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INVARIANCE PRINCIPLES FOR IMPULSIVE SWITCHED SYSTEMS

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Abstract. The classical LaSalle's invariance principle is extended to a large class of impulsive switched systems. By introducing the notion of persistent limit set and persistent mode, some weak invariance principles are established, which generalize some known results to impulsive switched systems under weak dwell-time switchings. These weak invariance principles are then applied to derive several asymptotic stability criteria for impulsive switched systems. It is shown, by several examples, that even special cases of our results improve some known results in the literature.

Keywords. Switched systems; Impulsive Systems; Invariance principles; Stability; Lyapunov method; Multiple Lyapunov functions; Hybrid systems.

AMS (MOS) subject classification: 93B12, 93D20.

1 Introduction

Recently, the classical LaSalle's invariance principle [16, 17] has been extended to hybrid and switched systems by various authors (see, e.g., [1, 5, 7, 9, 10, 21, 26]). In [9], under rather general switchings (including weak dwell-time switchings), an extension of LaSalle's principle is obtained for switched linear systems. In [1], a more traditional approach is taken and the results there cover general switched nonlinear systems, while a positive dwell-time condition is assumed. In [21], the results in [1, 9] are extended and improved such that the results can deal with switched nonlinear systems with average dwell-time switching. Moreover, the weak invariance notion, which is essential to develop invariance principles for switched systems, is different from that of [1] and a more comprehensive property of the limit sets of a switched system is proved (Proposition 4.1 in [21]). The weak invariance principle improves the results in [1] (see how $\mathcal{T}_V^*$ plays a role in the argument of Example 1 in [21]). However, the results in [21] still impose a moderate restriction on the switching rules, i.e., the average dwell-time restriction. More recently, the work of [26] investigates asymptotic stability of switched linear systems with dwell-time switchings using invariance-like ideas. It is shown that, under additional assumption on the switching signals (i.e. the ergodicity assumption proposed there), sharper results can be obtained regarding the asymptotic stability of switched linear systems. The idea used there is to exploit the geometric property of the union of the null space corresponding to a common quadratic Lyapunov function. However, the result in [26] only applies to switched linear systems with dwell-time switchings satisfying the ergodicity assumption (i.e. each mode of the switched system is a persistent mode according to the definitions to be given later in the current paper), and moreover, it requires the existence of a common joint quadratic Lyapunov function (see [26] for details). Following a hybrid invariance principle derived for general hybrid systems formulated on hybrid time domains [23], the work of [7] investigates invariance principles for switched systems. The results obtained there apply to quite general switched systems under weak dwell-time switchings.

On the other hand, some invariance-like principles for switched systems are obtained in [10] by exploiting the norm-observability notions, where only a weak dwell-time condition on the switching signals is imposed. It is interesting to note that, although the results in [21] require a stronger assumption on the switching signal, a stability criterion based on zero-state small-time observability hypothesis is shown by the invariance principles developed there. Therefore, the results partially generalize those of [10] in that the zero-state small-time observability hypothesis is weaker than the small-time norm-observability assumption in [10].

Although there are various versions of invariance principles established in the literature for ordinary differential systems, similar invariance-like principles have not yet been well addressed for differential systems with impulse effects, which is in contrast with the fact that Lyapunov stability results on dynamical systems with impulse effects are extensively studied in the literature (e.g., [2, 12–15, 19, 20, 22, 25, 27]). In [5], the authors establish an invariance principle for dynamical systems with left-continuous flows, which applies to state-dependent impulsive systems as a special case. The authors point out, in the introduction of [5], that there appear to be (at least) two reasons for this situation. Namely, solutions of impulsive dynamical systems are *not* continuous in time and are *not* continuous functions of the system's initial conditions, whereas these two continuity properties are essential to establish invariance principles for ordinary differential equations. However, inspired by the weak invariance principles established in [1] and [21] for switched systems using multiple Lyapunov functions, we note that, under the notion of weak invariance, those continuity properties are not essential and, therefore, it becomes possible to establish weak invariance principles for impulsive switched systems, using a different approach from that of [5].

The main contributions of the current paper are as follows:

(i) invariance principles are established for a general class of impulsive switched systems and asymptotic stability criteria are derived as important applications of these invariance principles;

(ii) even special cases of these results are applicable to general switched nonlinear systems and improve the known results from [1, 21], while assuming rather mild restriction on the switching signals; and

(iii) the invariance principles developed here, by using a different approach from that of [5], will also cover impulsive differential systems as a special case, while, to the best knowledge of the authors, the only result available in the literature on invariance principles for impulsive systems remains the one proposed in [5].

The rest of this paper is organized as follows. In Section 2, the necessary notations and definitions are given to formulate a general impulsive switched nonlinear system. Particularly, several characterizations of the weak dwell-time notion are presented. Section 3 gives the preliminary definitions and lemmas which are necessary to prove weak invariance principles for impulsive switched systems under weak dwell-time conditions. The main results are presented in Section 4, in which Section 4.1 is devoted to prove two weak invariance principles and Section 4.2 is on applying the weak invariance principles to asymptotic stability analysis. Finally, applications of the main results are shown by several examples in Section 5. Comparisons with the previous results in the literature show that our results are less conservative and are applicable to a larger class of switched systems and impulsive systems. The paper is summarized and concluded with Section 6.

2 Notations and Definitions

Let $\mathbb{R}^n$ denote the n-dimensional real Euclidean space and $|x|$ the norm of a vector x in $\mathbb{R}^n$. Let $\mathcal{P}$ be a finite index set. By a *switching signal*, we mean a piecewise constant and right-continuous function σ from $[0,\infty)$ to $\mathcal{P}$, with only finitely many discontinuities on each bounded subinterval of $[0,\infty)$. Such points of discontinuity are called *switching times* of σ. Given an arbitrary switching signal σ, let $\varpi(\sigma)$ be the set of all switching times for σ. Without ambiguity, we may also write $\varpi(\sigma) = \{t_k : t_k < t_{k+1}, k = 1, 2, \cdots\}$. A *switched system* can be written as

$$x' = f_\sigma(x), \quad t \geq 0, \quad \sigma \in \mathcal{S}, \tag{2.1}$$

where $\{f_p : p \in \mathcal{P}\}$ is a family of vector fields and $\mathcal{S}$ denotes a certain admissible set of switchings. For a switching signal σ in $\mathcal{S}$, a vector $x_0 \in \mathbb{R}^n$, and a continuous function $x(t)$ from $[0,\infty)$ to $\mathbb{R}^n$, we call the pair (x,σ) a *solution* for (2.1) starting from x_0, if $x(t)$ is differentiable for $t \in (0,\infty)\backslash\varpi(\sigma)$ and has right-hand derivatives for $t \in \varpi(\sigma)\cup\{0\}$, and the pair (x,σ) satisfies

$$x'(t) = f_{\sigma(t)}(x(t)), \quad t \geq 0, \quad x(0) = x_0,$$

where $'$ denotes right-hand derivatives for $t \in \varpi(\sigma)\cup\{0\}$ and ordinary derivatives otherwise. It should be noted that the switching time sequence is not *a priori* known, which may depend on the evolution of the state variable.

Instead of the usual dwell-time conditions or average dwell-time conditions on the switching signals (see [8, 18]), only a rather mild condition, i.e., the weak dwell-time condition as in [9, 10], is assumed throughout this paper and formulated as follows.

