)|^2 - 2\,|\varphi(0)|^4 - 2\,|\varphi(0)|^6\\ &\quad + \frac{4}{3}\int_{-r}^0 |\varphi(\theta)|^4\,d\mu_1(\theta) + \frac{4}{3}\int_{-r}^0 |\varphi(\theta)|^6\,d\mu_2(\theta)\\ &\quad + \int_{-r}^0 |\varphi(\theta)|^2\,d\mu_3(\theta).\end{aligned} \tag{1.5}$$

It can again be seen that the existing results by Luo et al. (2011) cannot be applied here due to the coexistence of multiple nonlinear terms and multiple distributed delay terms on the right-hand side of the above estimate for $\mathcal{L}V(t, \varphi)$. In Section 4, we will show that results of the current paper can be applied to conclude both moment and almost sure asymptotic convergence and boundedness results for the above system.

Finally, another motivation for the results in this paper is that there are many practical systems whose solutions only tend to zero asymptotically but not exponentially. While the results by Luo et al. (2011) cover exponential stability and convergence, it is still very useful to obtain additional criteria on the asymptotic convergence of stochastic systems with time-delay.

Motivated by the discussions above, the main aim of this paper is to develop a theory for checking asymptotic convergence and boundedness properties of stochastic systems with time-delay. In particular, the results should cover stochastic systems with time-varying coefficients, can be applied to study systems with multiple orders of nonlinearities and multiple delays, and can be applied to study asymptotic convergence that is not necessarily exponential. We note that the emphasis of the current paper is on generalizing Lyapunov-type methods for proving asymptotic convergence and boundedness properties for a larger class of stochastic delay systems. There exist other methods, including the method of integral transforms originally developed for deterministic linear functional differential equations Azbelev and Simonov (2002), that have proven to be efficient in studying stability properties of specific SFDEs (both linear and nonlinear). Interested readers are referred to Berezansky and Braverman (2006); Kadiev and Ponosov (2004) and references therein for more details.

2. Existence and uniqueness theorem

Let $\mathbb{R}^n$ denote the n-dimensional real Euclidean space and $\mathbb{R}^{n\times m}$ the space of $n \times m$ real matrices. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x. For $A = (a_{ij}) \in \mathbb{R}^{n\times m}$, $|A| := \sqrt{\mathrm{trace}(A^TA)} = \sqrt{\sum_{i=1}^n\sum_{j=1}^m |a_{ij}|^2}$ denotes the Frobenius norm of the matrix A. Given $r > 0$, let $\mathcal{C}([-r, 0]; \mathbb{R}^n)$ denote the family of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$. A norm on $\mathcal{C}([-r, 0]; \mathbb{R}^n)$ is defined as $\|\varphi\| := \sup_{-r\le s\le 0} |\varphi(s)|$ for $\varphi \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$.

Let $(\Omega, \mathcal{F}, P)$ be a given complete probability space with $\{\mathcal{F}_t\}_{t\ge0}$ as a filtration satisfying the usual conditions and $W(t)$ be an m-dimensional standard Wiener process defined on $(\Omega, \mathcal{F}, P)$ and adapted to $\{\mathcal{F}_t\}_{t\ge0}$.

Consider the following stochastic functional differential equation (SFDE):

$$dx(t) = f(t, x_t)dt + g(t, x_t)dW(t), \quad t \ge 0, \tag{2.1}$$

where x_t is defined by $x_t(s) = x(t+s)$, for $-r \le s \le 0$ and can be treated as a $\mathcal{C}([-r, 0]; \mathbb{R}^n)$-valued stochastic process. Both $f : \mathbb{R}^+ \times \mathcal{C}([-r, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ and $g : \mathbb{R}^+ \times \mathcal{C}([-r, 0]; \mathbb{R}^n) \to \mathbb{R}^{n\times m}$ are Borel measurable functionals.

Let $\mathcal{C}^{1,2}([-r, \infty) \times \mathbb{R}^n; \mathbb{R}^+)$ denote the set of all functions from $[-r, \infty) \times \mathbb{R}^n$ to $\mathbb{R}^+$ that are continuously differentiable in t and twice continuously differentiable in x. For each $V \in \mathcal{C}^{1,2}([-r, \infty) \times \mathbb{R}^n; \mathbb{R}^+)$, define an operator from $\mathbb{R}^+ \times \mathcal{C}([-r, 0]; \mathbb{R}^n)$ to $\mathbb{R}$ by

$$\begin{aligned}\mathcal{L}V(t, \varphi) &:= V_t(t, \varphi(0)) + V_x(t, \varphi(0))f(t, \varphi)\\ &\quad + \frac{1}{2}\mathrm{trace}\big[g^T(t, \varphi)V_{xx}(t, \varphi(0))g(t, \varphi)\big],\end{aligned}$$

where $V_x(t, x) = (V_{x_1}(t, x), \ldots, V_{x_n}(t, x))$ and $V_{xx}(t, x) = (V_{x_ix_j}(t, x))_{n\times n}$.

Theorem 2.1. *Suppose that the following two conditions are satisfied:*

(i) *There exist functions $V \in \mathcal{C}^{1,2}([-r, \infty) \times \mathbb{R}^n; \mathbb{R}^+)$ and $U_i \in \mathcal{C}([-r, \infty) \times \mathbb{R}^n; \mathbb{R}^+)$, probability measures μ_0 and μ_i on $[-r, 0]$, where $i = 1, \ldots, N$, and a piecewise continuous functions α from $[-r, \infty)$ to $\mathbb{R}^+$*

$$\lim_{|x|\to\infty}\ \inf_{-r\le t<\infty} V(t, x) = \infty, \tag{2.2}$$

and

$$\begin{aligned}&\mathcal{L}V(t, \varphi)\\ &\quad\le \alpha(t)\left[1 + V(t, \varphi(0)) + \int_{-r}^0 V(t+\theta, \varphi(\theta))d\mu_0(\theta)\right]\\ &\qquad + \sum_{i=1}^N\left[-U_i(t, \varphi(0)) + \int_{-r}^0 U_i(t+s, \varphi(\theta))d\mu_i(\theta)\right],\\ &\quad\forall(t, \varphi) \in \mathbb{R}^+ \times \mathcal{C}([-r, 0]; \mathbb{R}^n).\end{aligned} \tag{2.3}$$

(ii) *For each integer $i \ge 1$, there exists a positive constant K_i such that*

$$\begin{aligned}&|f(t, \varphi) - f(t, \psi)|^2 \vee |g(t, \varphi) - g(t, \psi)|^2\\ &\quad\le K_i\|\varphi - \psi\|^2,\end{aligned} \tag{2.4}$$

holds for all $t \in \mathbb{R}^+$ and all $\varphi, \psi \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$ such that $\|\varphi\| \vee \|\psi\| \le i$.

Then, for any given initial data $\xi \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$, there is a unique global solution $x(t)$ for (2.1) on $[t_0 - r, \infty)$.

This theorem gives a Lyapunov-based non-explosion test for SFDEs without a linear growth condition and with a more relaxed condition on the diffusion operator as given by (2.3). The proof for this theorem is essentially similar to that of Theorem 2 of Luo et al. (2011). The main differences are that we incorporate a time-varying function α and allow multiple auxiliary functions U_i, instead of a single one. We omit the proof here, since the main aim of this paper is to establish new criteria for asymptotic convergence and boundedness.

3. Asymptotic convergence and boundedness

We consider a stronger version of (2.3) in Theorem 2.1, and use it as a standing assumption to prove several results on asymptotic convergence and boundedness for solutions of (2.1).

Assumption 3.1. Let γ, c, and η be piecewise continuous functions from $[-r, \infty)$ to $\mathbb{R}^+$ satisfying the following conditions: (a) $\int_0^\infty \gamma(s)ds < \infty$; (b) $\int_0^\infty c(s)ds = \infty$; (c) both c and η are monotonically non-increasing and bounded by 1. Let condition (i) of Theorem 2.1 hold, except that (2.3) is replaced by

$$\begin{aligned}&\mathcal{L}V(t, \varphi)\\ &\quad \le \gamma(t) - \nu c(t)V(t, \varphi(0)) + \lambda c(t)\int_{-r}^0 V(t+\theta, \varphi(\theta))d\mu_0(\theta)\\ &\qquad + \sum_{i=1}^N \eta(t)\Big[-U_i(t, \varphi(0)) + \beta_i\int_{-r}^0 U_i(t+\theta, \varphi(\theta))d\mu_i(\theta)\Big],\\ &\quad \forall (t, \varphi) \in \mathbb{R}^+ \times \mathcal{C}([-r, 0]; \mathbb{R}^n),\end{aligned} \tag{3.1}$$

where ν, λ, and β_i are constants satisfying $\nu \ge \lambda \ge 0$ and $0 \le \beta_i \le 1, i = 1, \ldots, N$.

3.1. Moment convergence and boundedness

We will start with a result on moment convergence and boundedness of the solutions of (2.1) under Assumption 3.1.

Theorem 3.1. *Suppose that conditions of Theorem 2.1 hold, except that (2.3) is replaced by Assumption 3.1. Then for any given initial data $\xi \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$, there is a unique global solution $x(t)$ for (2.1) on $[t_0 - r, \infty)$, which satisfies the following:*

(A)

$$\sup_{-r\le t<\infty} EV(t, x(t)) < \infty. \tag{3.2}$$

(B) *If $\nu > \lambda$, then*

$$\int_0^\infty c(s)EV(s, x(s))ds < \infty. \tag{3.3}$$

(C) *If $\beta_i < 1$ for some $i \in \{1, \ldots, N\}$, then*

$$\int_0^\infty \eta(s)EU_i(s, x(s))ds < \infty. \tag{3.4}$$

(D) *If $\nu > \lambda$ and $0 < \beta_i < 1$ for all $i \in \{1, \ldots, N\}$, then*

$$EV(t, x(t)) \le C_0 e^{-\rho\int_0^t c(\tau)d\tau} + \int_0^t e^{-\rho\int_s^t c(\tau)d\tau}\gamma(s)ds, \tag{3.5}$$

for all $t \ge 0$, where $C_0 > 0$ is a finite constant, and ρ is the minimum of $\{\rho_0, \rho_1, \ldots, \rho_N\}$, where $\rho_0 > 0$ is the unique root of $\rho_0 - \nu + \lambda e^{\rho_0 r} = 0$ and $\rho_i = -\ln(\beta_i)/r$, for $i = 1, \ldots, N$. In particular,

$$\lim_{t\to\infty} EV(t, x(t)) = 0. \tag{3.6}$$

Proof. Existence of the global solution $x(t)$ for any given initial data $\xi \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$ follows from Theorem 2.1. For any given $\varepsilon \ge 0$, by Itô's formula and (3.1), we have

$$\begin{aligned}&E\big(e^{\varepsilon\int_0^t c(\tau)d\tau}V(t, x(t))\big) - E\big(V(0, x(0))\big)\\ &\quad = E\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}\big[\varepsilon c(s)V(s, x(s)) + \mathcal{L}V(s, x_s)\big]ds\\ &\quad \le \int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}\gamma(s)ds + (\varepsilon - \nu)E\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}c(s)V(s, x(s))ds\\ &\qquad + \lambda E\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}c(s)\int_{-r}^0 V(s+\theta, x(s+\theta))d\mu_0(\theta)ds\\ &\qquad - \sum_{i=1}^N E\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}\eta(s)U_i(s, x(s))\\ &\qquad + \sum_{i=1}^N \beta_i E\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}\eta(s)\int_{-r}^0 U_i(s+\theta, x(s+\theta))d\mu_i(\theta)ds.\end{aligned}$$