Definition 2.1. A switching signal σ is said to have *weak dwell-time* $\tau > 0$ if one of the following is satisfied:

(i) for every $T \geq 0$, we can find a positive integer m such that $t_{m+1} - t_m \geq \tau$ and $t_m \geq T$, where t_m and t_{m+1} belong to $\varpi(\sigma) = \{t_k : t_k < t_{k+1}, k = 1, 2, \cdots\}$ (same for (2) below);

(ii) if there exists some $p \in \mathcal{P}$ such that, for every $T \geq 0$, we can find a positive integer m such that $t_{m+1} - t_m \geq \tau$ with $t_m \geq T$ and $\sigma(t_m) = p$ (the pth mode is called a *persistent mode* of σ); or

(iii) if there exists some $p \in \mathcal{P}$ such that the union of all the intervals of length greater than τ on which $\sigma = p$, denoted by $\mathcal{I}_p$, has an infinite Lebesgue measure (we call $\mathcal{I}_p$ a *τ-persistent domain* for σ).

In other words, the resulting switched system persistently encounters intervals of length greater than $\tau > 0$ on which the system maintain the same mode. It is easy to see that, if a switching signal has dwell-time $\tau_D > 0$, it has weak dwell-time $\tau = \tau_D$. It is also shown in [10] that, if a switching signal has average dwell-time $\tau_{AD} > 0$, then it has weak dwell-time τ for every $\tau \in (0, \tau_{AD})$. Therefore, the weak dwell-time notion is indeed weaker than dwell-time and average dwell-time.

To consider both impulses and switchings in the same framework, we introduce the notion of impulsive switching laws. By an *impulsive switching law*, we mean a pair (σ, I), where σ denotes a switching signal and $I := \{I_k : k = 1, 2, \cdots\}$ is an associated sequence of *impulse functions* from $\mathbb{R}^n$ to $\mathbb{R}^n$, which act on the state of the system at each switching time of σ. An *impulsive switched system* is defined by a family of vector fields $\{f_p : p \in \mathcal{P}\}$ and an admissible set of impulsive switching laws $\mathcal{L}$, which can be written as

$$x' = f_\sigma(x), \quad t \geq 0, \tag{2.2}$$

$$\Delta x = I_{m(t)}(x(t^-)), \quad t \in \varpi(\sigma), \tag{2.3}$$

where $(\sigma, I) \in \mathcal{L}$, $\Delta x(t) = x(t) - x(t^-)$ with as the left limit of x at t, and $m(t)$ is the position of t in the increasing sequence representation of $\varpi(\sigma)$. For example, given a switching signal σ and $\varpi(\sigma) = \{t_k : t_k < t_{k+1}, k = 1, 2, \cdots\}$, we have $m(t_k) = k$, for $k = 1, 2, \cdots$. Roughly speaking, we can say that the impulsive switched system consists of the switched system (2.2) and the difference equation (2.3).

Definition 2.2. Given an impulsive switching law $(\sigma, I) \in \mathcal{L}$, a vector $x_0 \in \mathbb{R}^n$, and a function $x(t)$ from $[0, \infty)$ to $\mathbb{R}^n$, the triple (x, σ, I) is called a *solution* to (2.2) and (2.3) starting from x_0 if (x, σ, I) satisfies

(i) $x(0) = x_0$;

(ii) $x(t)$ is differentiable on $(0, \infty) \backslash \varpi(\sigma)$ and has right-hand derivatives at $\varpi(\sigma) \cup \{0\}$;

(iii) (x, σ) satisfies (2.2) for all $t \geq 0$, where $'$ denotes right-hand derivatives for $t \in \varpi(\sigma) \cup \{0\}$ and ordinary derivatives otherwise; and

(iv) (x, I) satisfies the difference equation (2.3) for all $t \in \varpi(\sigma)$.

Remark 2.1. It should be noted that $x(t)$ is only required to be right-continuous at switching times. Moreover, the vector fields f_p are only assumed to be continuous throughout this paper. Therefore, uniqueness of solutions is not guaranteed. While we assume each sequence I of impulse functions is associated with a switching signal σ, the impulse functions can be arbitrary and obey no restrictions at this time. However, an assumption on the solution triples (x, σ, I) is proposed later in this section, which essential imposes some conditions on the impulse functions (see Assumption 2.1 and Remark 2.6).

Remark 2.2. While switching in a dynamic system models abrupt changes in the right-hand side of the system, impulses represent sudden changes in the state valuable itself, at certain discrete times. Impulse effects either naturally exist or can be intentionally added to impulsively stabilize an otherwise unstable system.

Remark 2.3. To formulate an impulsive switched system, one may either start with an impulsive differential system without switching and allow the right-hand side of the continuous evolution to switch among a family of vector fields, while the switching obeys certain regularity properties, or one can start with a switched system and add impulse effects to the switched system at certain times. In this paper, we choose the latter approach. To simplify the situation, we assume that the impulses are added only at the switching times. That is, while the switching signal effectively selects the current mode of the switched system in real-time, impulses occur at each switching time and reset the state variable. This simpler formulation can easily accommodate the situations where impulses are required more often than switchings or vice versa. If an impulse is needed between two switching times of a signal σ, an artificial switching time can be added to the set $\varpi(\sigma)$, i.e., a time that is not an actual discontinuity point of σ is added to $\varpi(\sigma)$. If a switching occurs between two impulse times, a trivial impulse can be added, i.e., a (trivial) function that does reset the state variable is added at this switching time.

Now let $\mathcal{S}$ denote an admissible set of switching signals. The set of all solution triples for (2.2) and (2.3) (or *trajectories*), while σ ranges over $\mathcal{S}$, is denoted by $\mathcal{T}$. State-dependent switchings are usually defined by a covering of the state space $\mathbb{R}^n$, which is a family of sets $\mathcal{X} := \{\mathcal{X}_p : p \in \mathcal{P}\}$ such that

$\bigcup_{p\in\mathcal{P}} \mathcal{X}_p = \mathbb{R}^n$. Let $\mathcal{T}_{\text{cover}}[\mathcal{X}]$ denote the set of triples (x, σ, I) such that

$$x(t) \in \mathcal{X}_{\sigma(t)}, \quad t \geq 0,$$

i.e., for solution triples in $\mathcal{T}_{\text{cover}}[\mathcal{X}]$, a certain mode $p \in \mathcal{P}$ can be activated at time t, only if the state valuable $x(t)$ lies in the set $\mathcal{X}_p$. This notion of covering-induced state-dependent switching can easily accommodate purely time-dependent switchings if we use the trivial covering $\mathcal{X}_p = \mathbb{R}^n$, $\forall p \in \mathcal{P}$. Now let $\mathcal{S}_{\text{weak}}(\tau)$ $(\tau > 0)$ denote all the switching signals that have weak dwell-time τ. The corresponding set of solution triples is denoted by $\mathcal{T}_{\text{weak}}(\tau)$ or $\mathcal{T}_{\text{weak}}(\tau) \cap \mathcal{T}_{\text{cover}}[\mathcal{X}]$ if state-dependent switchings are also considered. In the remaining of this paper, it is assumed that

$$\mathcal{T} \subset \mathcal{T}_{\text{weak}}(\tau) \cap \mathcal{T}_{\text{cover}}[\mathcal{X}],$$

for some $\tau > 0$ and some covering $\mathcal{X}$ of $\mathbb{R}^n$.

If we assume $f_p(0) = 0$ for all $p \in \mathcal{P}$ such that $0 \in \mathcal{X}_p$, then (2.2) and (2.3) have trivial solutions $(0, \sigma, 0)$ in $\mathcal{T}$, where the last 0 in the triple denotes a sequence of trivial impulse functions which map $\mathbb{R}^n$ to 0.