Using Fubini's theorem, we can compute

$$\begin{aligned}&E\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}c(s)\int_{-r}^0 V(s+\theta, x(s+\theta))d\mu_0(\theta)ds\\ &\quad = E\int_{-r}^0\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}c(s)V(s+\theta, x(s+\theta))dsd\mu_0(\theta)\\ &\quad \le e^{\varepsilon r}E\int_{-r}^0\int_{-r}^t e^{\varepsilon\int_0^s c(\tau)d\tau}c(s)V(s, x(s))dsd\mu_0(\theta)\\ &\quad = e^{\varepsilon r}E\int_{-r}^t e^{\varepsilon\int_0^s c(\tau)d\tau}c(s)V(s, x(s))ds,\end{aligned}$$

where we have used the fact that $c(t) \le 1$ and $c(t)$ is monotonically non-increasing for all $t \ge 0$. Similarly, we can show that

$$\begin{aligned}&E\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}\eta(s)\int_{-r}^0 U_i(s+\theta, x(s+\theta))d\mu_i(\theta)ds\\ &\quad \le e^{\varepsilon r}E\int_{-r}^t e^{\varepsilon\int_0^s c(\tau)d\tau}\eta(s)U_i(s, x(s))ds. \quad i = 1, \ldots, N.\end{aligned}$$

Putting all the three estimates above together, we obtain

$$\begin{aligned}&E\big(e^{\varepsilon\int_0^t c(\tau)d\tau}V(t, x(t))\big) \le C_0 + \int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}\gamma(s)ds\\ &\quad + (\varepsilon - \nu + \lambda e^{\varepsilon r})E\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}c(s)V(s, x(s))ds\\ &\quad + \sum_{i=1}^N(-1 + \beta_i e^{\varepsilon\tau})E\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}\eta(s)U_i(s, x(s))ds,\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}C_0 &= E\big(V(0, x(0))\big) + \lambda e^{\varepsilon r}E\int_{-r}^0 e^{\varepsilon\int_0^s c(\tau)d\tau}c(s)V(s, x(s))ds\\ &\quad + \sum_{i=1}^N e^{\varepsilon r}E\int_{-r}^0 e^{\varepsilon\int_0^s c(\tau)d\tau}\eta(s)U_i(s, x(s))ds < \infty.\end{aligned}$$

Letting $\varepsilon = 0$ in (3.7), we obtain

$$\begin{aligned}&EV(t, x(t)) + (\nu - \lambda)\int_0^\infty c(s)EV(s, x(s))ds\\ &\quad + \sum_{i=1}^N(1 - \beta_i)\int_0^t \eta(s)EU_i(s, x(s))ds \le C_0 + \int_0^t \gamma(s)ds,\end{aligned}$$

for all $t \geq 0$. Conclusions of (A), (B), and (C) all follow from this inequality and the assumption that $\int_0^\infty \gamma(s)ds < \infty$.

Now suppose that $\nu > \lambda$ and $0 < \beta_i < 1$ for all $i \in \{1, \ldots, N\}$. By the definition of ρ, we have $\rho - \nu + \lambda e^{\rho r} \leq 0$ and $-1 + \beta_i e^{\rho\tau} \leq 0$ for all $i = 1, \ldots, N$. Therefore, choosing $\varepsilon = \rho$ in (3.7) gives (3.5). To show $\lim_{t\to\infty} EV(t, x(t)) = 0$, first note that, in (3.5),

$$C_0 e^{-\rho\int_0^t c(\tau)d\tau} \to 0, \quad \text{as } t \to \infty, \tag{3.8}$$

since $\int_0^\infty c(s)ds = \infty$. On the other hand, note that, for all $t \geq T_0$, we have

$$\begin{aligned}\int_0^t & e^{-\rho\int_s^t c(r)dr}\gamma(s)ds \\ &= \int_0^T e^{-\rho\int_s^t c(r)dr}\gamma(s)ds + \int_T^t e^{-\rho\int_s^t c(r)dr}\gamma(s)ds \\ &\leq e^{-\rho\int_T^t c(r)dr}\int_0^\infty \gamma(s)ds + \int_T^\infty \gamma(s)ds. \end{aligned}\tag{3.9}$$

For fixed T, we have $e^{-\rho\int_T^t c(r)dr} \to 0$, as $t \to \infty$. Then letting $T \to \infty$, $\int_T^\infty \gamma(s)ds \to 0$ as well. Therefore,

$$\int_0^t e^{-\rho\int_s^t c(\tau)d\tau}\gamma(s)ds \to 0, \quad \text{as } t \to \infty,$$

which implies that $\lim_{t\to\infty} EV(t, x(t)) = 0$ in view of (3.5) and (3.8). The proof is complete. □

Both (3.3) and (3.4) give certain boundedness results of $EV(t, x(t))$ and $EU_i(t, x(t))$ in an integral form. With some additional conditions, we can conclude moment asymptotic convergence of $x(t)$ from (3.3) and (3.4). For this purpose, we invoke the following assumption, called the *linear growth condition*: there exists a constant $K > 0$ such that

$$|f(t,\varphi)| \vee |g(t,\varphi)| \leq K\left[1 + \int_{-r}^0 |\varphi(\theta)|\, d\mu(\theta)\right], \tag{3.10}$$

for all $t \geq 0$ and $\varphi \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$, where μ is some probability measure on $[-r, 0]$. Let $\mathcal{K}_\infty$ denote the class of functions $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ such that α is continuous, strictly increasing, and $\alpha(t) \to \infty$ as $t \to \infty$.