Definition 2.3. The trivial solutions in $\mathcal{T}$ of (2.2) and (2.3) are said to be

($\mathbb{S}_1$) *stable* if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that, for each $(x, \sigma, I) \in \mathcal{T}$ starting from x_0, $|x_0| < \delta$ implies that $|x(t)| < \varepsilon$ for all $t \geq 0$;

($\mathbb{S}_2$) *asymptotically stable* if ($\mathbb{S}_1$) is satisfied and there exists some $\rho > 0$ such that $|x_0| < \rho$ implies that $\lim_{t\to\infty} x(t) = 0$;

($\mathbb{S}_3$) *globally asymptotically stable* if ($\mathbb{S}_2$) is satisfied with arbitrary $\rho > 0$;

($\mathbb{S}_4$) *unstable* if ($\mathbb{S}_1$) fails.

Definition 2.4. A family of functions $\{V_p : p \in \mathcal{P}\}$ from $\mathbb{R}^n$ to $\mathbb{R}$ are called *multiple Lyapunov functions* for (2.2) and (2.3) on a set $G \subset \mathbb{R}^n$ if

(i) V_p is continuously differentiable at each point in $G \cap \mathcal{X}_p$ and is continuous on $\overline{G \cap \mathcal{X}_p}$, the closure of $G \cap \mathcal{X}_p$;

(ii) the derivative of each V_p along the trajectories of the pth mode of (2.2) and (2.3) satisfies

$$V_p'(x) := \nabla V_p(x) \cdot f_p(x) \leq 0,$$

for all $x \in G \cap \mathcal{X}_p$, where ∇ is the gradient.

Remark 2.4. If the covering is not a trivial one, defining the multiple Lyapunov function conditions based on a covering $\mathcal{X}$ actually relaxes the assumptions on each function V_p (see Remark 5.4 after Example 5.6 in Section 5).

Definition 2.5. A family of functions $\{V_p : p \in \mathcal{P}\}$ from $\mathbb{R}^n$ to $\mathbb{R}$ is called *positive definite* on $G \subset \mathbb{R}^n$ if

(i) for each $p \in \mathcal{P}$, $V_p(x) \geq 0$ for all $x \in G \cap \mathcal{X}_p$;

(ii) $V_p(x) = 0$ if and only if $x = 0$ and $0 \in \mathcal{X}_p$.

The family of function is called *nonnegative definite* if only (i) is satisfied.

Definition 2.6. A family of functions $\{V_p : p \in \mathcal{P}\}$ from $\mathbb{R}^n$ to $\mathbb{R}$ is called *radially unbounded* if $V_p(x) \to \infty$ as $|x| \to \infty$ and $x \in \mathcal{X}_p$.

The following assumption imposes a condition on the evolution of the functions V_p along a solution at switching instants. This type of conditions are typically encountered in results involving multiple Lyapunov functions (see [4] and [1, 9, 10, 18, 21]).

Assumption 2.1. The following property holds for all $(x, \sigma, I) \in \mathcal{T}$: for every pair of switching instants $t_j < t_k$ of σ such that $\sigma(t_j) = \sigma(t_k) = p \in \mathcal{P}$, we have

$$V_p(x(t_k)) \leq V_p(x(t_{j+1})). \tag{2.4}$$

Remark 2.5. In other words, Assumption 2.1 says that the value of $V_{\sigma(t)}(x(t))$ at the beginning of each interval on which $\sigma = p$ does not exceed the value of V_p at the end of previous such interval (if one exists). Therefore, inequality (2.4), together with the fact that $\{V_p\}$ is a family of multiple Lyapunov functions for (2.2) and (2.3), ensures that $V_{\sigma(t)}(x(t))$ is nonincreasing on the union of all the intervals where the pth mode of (2.2) and (2.3) is activated, which can guarantee local stability of (2.2) and (2.3) (see [4] for the case of switched systems without impulses). Assumption 2.1 is trivially satisfied in the case when all the V_p are equal (common Lyapunov function) and there are no impulse effects.

Remark 2.6. It is worth noting that, with the presence of impulse effects, Assumption 2.1 is *not* trivially satisfied for a common Lyapunov function V. Additional conditions have to be imposed, e.g., $V(x(t_k)) \leq V(x(t_k^-))$ at all switching instants t_k as in [5]. This kind of conditions can be easily satisfied by impulse control, which, in the case of multiple Lyapunov functions, can also contribute to relax the conditions on V_p imposed by Assumption 2.1.

If there exists a common Lyapunov function V for (2.2) and (2.3), the following condition on the common Lyapunov function V and the impulse functions affiliated with the set $\mathcal{T}$ (i.e., all the impulse functions included in the solution triples from $\mathcal{T}$) will ensure that Assumption 2.1 is satisfied.

Condition 2.1. Each impulse function δ affiliated with the set $\mathcal{T}$ verifies that

$$V(x + \delta(x)) \leq V(x), \quad \forall x \in \mathbb{R}^n.$$

Condition 2.1 is easy to check in that it requires no information on the solution function x. It is obvious that Condition 2.1 is trivially satisfied when there are no impulse effects.

3 Preliminaries Results

In [9], the author showed, by an example, that LaSalle's invariance principle for ordinary differential equations cannot be applied directly to switched systems. One has to impose some restrictions on the switching property of the systems. In [1], the invariance principles are established for switched systems with nonvanishing dwell-time. In [21], the dwell-time restriction is further weakened to average dwell-time switching. Along this line, we are interested in establishing some invariance principles for switched systems under an even larger class of switching, i.e., weak dwell-time switching as defined in Definition 2.1. To this end, we need to formulate some preliminaries for developing invariance principles for impulsive switched systems, which is the purpose of this section.

Now let $(x, \sigma, I) \in \mathcal{T}$ be a specific solution triple of (2.2), and (2.3) and recall that $\mathcal{T} \subset \mathcal{T}_{\text{weak}}(\tau)$ for some $\tau > 0$, i.e., the switching signal σ has a positive weak dwell-time τ. Let $p \in \mathcal{P}$ be a persistent mode and $\mathcal{I}_p$ denote the corresponding τ-persistent domain.

Definition 3.1. Given $p \in \mathcal{P}$, a point $\eta \in \mathbb{R}^n$ is said to be a *persistent limit point* of (x, σ, I) in the pth mode, if p is a persistent mode and there exists a sequence of $s_n \in \mathcal{I}_p$, with $s_n \to \infty$ as $n \to \infty$, such that $x(s_n) \to \eta$ as $n \to \infty$. The set of all such points is called the *persistent limit set* of (x, σ, I) in the pth mode and is denoted by $\Omega_p(x, \sigma, I)$.

Definition 3.2. A set $M \subset \mathbb{R}^n$ is called a *weakly invariant set* with respect to the pth mode of (2.2), if, for any $\xi \in M$, there exist a positive number r and a continuously differentiable function ϕ defined on some interval $[\alpha, \beta]$, with $\alpha \le 0 \le \beta$ and $\beta - \alpha \ge r$, such that

(i) $\phi'(t) = f_p(\phi(t))$, $\forall t \in [\alpha, \beta]$,

(ii) $\phi(0) = \xi$,

(iii) $\phi(t) \in M$, $\forall t \in [\alpha, \beta]$.

Definition 3.3. The solution triple (x, σ, I) is said to *weakly approach* a set $M \subset \mathbb{R}^n$ in the pth mode as $t \to \infty$, if the pth mode persists and

$$\lim_{\substack{t \to \infty \\ t \in \mathcal{I}_p}} \operatorname{dist}(x(t), M) = 0,$$

where $\operatorname{dist}(y, M)$ for $y \in \mathbb{R}^n$ is defined by

$$\operatorname{dist}(y, M) = \inf_{z \in M} |y - z| .$$

Remark 3.1. The convergence in Definition 3.3 is called "weak approaching" because the limit is only taken for $t \in \mathcal{I}_p$, not the entire real line.