Theorem 3.2. *Suppose that conditions of Theorem 2.1 hold, except that (2.3) is replaced by Assumption 3.1. Then for any given initial data $\xi \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$, there is a unique global solution $x(t)$ for (2.1) on $[t_0 - r, \infty)$, which satisfies the following:*

(A) If $\nu > \lambda$ and, in addition, (3.10) holds, $\lim_{t\to\infty} c(t) > 0$, and there exists a convex function $\alpha \in \mathcal{K}_\infty$ and some $p \geq 2$ such that

$$\alpha(|x|^p) \leq V(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \tag{3.11}$$

then

$$\lim_{t\to\infty} E\,|x(t)|^p = 0. \tag{3.12}$$

(B) If $\beta_i < 1$ for some $i \in \{1, \ldots, N\}$, and, in addition, (3.10) holds, $\lim_{t\to\infty} \eta(t) > 0$, and there exists a convex function $\alpha \in \mathcal{K}_\infty$ and some $p \geq 2$ such that

$$\alpha(|x|^p) \leq U_i(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

then (3.12) also holds.

(C) If $\nu > \lambda$ and $0 < \beta_i < 1$ for all $i \in \{1, \ldots, N\}$, and, in addition, there exist positive constants α and p such that

$$\alpha\,|x|^p \leq V(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

then

$$E\,|x(t)|^p \leq \alpha^{-1}C_0 e^{-\rho\int_0^t c(\tau)d\tau} + \alpha^{-1}\int_0^t e^{-\rho\int_s^t c(\tau)d\tau}\gamma(s)ds, \quad t \geq 0,$$

where C_0 and ρ are positive constants specified in Theorem 3.1. In particular, (3.12) holds.

Proof. Part (C) is a straightforward corollary of part (D) of Theorem 3.1. The proofs for parts (A) and (B) are essentially the same. Therefore, we will only show the proof for part (A).

Note that (3.2) and (3.11), together with Jensen's inequality, imply that

$$\sup_{0\leq t<\infty} \alpha(E\,|x(t)|^p) < \infty. \tag{3.13}$$

Similarly, (3.3) implies that

$$\int_0^\infty c(s)\alpha(E\,|x(s)|^p)ds < \infty. \tag{3.14}$$

Since $\alpha \in \mathcal{K}_\infty$, (3.13) implies that there exists a constant $C > 0$ such that

$$\sup_{-r\leq t<\infty} E\,|x(t)|^p \leq C. \tag{3.15}$$

We claim that there exists a constant $\hat{C} > 0$ such that

$$\left|E\,|x(t)|^p - E\,|x(s)|^p\right| \leq \hat{C}(t - s), \tag{3.16}$$

for all $0 \leq s < t < \infty$. Actually, using Itô's formula, we can show that

$$\begin{aligned}\left|E\,|x(t)|^p - E\,|x(s)|^p\right| \leq E\int_s^t \Bigg[& \frac{p(p-1)}{2}\,|x(\tau)|^{p-2}\,|g(\tau, x_\tau)|^2 \\ &+ p\,|x(\tau)|^{p-1}\,|f(\tau, x_\tau)|\Bigg]d\tau. \end{aligned}\tag{3.17}$$

Invoking several elementary inequalities and by (3.10), we have

$$|x(\tau)|^{p-1}\,|f(\tau, x_\tau)| \leq \frac{p-1}{p}\,|x(\tau)|^p + \frac{1}{p}\,|f(\tau, x_\tau)|^p,$$

$$|x(\tau)|^{p-2}\,|g(\tau, x_\tau)|^2 \leq \frac{p-2}{p}\,|x(\tau)|^p + \frac{2}{p}\,|g(\tau, x_\tau)|^p,$$

and

$$\begin{aligned} E\,&|f(\tau, x_\tau)|^p \vee E\,|g(\tau, x_\tau)|^p \\ &\leq K^p 2^{p-1} + K^p 2^{p-1}\int_{-r}^0 E\,|x(\tau+\theta)|^p\, d\mu(\theta). \end{aligned}$$

In view of (3.15), putting all the above inequalities into (3.17) shows that the claim (3.16) is true. In particular, (3.16) implies that $E\,|x(t)|^p$ is uniformly continuous in t on $\mathbb{R}^+$.

Now suppose that (3.12) is not true. It follows that there exist some $\varepsilon > 0$ and a sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$, $t_{k+1} - t_k \geq 1$ for all k, and $E\,|x(t_k)|^p \geq 2\varepsilon$, for all $k \geq 1$. Since $E\,|x(t)|^p$ is uniformly continuous in t on $\mathbb{R}^+$, we can choose $\delta < 1$ such that $E\,|x(t)|^p \geq \varepsilon$, for all $t \in [t_k, t_k + \delta]$ and all $k \geq 1$, which implies that

$$\begin{aligned}\int_0^\infty c(s)\alpha(E\,|x(s)|^p)ds &\geq \sum_{k=1}^\infty \int_{t_k}^{t_k+\delta} c(s)\alpha(E\,|x(s)|^p)ds \\ &\geq \alpha(\varepsilon)\delta\sum_{k=1}^\infty c(t_k)ds = \infty, \end{aligned}$$

where the last line follows from the fact that $c(t)$ is monotonically non-increasing and $\lim_{t\to\infty} c(t) > 0$. This is a contradiction with (3.14). Therefore, (3.12) must be true. The proof is complete. □

3.2. Almost sure convergence and boundedness

In Section 3.2, we investigate the almost sure sample convergence and boundedness of the solutions of (2.1).

Theorem 3.3. *Suppose that conditions of Theorem 2.1 hold, except that (2.3) is replaced by Assumption 3.1. Then for any given initial data $\xi \in \mathcal{C}([-r,0];\mathbb{R}^n)$, there is a unique global solution $x(t)$ for (2.1) on $[t_0-r,\infty)$, which satisfies the following:*

(A)

$$\sup_{-r\le t<\infty} V(t,x(t)) < \infty, \quad a.s. \tag{3.18}$$

(B) *If $\nu > \lambda$, then*

$$\int_0^\infty c(s)V(s,x(s))ds < \infty, \quad a.s. \tag{3.19}$$

(C) *If $\beta_i < 1$ for some $i \in \{1,\dots,N\}$, then*

$$\int_0^\infty \eta(s)U_i(s,x(s))ds < \infty, \quad a.s. \tag{3.20}$$

(D) *If $\nu > \lambda$ and $0 < \beta_i < 1$ for all $i \in \{1,\dots,N\}$, and, in addition, that, for each $h > 0$, there exists a function $\zeta_h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\int_0^\infty \zeta_h(s)ds < \infty$ and*

$$|V_x(t,\varphi(0))g(t,\varphi)|^2 \le \zeta_h(t), \tag{3.21}$$

for all $t \ge 0$ and $\varphi \in \mathcal{C}([-r,0];\mathbb{R}^n)$ satisfying $\|\varphi\| \le h$, then

$$\lim_{t\to\infty} V(t,x(t)) = 0, \quad a.s. \tag{3.22}$$

Proof. To investigate almost sure sample convergence and boundedness, we apply Itô's formula to obtain

$$\begin{aligned}
&e^{\varepsilon\int_0^t c(\tau)d\tau}V(t,x(t)) - V(0,x(0)) \\
&= \int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}\left[\varepsilon c(s)V(s,x(s)) + \mathcal{L}V(s,x_s)\right]ds + M(t) \\
&\le \int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}\gamma(s)ds + (\varepsilon-\nu)\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}c(s)V(s,x(s))ds \\
&\quad + \lambda\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}c(s)\int_{-r}^0 V(s+\theta,x(s+\theta))d\mu_0(\theta)ds \\
&\quad - \sum_{i=1}^N \int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}\eta(s)U_i(s,x(s)) + M(t) \\
&\quad + \sum_{i=1}^N \beta_i\int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}\eta(s)\int_{-r}^0 U_i(s+\theta,x(s+\theta))d\mu_i(\theta),
\end{aligned}$$

where $M(t) = \int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}V_x(t,x(s))g(s,x_s)dW(s)$, which is a local martingale with $M(0) = 0$. By using the same techniques we used in the proof of Theorem 3.1, we can show that

$$\begin{aligned}
e^{\varepsilon\int_0^t c(\tau)d\tau}V(t,x(t)) &\le \xi_0 + \int_0^t e^{\varepsilon\int_0^s c(\tau)d\tau}\gamma(s)ds \\
&\quad + (\varepsilon-\nu+\lambda e^{\varepsilon r})\int_0^t c(s)V(s,x(s))ds \\
&\quad + \sum_{i=1}^N(-1+\beta_i e^{\varepsilon r})\int_0^t \eta(s)U_i(s,x(s))ds + M(t),
\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}
\xi_0 &= V(0,x(0)) + \lambda e^{\varepsilon r}\int_{-r}^0 e^{\varepsilon\int_0^s c(\tau)d\tau}c(s)V(s,x(s))ds \\
&\quad + \sum_{i=1}^N e^{\varepsilon r}\int_{-r}^0 e^{\varepsilon\int_0^s c(\tau)d\tau}\eta(s)U_i(s,x(s))ds < \infty.
\end{aligned}$$

Letting $\varepsilon = 0$ in (3.23), we obtain

$$\begin{aligned}
V(t,x(t)) &\le \xi_0 + \int_0^t \gamma(s)ds - (\nu-\lambda)\int_0^\infty c(s)V(s,x(s))ds \\
&\quad - \sum_{i=1}^N(1-\beta_i)\int_0^t \eta(s)U_i(s,x(s))ds + M_0(t),
\end{aligned}$$

for all $t \ge 0$, where $M_0(t) = \int_0^t V_x(t,x(s))g(s,x_s)dW(s)$ is a local martingale with $M_0(0) = 0$. The conclusions of (A)–(C) follow from the non-negative semimartingale convergence theorem (see, e.g., Liptser and Shiryayev (1989) or Mao (1997)). It remains to show almost sure convergence in (D).

By the definition of ρ, we have $\rho - \nu + \lambda e^{\rho r} \le 0$ and $-1 + \beta_i e^{\rho\tau} \le 0$ for all $i = 1,\dots,N$. Therefore, choosing $\varepsilon = \rho$ in (3.23) gives

$$\begin{aligned}
V(t,x(t)) &\le \xi_0 e^{-\rho\int_0^t c(\tau)d\tau} + \int_0^t e^{-\rho\int_s^t c(\tau)d\tau}\gamma(s)ds \\
&\quad + M(t)e^{-\rho\int_0^t c(\tau)d\tau},
\end{aligned} \tag{3.24}$$

where both $\xi_0 e^{-\rho\int_0^t c(\tau)d\tau}$ and $\int_0^t e^{-\rho\int_s^t c(\tau)d\tau}\gamma(s)ds$ tends to 0 as $t \to \infty$, as shown in the proof of Theorem 3.1. We only need to show that $M(t)e^{-\rho\int_0^t c(\tau)d\tau} \to 0$ a.s., as $t \to \infty$. Note that $M(t)$ is a local martingale with its quadratic variation process given by

$$[M]_t = \int_0^t e^{2\rho\int_0^s c(\tau)d\tau}\left|V_x(s,x(s))g(s,x_s)\right|^2 ds, \quad \text{a.s.} \tag{3.25}$$