Lemma 3.1. *If* $(x, \sigma, I) \in \mathcal{T}$ *is such that* $x(t)$ *is bounded on* $[0, \infty)$ *and* $p \in \mathcal{P}$ *denotes a persistent mode of* σ*, then* $\Omega_p(x, \sigma, I)$ *is a nonempty, compact, and weakly invariant set w.r.t. the* p*th mode of (2.2) and (2.3). Moreover,* $x(t)$ *weakly approaches* $\Omega_p(x, \sigma, I)$ *in the* p*th mode as* $t \to \infty$*.*

Proof. Since the persistent domain $\mathcal{I}_p$ has an infinite Lebesgue measure, one can pick up a sequence $\{s_n\}$ in $\mathcal{I}_p$ such that $s_n \to \infty$ as $n \to \infty$. Since $x(t)$ is bounded on $[0, \infty)$, it follows that $\{x(s_n)\}$ is a bounded sequence and therefore has a subsequence which converges to some limit point p. By definition, p is a persistent limit point in the pth mode, which shows $\Omega_p(x, \sigma, I)$ is nonempty.

Since $x(t)$ is bounded, it follows that $\Omega_p(x, \sigma, I)$ is bounded. To show closedness of $\Omega_p(x, \sigma, I)$, suppose $\xi_n \in \Omega_p(x, \sigma, I)$ approaches ξ as $n \to \infty$. Since $\xi_n \in \Omega_p(x, \sigma, I)$ for each n, by Definition 3.1, one can choose $s_n \in \mathcal{I}_p$, for each n, large enough such that $|x(s_n) - \xi_n| < 1/n$. Now given any $\varepsilon > 0$, choose n large enough so that $|\xi_n - \xi| < \varepsilon/2$ and $|x(s_n) - \xi_n| < \varepsilon/2$. Then $|x(s_n) - \xi| < \varepsilon$ for n large enough, which shows $\xi \in \Omega_p(x, \sigma, I)$ and therefore $\Omega_p(x, \sigma, I)$ is closed. It follows that $\Omega_p(x, \sigma, I)$ is compact.

The last assertion of the lemma can be shown by contradiction. Suppose that there exists an increasing sequence of s_n in $\mathcal{I}_p$, with $s_n \to \infty$ as $n \to \infty$, and a $\delta > 0$ such that $|x(s_n) - \xi| \geq \delta$ for all $\xi \in \Omega_p(x, \sigma, I)$. Now since $x(s_n)$ is a bounded sequence, there exists a subsequence of $x(s_n)$ which converges to some $\xi \in \Omega_p(x, \sigma, I)$. This contradicts with the inequality above and shows that the last assertion of the lemma holds.

Finally, we show that $\Omega_p(x, \sigma, I)$ is weakly invariant w.r.t. the pth mode of (2.2) and (2.3), i.e., for any ξ in $\Omega_p(x, \sigma, I)$, there exist a positive number r and a continuously differentiable function ϕ defined on some interval $[\alpha, \beta]$, with $\alpha \leq 0 \leq \beta$ and $\beta - \alpha \geq r$, such that (i) $\phi'(t) = f_p(\phi(t))$, $\forall t \in [\alpha, \beta]$, (ii) $\phi(0) = \xi$, (iii) $\phi(t) \in \Omega_p(x, \sigma, I)$, $\forall t \in [\alpha, \beta]$.

Since $\xi \in \Omega_p(x, \sigma, I)$, there exists an increasing sequence of $s_n \in \mathcal{I}_p$ such that $s_n \to \infty$ and $x(s_n) \to \xi$ as $n \to \infty$. Moreover, we can pick s_n so that there exists a sequence of intervals $[\tau_{2n-1}, \tau_{2n}]$ which verifies that, for all n,

(i) $\tau_{2n} - \tau_{2n-1} \geq \tau$,

(ii) $s_n \in [\tau_{2n-1}, \tau_{2n}]$,

(iii) $\sigma = p$ on $[\tau_{2n-1}, \tau_{2n}]$.

By this choice, $x(t)$ satisfies the pth subsystem on $[\tau_{2n-1}, \tau_{2n}]$ for all n, i.e., $x'(t) = f_p(x(t))$, $\forall t \in [\tau_{2n-1}, \tau_{2n}]$. Moreover, we have that $x(t)$ is continuously differentiable on (τ_{2n-1}, τ_{2n}). Putting

$$\alpha_n = \tau_{2n-1} - s_n \quad \text{and} \quad \beta_n = \tau_{2n} - s_n, \tag{3.1}$$

then $\beta_n - \alpha_n \geq \tau$ and $\alpha_n \leq 0 \leq \beta_n$. Define

$$\phi_n(t) := x(t + s_n), \quad t \in [\alpha_n, \beta_n]. \tag{3.2}$$

It follows that ϕ_n satisfies

$$\phi_n'(t) = f_p(\phi_n(t)), \quad t \in [\alpha_n, \beta_n], \tag{3.3}$$

$\phi_n(0) = x(s_n) \to \xi$ as $n \to \infty$, and ϕ_n is continuously differentiable on (α_n, β_n).

We claim that the sequence of intervals $[\alpha_n, \beta_n]$ has a subsequence, still designated by $[\alpha_n, \beta_n]$, which has a common subinterval $[\alpha, \beta]$, i.e., $[\alpha, \beta] \subset [\alpha_n, \beta_n]$ for all n, with $\beta - \alpha \geq \tau/2$ and $\alpha \leq 0 \leq \beta$. Actually, it is clear that either $\{\alpha_n\}$ has a subsequence, which we can keep the same designation, that lies in $(-\infty, -\tau/2]$ or it has a subsequence in $[-\tau/2, 0]$. In the latter case, since $\beta_n - \alpha_n \geq \tau$, one must have $\beta_n \geq \tau/2$ and therefore letting $\alpha = 0$, $\beta = \tau/2$ will give the required common subinterval; in the former case, since $\beta_n \geq 0$, letting $\alpha = -\tau/2$, $\beta = 0$ gives the required common subinterval. Now according to (3.2) and (3.3), what we have obtained is a sequence of functions ϕ_n defined on a common interval $[\alpha, \beta]$, with $\beta - \alpha \geq \tau/2$ and $\alpha \leq 0 \leq \beta$, such that

$$\phi_n'(t) = f_p(\phi_n(t)), \quad t \in [\alpha, \beta]. \tag{3.4}$$

We proceed to show that ϕ_n has a subsequence that uniformly converges to a function ϕ on $[\alpha, \beta]$. Since $x(t)$ is bounded, it follows that ϕ_n is uniformly bounded on $[\alpha, \beta]$. Since f_p is continuous, it follows that $\phi_n'(t) = f_p(\phi_n(t))$ is uniformly bounded on $[\alpha, \beta]$. By the mean-value theorem, this implies that the sequence ϕ_n is equicontinuous on $[\alpha, \beta]$. By the Arzela-Ascoli Theorem, there exists a subsequence of ϕ_n, still designated by ϕ_n, uniformly converges to some function ϕ on $[\alpha, \beta]$ and ϕ is continuously differentiable on (α, β). Passing the limit in (3.4) (or its equivalent integral form), one is able to see that ϕ satisfies $\phi'(t) = f_p(\phi(t))$, $t \in [\alpha, \beta]$. Moreover, $\phi(0) = \lim_{n\to\infty} \phi_n(0) = \xi$. Finally, we have $\phi(t) = \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} x(t + s_n)$ for any fixed $t \in [\alpha, \beta]$. For a fixed $t \in [\alpha, \beta]$, put $s_n' = t + s_n$. According to (3.1), $s_n' \in [\tau_{2n-1}, \tau_{2n}]$ for each n. Since $\sigma = p$ on $[\tau_{2n-1}, \tau_{2n}]$ and $\tau_{2n} - \tau_{2n-1} \geq \tau$, by Definition 3.1, it follows that $\phi(t) \in \Omega_p(x, \sigma, I)$ for all $t \in [\alpha, \beta]$ as required. Therefore $\Omega_p(x, \sigma, I)$ is shown to be weakly invariant w.r.t. the pth mode of (2.2) and (2.3) and the proof is complete. □

4 Main Results

4.1 Weak invariance principles

Let $\{V_p : p \in \mathcal{P}\}$ be a family of multiple Lyapunov functions for (2.2) and (2.3) on $G \subset \mathbb{R}^n$ and define

$$E_p := \left\{ x \in \overline{G \cap \mathcal{X}_p} : V_p'(x) = 0 \right\}.$$

Let M_p denote the largest weakly invariant set w.r.t. the pth mode of (2.2) and (2.3) in E_p.