In view of (2.2), which essentially says that $V(t,x)$ is radially unbounded in x, uniformly in t, it can be shown that (3.18) implies $\sup_{-r\le t<\infty}|x(t)| < \infty$, a.s. Equivalently, there exists a random variable $h(\omega)$ such that $\sup_{-r\le t<\infty}|x(t)| \le h < \infty$, a.s. Let $A(t) = e^{\rho\int_0^t c(\tau)d\tau}$. It is clear that $A(t)$ is increasing in t and $\lim_{t\to\infty}A(t) = \infty$. By (3.21) and (3.25), we have

$$\int_0^\infty \frac{d[M]_t}{(1+A(t))^2} \le \int_0^\infty |V_x(t,x(t))g(t,x_t)|^2 dt < \infty, \quad \text{a.s.}$$

It follows from the strong law of large numbers for local martingales (see, e.g., Theorem 1.3.4 of Mao (1997) or Theorem 2.10 of Liptser and Shiryayev (1989)) that

$$\lim_{t\to\infty} M(t)e^{-\rho\int_0^t c(\tau)d\tau} = \lim_{t\to\infty}\frac{M(t)}{A(t)} = 0, \quad \text{a.s.}$$

The proof is complete. □

Both (3.19) and (3.20) give certain boundedness results of $V(t,x(t))$ and $U_i(t,x(t))$ in an integral form. With some additional conditions, we can conclude almost sure asymptotic convergence of $x(t)$ from (3.19) and (3.20). For this purpose, we invoke the following weak assumption:

$$\sup_{0\le t<\infty}\big(|f(t,0)| \vee |g(t,0)|\big) < \infty, \tag{3.26}$$

which, together with the local Lipschitz condition (2.4), essentially guarantees that both f and g map bounded sets in $\mathcal{C}([-r,0];\mathbb{R}^n)$ to bounded sets in $\mathbb{R}^n$, uniformly in t.

Theorem 3.4. *Suppose that conditions of Theorem 2.1 hold, except that (2.3) is replaced by Assumption 3.1. Then for any given initial data $\xi \in \mathcal{C}([-r,0];\mathbb{R}^n)$, there is a unique global solution $x(t)$ for (2.1) on $[t_0-r,\infty)$, which satisfies the following:*

(A) *If $\nu > \lambda$ and, in addition, (3.26) holds, $\lim_{t\to\infty}c(t) > 0$, and there exists a continuous positive definite function $W : \mathbb{R}^n \to \mathbb{R}^+$ such that $W(x) \le V(t,x)$ for all $(t,x) \in [-r,\infty)\times\mathbb{R}^n$, then*

$$\lim_{t\to\infty} x(t) = 0, \quad a.s. \tag{3.27}$$

(B) *If* $\beta_i < 1$, *for some* $i \in \{1, \ldots, N\}$, *and, in addition,* (3.26) *holds,* $\lim_{t\to\infty} \eta_i(t) > 0$, *and there exists a continuous positive definite function* $W : \mathbb{R}^n \to \mathbb{R}^+$ *such that* $W(x) \le U_i(t, x)$ *for all* $(t, x) \in [-r, \infty) \times \mathbb{R}^n$, *then* (3.27) *holds.*

Proof. We shall only show the proof for case (A). The proof for case (B) is similar. Note that, since W is positive definite, (3.27) is equivalent to $\lim_{t\to\infty} W(x(t)) = 0$, a.s. By (3.19) and $\int_0^\infty c(s)ds = \infty$, we must have

$$\liminf_{t\to\infty} W(x(t))ds = 0, \quad \text{a.s.} \tag{3.28}$$

Suppose, for the sake of contradiction, that

$$P\left\{\omega : \limsup_{t\to\infty} W(x(t))ds > 0\right\} > 0. \tag{3.29}$$

Therefore, we can find $\varepsilon > 0$ such that

$$P\left\{\omega : \limsup_{t\to\infty} W(x(t))ds > 2\varepsilon\right\} > 3\varepsilon. \tag{3.30}$$

In view of (2.2), which essentially says that $V(t, x)$ is radially unbounded in x, uniformly in t, it can be shown that (3.18) implies $\sup_{-r\le t<\infty} |x(t)| < \infty$, a.s. Therefore, we can find $h = h(\varepsilon) > 0$ such that

$$P\left\{\omega : \sup_{-r\le t<\infty} |x(t)| < h\right\} \ge 1 - \varepsilon. \tag{3.31}$$

Denoting $A := \{\omega : \limsup_{t\to\infty} W(x(t))ds > 2\varepsilon\}$, and $B := \left\{\omega : \sup_{-r\le t<\infty} |x(t)| < h\right\}$, (3.30) and (3.31) imply that $P(A \cap B) > 2\varepsilon$. In view of (3.28) and (3.30), define a sequence of stopping times by

$$s_1 = \inf\{t \ge 0 : W(x(t)) \ge 2\varepsilon\},$$
$$s_{2i} = \inf\{t \ge s_{2i-1} : W(x(t)) \le \varepsilon\}, \quad i = 1, 2, \ldots,$$
$$s_{2i+1} = \inf\{t \ge s_{2i} : W(x(t)) \ge 2\varepsilon\}, \quad i = 1, 2, \ldots.$$