Theorem 4.1. *Let $\{V_p : p \in \mathcal{P}\}$ be a family of multiple Lyapunov functions for (2.2) and (2.3) on G, $(x, \sigma, I) \in \mathcal{T}$ be a bounded solution of (2.2) and (2.3) such that $x(t)$ remains in G for $t \geq 0$, and $p \in \mathcal{P}$ be a persistent mode of σ. Suppose, in addition, Assumption 2.1 is satisfied. Then (x, σ, I) weakly approaches $M_p \cap V_p^{-1}(c)$, for some c, in the pth mode as $t \to \infty$.*

Proof. By Lemma 3.1, (x, σ, I) has a nonempty persistent limit set in the pth mode $\Omega_p(x, \sigma, I)$. We proceed to show that $\Omega_p(x, \sigma, I) \subset E_p$. Let $\mathcal{I}_p$ denote the union of all the intervals of length greater than τ such that the pth mode is active. Since the pth mode persists, $\mathcal{I}_p$ must have an infinite Lebesgue measure. The conditions on V_p imply that $V_p(x(t))$ is nonincreasing on $\mathcal{I}_p$. Moreover, $V_p(x(t))$ is bounded below since $x(t)$ is bounded. Therefore, as $t \to \infty$ in $\mathcal{I}_p$, $V_p(x(t))$ yields a limit as

$$\lim_{\substack{t\to\infty \\ t\in\mathcal{I}_p}} V_p(x(t)) = c.$$

For any $\xi \in \Omega_p(x, \sigma, I)$, there exists a sequence $s_n \in \mathcal{I}_p$ such that $s_n \to \infty$ and $x(s_n) \to \xi$ as $n \to \infty$. It follows by the continuity of V_p that

$$V_p(\xi) = \lim_{n\to\infty} V_p(x(s_n)) = \lim_{\substack{t\to\infty \\ t\in\mathcal{I}_p}} V_p(x(t)) = c.$$

Hence, $V_p(\xi) = c$ for all $\xi \in \Omega_p(x, \sigma, I)$ and $\Omega_p(x, \sigma, I) \subset V_p^{-1}(c)$. According to Lemma 3.1, $\Omega_p(x, \sigma, I)$ is weakly invariant w.r.t. the pth mode, that is, for each $\xi \in \Omega_p(x, \sigma, I)$, there exist a positive number r and a continuous differentiable function ϕ defined on some interval $[\alpha, \beta]$, with $\alpha \leq 0 \leq \beta$ and $\beta - \alpha \geq r$, such that (i) $\phi'(t) = f_p(\phi(t))$, $\forall t \in [\alpha, \beta]$, (ii) $\phi_0 = \xi$, (iii) $\phi(t) \in \Omega_p(x, \sigma, I)$, $\forall t \in [\alpha, \beta]$. Hence, $V_p(\phi(t)) = c$ for all $t \in [\alpha, \beta]$. Differentiating $V_p(\phi(t))$ at $t = 0$ gives

$$V_p'(\xi) = \nabla V_p(\xi) \cdot f_p(\xi) = 0.$$

It follows that $\Omega_p(x, \sigma, I) \subset E_p$. By the definition of M_p and because $\Omega_p(x, \sigma, I)$ is weakly invariant w.r.t. the pth mode of (2.2), we have $\Omega_p(x, \sigma, I) \subset M_p \subset E_p$. From Lemma 3.1, (x, σ, I) weakly approaches $\Omega_p(x, \sigma, I)$ in the pth mode and therefore it weakly approaches $M_p \cap V_p^{-1}(c)$ in the pth mode. This completes the proof. □

Note that Theorem 4.1 does not require the functions $\{V_p : p \in \mathcal{P}\}$ to be positive definite. However, it does require that the solutions are bounded and remain in the set G for all $t \geq 0$. These assumptions on the solutions can be avoided, if we make additional assumptions on the Lyapunov functions $\{V_p : p \in \mathcal{P}\}$. Some notations are introduced before we state the second invariance-like principle.

Now for any given real number l, define

$$\Lambda_l := \left\{x \in \mathbb{R}^n : V_p(x) < l \text{ for all } p \in \mathcal{P} \text{ such that } x \in \mathcal{X}_p\right\}$$

and

$$\Upsilon_l := \bigcup_{p\in\mathcal{P}} \{x \in \mathcal{X}_p : V_p(x) < l\}.$$

Assume Υ_{l_0} is bounded for some l_0. Find $l_k \geq l_{k-1}$, $1 \leq k \leq N-1$, inductively such that

$$\Upsilon_{l_{k-1}} \subset \Lambda_{l_k}, \quad 1 \leq k \leq N-1, \tag{4.1}$$

where N is the cardinality of $\mathcal{P}$. It is necessary to assume Υ_{l_k}, $0 \leq k \leq N-2$, to be bounded to justify each inductive step. This procedure is guaranteed by the following assumption.

Assumption 4.1. For some l_0, there exists l_k, $1 \leq k \leq N-1$, such that the above procedure of constructing $\{\Upsilon_k\}$ and $\{\Lambda_k\}$ can be completed and (4.1) holds.

It is easy to see that Assumption 4.1 is readily verified if $\{V_p : p \in \mathcal{P}\}$ is radially unbounded.

Theorem 4.2. *Suppose that there exist a family of functions $\{V_p : p \in \mathcal{P}\}$ and a constant l_0 such that Assumption 4.1 holds and $\{V_p : p \in \mathcal{P}\}$ forms a family of multiple Lyapunov functions on $\Upsilon_{l_{N-1}}$ for (2.2) and (2.3). Suppose, in addition, Assumption 2.1 is satisfied and $\Upsilon_{l_{N-1}}$ is bounded. Let (x, σ, I) be a solution triple starting from $x_0 \in \Lambda_{l_0}$ and $p \in \mathcal{P}$ be a persistent mode of σ. Then $x(t)$ stays in $\Upsilon_{l_{N-1}}$ for all $t \geq 0$ and (x, σ, I) weakly approaches $M_p \cap V_p^{-1}(c)$, for some c, in the pth mode as $t \to \infty$.*

Proof. Given a solution triple (x, σ, I) satisfying the theorem conditions, we shall show that $x_0 \in \Lambda_{l_0}$ implies $x(t) \in \Upsilon_{l_{N-1}}$ for all $t \geq 0$. Let $\{t_k : t_k < t_{k+1}, k = 1, 2, \cdots\} = \varpi(\sigma)$ be the consecutive switching instants of σ. Assume that, on $[0, t_1]$, the jth mode is active, i.e. $\sigma(0) = j$. Conditions on V_j implies that $V_j(x(t))$ is decreasing on $[0, t_1]$. Since $x_0 \in \Lambda_{l_0}$ implies $V_j(x_0) < l_0 \leq l_{N-1}$, it follows that

$$V_j(x(t)) \leq V_j(x_0) < l_0 \leq l_{N-1}, \quad t \in [0, t_1], \tag{4.2}$$

which particularly implies $x(t_1) \in \Upsilon_{l_0}$. Suppose, on $[t_1, t_2]$, the ith $(i \neq j)$ mode is active. Since $x(t_1) \in \Upsilon_{l_0}$, by Assumption 4.1, it follows that $x(t_1) \in \Upsilon_{l_0} \subset \Lambda_{l_1}$ and, therefore, $V_i(x(t_1)) < l_1$. Since $V_i(x(t))$ decreases on $[t_1, t_2]$, we have

$$V_i(x(t)) \leq V_i(x(t_1)) < l_1 \leq l_{N-1}, \quad t \in [t_1, t_2]. \tag{4.3}$$

On $[t_2, t_3]$, there are two possible cases: either the jth mode is active again or some mth $(m \neq j, i)$ mode is active on $[t_2, t_3]$.