It is clear that $A \cap B \subset \{s_i < \infty, \forall i \ge 1\}$. Using (3.26) and the local Lipschitz condition on f and g (i.e., condition (ii) of Theorem 2.1), there exists a constant $C_h > 0$ such that $|f(t, \varphi)|^2 \vee |g(t, \varphi)|^2 \le C_h$, for $t \ge 0$, provided that $\|\varphi\| \le h$. Let $\mathbf{1}_{A\cap B}$ denote the indicator function of the set $A \cap B$. By use of Hölder's inequality and Doob's martingale inequality, we can derive that, for any given $\tau > 0$,

$$\begin{aligned}
&E\left(\mathbf{1}_{A\cap B} \sup_{i\ge1}\sup_{0\le t\le\tau} |x(s_{2i-1} + t) - x(s_{2i-1})|^2\right)\\
&\le 2E\left(\mathbf{1}_{A\cap B} \sup_{i\ge1}\sup_{0\le t\le\tau} \left|\int_{s_{2i-1}}^{t} f(s, x_s)ds\right|^2\right)\\
&\quad + 2E\left(\mathbf{1}_{A\cap B} \sup_{i\ge1}\sup_{0\le t\le\tau} \left|\int_{s_{2i-1}}^{t} g(s, x_s)dW(s)\right|^2\right)\\
&\le 2\tau E\left(\mathbf{1}_{A\cap B} \sup_{i\ge1} \int_{s_{2i-1}}^{s_{2i-1}+\tau} |f(s, x_s)|^2\, ds\right)\\
&\quad + 8E\left(\mathbf{1}_{A\cap B} \sup_{i\ge1} \int_{s_{2i-1}}^{s_{2i-1}+\tau} |g(s, x_s)|^2\, ds\right)\\
&\le 2\tau^2 C_h + 8\tau C_h,
\end{aligned}$$

which by Chebyshev's inequality, implies that

$$\begin{aligned}
&P\left\{\omega \in A \cap B : \sup_{i\ge1}\sup_{0\le t\le\tau} |x(s_{2i-1} + t) - x(s_{2i-1})| \ge \delta\right\}\\
&\le \frac{2\tau^2 C_h + 8\tau C_h}{\delta^2}.
\end{aligned}$$

Denoting

$$\Omega_0 := A \cap B \cap \left\{\omega : \sup_{i\ge1}\sup_{0\le t\le\tau} |x(s_{2i-1} + t) - x(s_{2i-1})| \le \delta\right\},$$

it follows that

$$P(\Omega_0) \ge 2\varepsilon - \frac{2\tau^2 C_h + 8\tau C_h}{\delta^2}.$$

By the fact that $W(x)$ is uniformly continuous in x for $|x| \le h$, we can choose $\delta > 0$ such that $|W(x) - W(y)| < \varepsilon/2$, whenever $|x| \vee |y| \le h$ and $|x - y| \le \delta$. It follows that

$$\begin{aligned}
&\left\{\omega : \sup_{i\ge1}\sup_{0\le t\le\tau} |x(s_{2i-1} + t) - x(s_{2i-1})| \le \delta\right\}\\
&\subset \left\{\omega : \sup_{i\ge1}\sup_{0\le t\le\tau} |W(x(s_{2i-1} + t)) - W(x(s_{2i-1}))| \le \varepsilon/2\right\}\\
&\subset \left\{\omega : \inf_{i\ge1}\{s_{2i} - s_{2i-1}\} \ge \tau\right\}.
\end{aligned}$$

We then choose $\tau > 0$ sufficiently small that $\frac{2\tau^2 C_h + 8\tau C_h}{\delta^2} < \varepsilon$, which would imply $P(\Omega_0) \ge \varepsilon$. For all $\omega \in \Omega_0$, i.e. samples within Ω_0, we have

$$\begin{aligned}
\int_0^\infty c(s)V(s, x(s))ds &\ge \sum_{i=1}^{\infty} \int_{s_{2i-1}}^{s_{2i}} c(s)W(x(s))ds\\
&\ge \varepsilon\tau \sum_{i=1}^{\infty} c(s_{2i-1}) = \infty,
\end{aligned}$$

where the last line follows from the fact that $c(t)$ is monotonically non-increasing and $\lim_{t\to\infty} c(t) > 0$. This is a contradiction with (3.19). Therefore, (3.29) must be false. The proof is complete. □

Let us conclude this main section by highlighting a few contributions of this paper. Compared with the known results (e.g., Luo et al. (2011)), the results of this section can be applied to stochastic systems with time-varying coefficients and multiple delays and/or nonlinearities. We are able to conclude both moment and almost sure convergence and boundedness of the solutions of (2.1) under a more relaxed Lyapunov-based condition on the diffusion operator, and the convergence is not limited to exponential convergence.

We would also like to point out a novel approach we have employed to conclude almost sure convergence, which is part (D) of Theorem 3.3, where a strong law of large numbers for local martingales (see, e.g., Theorem 1.3.4 of Mao (1997) or Theorem 2.10 of Liptser and Shiryayev (1989)) has been used to reason about almost sure convergence involving a continuous local martingale term.

4. Examples

In this section, we will show by several examples that the main results presented in the previous section can be applied to investigate asymptotic convergence and boundedness of more general systems, including the two motivating examples presented in Section 1.

Example 4.1. Consider Eq. (1.2). Recalling (1.3), we see that Assumption 3.1 is satisfied with $U_i \equiv 0$, $\gamma(t) = \sigma^2 c^2(t)$, $\nu = 2-\alpha$, and $\lambda = \alpha$. Since $\nu > \mu$ and β_i can be chosen to be arbitrarily small (with $U_i \equiv 0$), it follows from Theorem 3.1 that

$$\sup_{-r\le t<\infty} E\,|x(t)|^2 < \infty, \qquad \int_0^\infty c(s)E\,|x(s)|^2\, ds < \infty,$$

and, moreover,

$$E\,|x(t)|^2 \le C_0 e^{-\rho\int_0^t c(\tau)d\tau} + \sigma^2 \int_0^t e^{-\rho\int_s^t c(\tau)d\tau} c^2(s)ds \to 0,$$

as $t \to \infty$, where $\rho > 0$ is the unique root of $2 - \alpha = \rho + \alpha e^{\rho r}$. Note that for $\|\varphi\| \le h$,

$$|V_x(t,\varphi(0))g(t,\varphi)|^2 \le 2\sigma^2 h^2 c^2(t).$$

Since $\int_0^\infty c^2(s)ds < \infty$, (3.21) is satisfied with $\zeta_h = 2\sigma^2 h^2 c^2(t)$. It follows from Theorem 3.3 such that

$$\sup_{-r\le t<\infty} |x(t)|^2 < \infty, \qquad \int_0^\infty c(s)\,|x(s)|^2\,ds < \infty, \quad \text{a.s.}$$

and $\lim_{t\to\infty} x(t) = 0$, a.s.

Example 4.2. Now we return to Eq. (1.4). Recalling (1.5), we can see that Assumption 3.1 is satisfied with $c(t) = \eta(t) \equiv 1$, $\nu = \frac{5}{2}$, $\lambda = 1$, $U_1 = 2x^4$, $U_2 = 2x^6$, $\beta_1 = \beta_2 = \frac{2}{3}$, and $\gamma(t) \equiv 0$. By Theorem 3.1, we can conclude that

$$\sup_{-r\le t<\infty} E\,|x(t)|^2 < \infty, \qquad \int_0^\infty E\,|x(s)|^i\,ds < \infty, \quad i = 2, 4, 6,$$

and $E\,|x(t)|^2 \le C_0 e^{-\rho t} \to 0$, as $t \to \infty$, where $\rho = \rho_0 \wedge \rho_1$, ρ_0 is the unique root of $5/2 = \rho_0 + e^{\rho_0 r}$ and $\rho_1 = \ln(3/2)/r$. From Theorem 3.3, we also see that

$$\sup_{-r\le t<\infty} |x(t)|^2 < \infty, \qquad \int_0^\infty |x(s)|^i\,ds < \infty, \quad \text{a.s.}, i = 2, 4, 6.$$

By Theorem 3.4, we can further conclude that $\lim_{t\to\infty} x(t) = 0$, a.s.

Example 4.3. Consider the scalar SFDE

$$dx(t) = c(t)\big[-x(t) - x^3(t) + \mathcal{D}^2(x_t)\big]dt + c(t)\sigma dW(t), \qquad (4.1)$$

where $c : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\int_0^\infty c(s)ds = \infty$ and $\int_0^\infty c^2(s)ds < \infty$, σ is some constant, and $\mathcal{D}$ is a bounded linear operator from $\mathcal{C}([-r,0];\mathbb{R}^n)$ to $\mathbb{R}$ satisfying $|\mathcal{D}(\varphi)| \le \int_{-r}^0 |\varphi(\theta)|\,d\mu(\theta)$, where μ is a probability measure on $[-r,0]$. Consider $V(t,x) = x^2$. Then

$$\begin{aligned}\mathcal{L}V(t,\varphi) &= 2c(t)\varphi(0)\big[-\varphi(0) - \varphi^3(0) + \mathcal{D}^2(\varphi)\big] + \sigma^2 c^2(t)\\ &\le -c(t)\,|\varphi(0)|^2 - 2c(t)\,|\varphi(0)|^4\\ &\quad + c(t)\int_{-r}^0 |\varphi(\theta)|^4\,d\mu(\theta) + \sigma^2 c^2(t).\end{aligned} \qquad (4.2)$$

Therefore, Assumption 3.1 is satisfied with $U_1 = 2x^4$, $\beta_1 = 1/2$, $\gamma(t) = \sigma^2 c^2(t)$, $\eta(t) \equiv c(t)$, $\nu = 1$, and $\lambda = 0$. It follows from Theorem 3.1 that

$$\sup_{-r\le t<\infty} E\,|x(t)|^2 < \infty, \qquad \int_0^\infty c(s)E\,|x(s)|^4\,ds < \infty,$$

and

$$E\,|x(t)|^2 \le C_0 e^{-\rho\int_0^t c(\tau)d\tau} + \sigma^2 \int_0^t e^{-\rho\int_s^t c(\tau)d\tau} c^2(s)ds \to 0,$$

as $t \to \infty$, where $\rho = \rho_0 \wedge \rho_1$, ρ_0 is the unique root of $\rho_0 = 1$ and $\rho_1 = \ln(2)/r$. From Theorem 3.3, we can conclude that

$$\sup_{-r\le t<\infty} |x(t)|^2 < \infty, \qquad \int_0^\infty c(s)\,|x(s)|^4\,ds < \infty, \quad \text{a.s.}$$

Moreover, for $\|\varphi\| \le h$, we have $|V_x(t,\varphi(0))g(t,\varphi)|^2 \le 2\sigma^2 h^2 c^2(t)$. Therefore, Theorem 3.3 guarantees that $\lim_{t\to\infty} x(t) = 0$, a.s.

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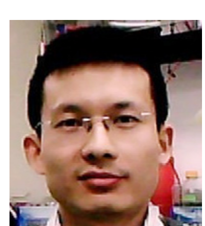

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