(i) In the former case, by Assumption 2.1, inequality (4.2), and the fact that $V_j(x(t))$ is deceasing on $[t_2, t_3]$, we have

$$V_j(x(t)) \leq V_j(x(t_2)) \leq V_j(x(t_1)) < l_0 \leq l_{N-1}, \quad t \in [t_2, t_3]. \tag{4.4}$$

(ii) In the latter case, (4.3) implies that $x(t_2) \in \Upsilon_{l_1}$ and, by Assumption 4.1, it follows that $x(t_2) \in \Upsilon_{l_1} \subset \Lambda_{l_2}$. Therefore, $V_m(x(t_2)) < l_2$. Since $V_m(x(t))$ decreases on $[t_2, t_3]$, we have

$$V_m(x(t)) \leq V_m(x(t_2)) < l_2 \leq l_{N-1}, \quad t \in [t_2, t_3]. \tag{4.5}$$

Based on (4.2), (4.3), (4.4), and (4.5), we propose the inductive assumption that

$$V_{\sigma(t_k)}(x(t)) < l_{n_k} \leq l_{N-1}, \quad t \in [t_k, t_{k+1}], \tag{4.6}$$

holds for $k = 1, 2, \cdots, k'$, where k' is a positive integer and n_k $(k = 1, 2, \cdots)$ is the number of different modes that have been activated up to the instant $t = t_k$ and are different from the jth mode activated at $t = 0$. It is easy to observe that $n_{k_1} \leq n_{k_2}$, for $k_1 \leq k_2$, and $n_k \leq N - 1$, for all k (since there are only $N - 1$ modes besides the one activated at $t = 0$). We shall show that (4.6) still holds for $k = k' + 1$. Actually, on $[t_{k'+1}, t_{k'+2}]$, there are two possible cases: either some mode activated before is activated at $t = t_{k'+1}$ again, i.e. $\sigma(t_{k'+1}) = \sigma(t_{k_0})$ for some $k_0 \leq k'$, or a new mode in the family $\mathcal{P}$ is activated for the first time, i.e. $\sigma(t_{k'+1}) \neq \sigma(t_k)$ for all $k \leq k'$.

(i) In the former case, by Assumption 2.1, the inductive assumption, and the fact that $V_{\sigma(t_{k'+1})}(x(t))$ is decreasing on $[t_{k'+1}, t_{k'+2}]$, we have

$$\begin{aligned} V_{\sigma(t_{k'+1})}(x(t)) &\leq V_{\sigma(t_{k'+1})}(x(t_{k'+1})) \leq V_{\sigma(t_{k_0})}(x(t_{k_0+1})) \\ &< l_{n_{k_0}} \leq l_{n_{k'+1}} \leq l_{N-1}, \quad t \in [t_{k'+1}, t_{k'+2}]. \end{aligned} \tag{4.7}$$

(ii) In the latter case, we have $n_{k'+1} = n_{k'}+1$ (since a new mode is activated at $t = t_{k'+1}$) and, by the inductive assumption,

$$V_{\sigma(t_{k'})}(x(t)) < l_{n_{k'}} \leq l_{N-1}, \quad t \in [t_{k'}, t_{k'+1}],$$

which particularly implies $x(t_{k'+1}) \in \Upsilon_{l_{n_{k'}}}$. By Assumption 4.1, it follows that $x(t_{k'+1}) \in \Lambda_{l_{n_{k'}+1}} = \Lambda_{l_{n_{k'+1}}}$ and therefore $V_{\sigma(t_{k'+1})}(x(t_{k'+1})) < l_{n_{k'+1}}$. Since $V_{\sigma(t_{k'+1})}(x(t))$ is decreasing on $[t_{k'+1}, t_{k'+2}]$, we have

$$V_{\sigma(t_{k'+1})}(x(t)) \leq V_{\sigma(t_{k'+1})}(x(t_{k'+1})) < l_{n_{k'+1}} \leq l_{N-1}, \quad t \in [t_{k'+1}, t_{k'+2}]. \tag{4.8}$$

Either (4.7) or (4.8) shows that (4.6) holds for $k = k' + 1$. By induction on k, (4.6) must hold for all positive integers k, which implies $x(t) \in \Upsilon_{l_{N-1}}$ for all $t \geq 0$. Since $\Upsilon_{l_{N-1}}$ is bounded, it follows that $x(t)$ is a bounded solution. Now use Theorem 4.1 and the proof is complete. □

Remark 4.1. It can be seen from the proof of Theorem 4.1 that Theorem 4.1 and Theorem 4.2 still hold, if Assumption 2.1 is satisfied only for a given $p \in \mathcal{P}$ such that p is a persistent mode of σ.

4.2 Stability criteria

Now we apply the invariance principles established in Section 4.1 to derive some results on asymptotic stability for impulsive switched systems under weak dwell-time conditions. Let

$$\mathcal{T}_p := \{(x, \sigma, I) \in \mathcal{T} \text{ such that } \sigma \text{ persists in the } p\text{th mode}\}.$$

Theorem 4.3. *Suppose that there exist a family of positive definite functions $\{V_p : p \in \mathcal{P}\}$ and a constant $l_0 > 0$ such that Assumption 4.1 holds and $\{V_p : p \in \mathcal{P}\}$ forms a family of multiple Lyapunov functions on $\Upsilon_{l_{N-1}}$ for (2.2) and (2.3). Suppose, in addition, Assumption 2.1 is satisfied and $\Upsilon_{l_{N-1}}$ is bounded. Then the trivial solutions of (2.2) and (2.3) in $\mathcal{T}_p \subset \mathcal{T} \subset \mathcal{T}_{weak}(\tau) \cap \mathcal{T}_{cover}[\mathcal{X}]$ are asymptotically stable, provided that $M_p = \{0\}$. If $\bigcup_{p\in\mathcal{P}} M_p = \{0\}$, then the trivial solutions of (2.2) and (2.3) in $\mathcal{T} \subset \mathcal{T}_{weak}(\tau) \cap \mathcal{T}_{cover}[\mathcal{X}]$ are asymptotically stable.*

Proof. We first show local stability of the solutions. Conditions on V_p imply that there exist two continuous and nondecreasing functions $u(s)$ and $v(s)$, satisfying that $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$, such that

$$u(|x|) \leq V_p(x) \leq v(|x|),$$

for all $p \in \mathcal{P}$ (see, e.g., Lemma 4.3 in [11]) in a certain neighborhood of the origin, say $B_\rho = \{|x| < \rho\}$ for some $\rho > 0$. Now given any $\varepsilon \in (0, \rho)$, we let $\delta_0 = \varepsilon$ and define $\delta_1, \delta_2, \cdots, \delta_N$ recursively such that $\delta_{j+1} < \delta_j$ and $v(\delta_{j+1}) < u(\delta_j)$, $j = 0, 1, 2, \cdots, N-1$. Stability follows from the following claim.

Claim: $|x_0| < \delta_N$ implies $|x(t)| < \varepsilon$ for all $t \geq 0$.

Proof of the Claim: Assume t_1 is the first switching instant and the ith subsystem is activated on $[t_0, t_1]$. Then conditions on V_i imply that, for $t \in [0, t_1]$,

$$u(|x(t)|) \leq V_i(x(t)) \leq V_i(x_0) \leq v(|x_0|) \leq v(\delta_N) < u(\delta_{N-1}) \leq u(\varepsilon).$$

This gives $|x(t)| < \delta_{N-1} \leq \varepsilon$, $t \in [0, t_1]$. The claim is proved on the first dwell interval $[0, t_1]$. Particularly, we have $|x(t_1)| < \delta_{N-1}$. Now assume t_2 is the second switching instant and jth subsystem is activated on $[t_1, t_2]$. Assume, with out loss of generality, that $N \geq 2$ and $j \neq i$. Then conditions on V_j imply that, on $[t_1, t_2]$,

$$u(|x(t)|) \leq V_j(x(t)) \leq V_j(x(t_1)) \leq v(|x(t_1)|) \leq v(\delta_{N-1}) < u(\delta_{N-2}) \leq u(\varepsilon).$$

This gives $|x(t)| < \delta_{N-2} \leq \varepsilon$, $t \in [t_1, t_2]$. The claim is proved on the first dwell interval $[t_1, t_2]$. Particularly, we have $|x(t_2)| < \delta_{N-2}$. The remaining of this proof is to repeat the above procedure over all switching instants. At each switching time, we may either encounter the case where a subsystem among the family is activated for the first time or a subsystem activated once before is activated again. In the former case, we can use the previous argument and show $|x(t)| < \delta_{N-m} \leq \varepsilon$ on the current dwell interval, where $m \leq N$ is such that $|x(t)| < \delta_{N-m+1} \leq \varepsilon$ holds on the previous dwell interval. In the latter case, Assumption 2.1 ensures that $|x(t)| < \delta_{N-n} \leq \varepsilon$ is conserved for the current dwell interval, where n is the number of subsystems that have been activated up to (and including) the current instant (not counting multiplicity). Since there exist only N subsystems in the family, the former case can occur at most N times. Therefore $|x(t)| < \delta_0 = \varepsilon$ is guaranteed for all $t \geq 0$. The claim is proved and local stability follows.

Then we show that there exists a small neighborhood of the origin such that any solution triple in $\mathcal{T}_p$ starting from this small neighborhood tends to zero as $t \to \infty$. Actually since all V_p are positive definite, this neighborhood, say B_ρ, can be chosen to be the one such that $B_\rho \subset \Lambda_{l_0}$. Then, by Theorem 4.2, any solution triple that starts from B_ρ remains in $\Upsilon_{l_{N-1}}$ and weakly approaches $M_p = \{0\}$ in the pth mode as $t \to \infty$. Now for any $\varepsilon > 0$, let $\delta(\varepsilon) > 0$ be the local stability constant, i.e., a constant δ such that any solution starting from this δ-neighborhood will stay in the ε-neighborhood for all future time. Let (x, σ, I) be an arbitrary solution starting from B_ρ. Since (x, σ, I) weakly approaches $M_p = \{0\}$ in the pth mode as $t \to \infty$, for any $\delta(\varepsilon) > 0$, we can find T large enough in $\mathcal{I}_p$ so that $|x(T)| \leq \delta$. By how we have chosen δ, it follows that $|x(t)| < \varepsilon$ for all $t \geq T$. Because ε can be arbitrarily chosen, this shows $x(t) \to 0$ as $t \to \infty$ and asymptotic stability follows. The last assertion of the theorem follows from the fact that $\mathcal{T} = \bigcup_{p \in \mathcal{P}} \mathcal{T}_p$. □

If we further assume that the multiple Lyapunov functions $\{V_p : p \in \mathcal{P}\}$ are radially unbounded, then we can show that the conditions of Theorem 4.3 are satisfied for any $l_0 > 0$ and hence global asymptotic stability can be obtained.

Theorem 4.4. *Let $\{V_p : p \in \mathcal{P}\}$ be a family of positive definite and radially unbounded multiple Lyapunov functions for (2.2) and (2.3) on $\mathbb{R}^n$. Suppose, in addition, Assumption 2.1 is satisfied. Then the trivial solutions of (2.2) and (2.3) in $\mathcal{T}_p \subset \mathcal{T} \subset \mathcal{T}_{weak}(\tau) \cap \mathcal{T}_{cover}[\mathcal{X}]$ are globally asymptotically stable, provided that $M_p = \{0\}$. If $\bigcup_{p \in \mathcal{P}} M_p = \{0\}$, then the trivial solutions of (2.2) and (2.3) in $\mathcal{T} \subset \mathcal{T}_{weak}(\tau) \cap \mathcal{T}_{cover}[\mathcal{X}]$ are globally asymptotically stable.*

Proof. Local stability remains the same. We proceed to show that the conditions of Theorem 4.3 are satisfied for any $l_0 > 0$. Global attraction of the trivial solutions will follow thereafter, since Λ_{l_0} is shown to be a domain of attraction in the proof for Theorem 4.3 and Λ_{l_0} contains any point in $\mathbb{R}^n$ for a sufficient large l_0. Fixing an arbitrary l_0 can give a fixed choice of l_{N-1}. Now the radial unboundedness of V_p implies that $\Upsilon_{l_{N-1}}$ is bounded. Suppose this is not true. Then there exists a sequence of $\xi_n \in \Upsilon_{l_{N-1}}$ such that $|\xi_n| \to \infty$ as $n \to \infty$. Since V_p is radially unbounded, this implies $V_p(\xi_n) \to \infty$ as $n \to \infty$ for all $p \in \mathcal{P}$, which contradicts the fact that $\xi_n \in \Upsilon_{l_{N-1}}$. Therefore

Theorem 4.3 is true for any $l_0 > 0$ and guarantees global attraction. This completes the proof. $\square$

Remark 4.2. Specialized to the switched system framework, two main features that make our results improve those in [1] and [21] are as follows. First, the notion of persistent limit set is introduced here, and it is therefore not required that the solution converges to its limit set (only weakly approaching is required). This weakly approaching notion together with local stability (guaranteed by the existence of multiple Lyapunov functions) actually suffices to guarantee asymptotic stability as shown in Theorem 4.3. Second, the largest invariant set M_p in each E_p is explicitly defined with respect to the pth mode, which makes the set $\bigcup_{p\in\mathcal{P}} M_p$ smaller (and in many cases remarkably smaller, as shown by several examples in Section 5) than the set M in [1], which is defined as the largest weakly invariant set (with respect to whatever mode) in the set $\bigcup_{p\in\mathcal{P}} E_p$ (according to our notation). This difference makes our results less conservative. Particularly, Theorem 4.4, specialized to switched systems without impulses, improves both Theorem 1 (in the case where a common Lyapunov function exists) and Theorem 2 of [1], the main results presented there, not only in that the dwell-time conditions are now replaced by weak dwell-time conditions, but also, more importantly, in that the set M is now remarkably smaller and, therefore, more precise convergence results can be obtained.

Remark 4.3. Even though $\bigcup_{p\in\mathcal{P}} M_p$ obtained in Theorem 4.3 and Theorem 4.4 can be remarkably smaller than the set M defined in [1], we may not yet, in some cases, be able to conclude that $\bigcup_{p\in\mathcal{P}} M_p = \{0\}$. It is easy to see, however, from Theorem 4.1 and Theorem 4.2 that if we know the pth mode of a switching signal persists and $M_p = \{0\}$, then we will be able to show that the solution converges to zero. By this observation, for each $p \in \mathcal{P}$, the subset $\mathcal{T}_p$ of $\mathcal{T}$ is defined and our main results cover the situation when $M_p = \{0\}$ can only be be claimed for a particular $p \in \mathcal{P}$ and, therefore, none of the results in $[1, 9, 10, 21]$ can give any useful information.

Theorem 2.3 of [21] gives an interesting stability criterion for switched systems, based on a zero-state small-time observability hypothesis, which partially generalizes Theorem 7 of [10] in that the zero-state small-time observability hypothesis is weaker than the small-time norm-observability assumption in [10]. However, Theorem 2.3 of [21] requires an average dwell-time condition, while Theorem 7 of [10] applies to weak dwell-time switchings. Now we apply our Theorem 4.4 to show the following stability criterion for impulsive switched systems, which is also based on a zero-state small-time observability hypothesis and generalizes both results from [10] and [21].

Corollary 4.1. *Let $\{V_p : p \in \mathcal{P}\}$ be a family of positive definite and radially unbounded functions and $\{W_p : p \in \mathcal{P}\}$ a family of nonnegative functions on $\mathbb{R}^n$. Suppose that* *(i)* $V_p'(x) \le -W_p(x)$*, for all* $x \in \mathcal{X}_p$*; and*

(ii) there exists some $p \in \mathcal{P}$ *such that the system*

$$x' = f_p(x), \quad y = W_p(x), \tag{4.9}$$

is zero-state small-time observable[1]*.*

Suppose, in addition, Assumption 2.1 is satisfied. Then the trivial solutions of (2.2) and (2.3) in $\mathcal{T}_p \subset \mathcal{T} \subset \mathcal{T}_{weak}(\tau) \cap \mathcal{T}_{cover}[\mathcal{X}]$ *are globally asymptotically stable. If system (4.9) is zero-state small-time observable for all* $p \in \mathcal{P}$*, then the trivial solutions of (2.2) and (2.3) in* $\mathcal{T} \subset \mathcal{T}_{weak}(\tau) \cap \mathcal{T}_{cover}[\mathcal{X}]$ *are globally asymptotically stable.*

Proof. It is easy to see that $\{V_p : p \in \mathcal{P}\}$ is a family of positive definite and radially unbounded functions for (2.2) and (2.3) on $\mathbb{R}^n$. According to Theorem 4.4, we only have to show $M_p = \{0\}$. Choose any $\xi \in M_p$. By definition, there exist a positive number r and a continuously differentiable function ϕ defined on some interval $[\alpha, \beta]$, with $\alpha \le 0 \le \beta$ and $\beta - \alpha \ge r$, such that: (i) $\phi'(t) = f_p(\phi(t))$, $\forall t \in [\alpha, \beta]$, (ii) $\phi(0) = \xi$, (iii) $\phi(t) \in M_p$, $\forall t \in [\alpha, \beta]$. By condition (i), we have $M_p \subset E_p \subset \{x \in \mathbb{R}^n : W_p(x) = 0\}$. Hence $W_p(\phi(t)) = 0$ for all $t \in [\alpha, \beta]$. If $\beta > 0$, then the zero-state small-time observability condition on (f_p, W_p) implies that $\phi(0) = \xi = 0$. If $\beta = 0$, we claim that $\phi(t) = 0$ for all $t \in [\alpha, 0)$. Therefore, we still have $\phi(0) = \xi = 0$ by continuity of ϕ. To show the claim, choose any $s_0 \in [\alpha, 0)$. Define $\psi(s) = \phi(s + s_0)$ for $s \in [0, -s_0)$. Then $\psi(s)$ satisfies $\psi'(s) = f_p(\psi(s))$ and $W_p(\psi(s)) = 0$ on $[0, -s_0)$. By the zero-state small-time observability condition, we have $\psi(0) = \phi(s_0) = 0$. Since s_0 is arbitrarily chosen from $[\alpha, 0)$, we have the claim proved and it follows that $\phi(0) = \xi = 0$. We have shown $M_p = \{0\}$. The conclusions of the corollary follow from Theorem 4.4. □

Remark 4.4. Corollary 4.1, specialized to switched systems, is general than Theorem 2.3 in [21], since it only requires a weak dwell-time condition and applies to the situation where only one of the pairs (f_p, W_p) satisfies a zero-state small-time observability condition. It is also more general than Theorem 7 from [10] in that the zero-state small-time observability condition is weaker than the small-time norm-observability assumption in [10], as shown by an example in [21].

5 Examples

In this section, we apply the main results to several examples. Unless otherwise specified, the set G in the definition of E_p in Section 4.1 is taken to

[1] According to the definition in [21], a system such as (4.9) is said to be zero-state small-time observable, if for every $\delta > 0$, we have $x(0) = 0$ whenever $W_p(x(t)) = 0$ for all $t \in [0, \delta)$.

be $\mathbb{R}^n$ and the covering $\{\mathcal{X}_p : p \in \mathcal{P}\}$ is such that $\mathcal{X}_p = \mathbb{R}^n$, $\forall p \in \mathcal{P}$, in the time-dependent switching case.

5.1 Switched systems with no impulse effects

In this subsection, we present comparisons of the main results obtained in this paper, now specialized on switched systems with no impulse effects, and those in [1, 9, 10, 21].

Example 5.1. [1, Example 4] Consider two subsystems given by

$$x' = f_1(x) = \begin{pmatrix} -x_1 - x_2 \\ x_1 \end{pmatrix},$$

and

$$x' = f_2(x) = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}.$$

Let $V_1 = V_2 = \frac{1}{2}|x|^2$. Then $E_1 \subset \{x_1 = 0\}$ and $E_2 = \{0\}$. It is easy to see that $M_1 = M_2 = \{0\}$. By Theorem 4.4, all the trivial solutions in $\mathcal{T} \subset \mathcal{T}_{\text{weak}}$ for the resulting switched system are globally asymptotically stable, whereas in [1] it can only be concluded, by Theorem 1 there, that all the solutions tend to the x_2 axis, even under dwell-time switching.

Example 5.2. [21, Example 2] Consider two subsystems given by

$$x' = f_1(x) = \begin{pmatrix} -x_1 - x_2 \\ x_1 \end{pmatrix},$$

and

$$x' = f_2(x) = -\frac{1}{1 + |x|^4}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The example is essentially similar to Example 5.1. By considering the common Lyapunov function $V_1 = V_2 = |x|^2$ and applying the same argument as in Example 5.1, it can be shown that all the trivial solutions in $\mathcal{T} \subset \mathcal{T}_{\text{weak}}$ for the resulting switched system are globally asymptotically stable, while in [21] the asymptotic stability can only be concluded under average dwell-time switchings.

Remark 5.1. It is worth noting that, as pointed out in [1], the switched systems in Examples 5.1 and 5.2 are both globally asymptotically stable under arbitrary switching, since $V_1 = V_2 = x_1^2 + x_1x_2 + x_2^2$ gives a common strict Lyapunov function for both systems.

Example 5.3. Consider two subsystems given by

$$x' = f_1(x) = \begin{pmatrix} -x_2 \\ x_1 - 2x_2^k \end{pmatrix},$$

and

$$x' = f_2(x) = \begin{pmatrix} x_2 \\ -x_1 - 2x_2^k \end{pmatrix},$$

where $k \geq 1$ is any odd integer. For $k = 1$, f_1 and f_2 are linear vector fields. It is shown in [9, Example 2] that the resulting switched system is asymptotically stable under weak dwell-time switchings. On the other hand, it is also shown that there exists a switching signal, which is not of weak dwell-time, such that the trajectory does not converge to zero. Now as a general case when $k \geq 1$, we can construct a common Lyapunov function $V_1 = V_2 = |x|^2$. Then $E_1 = E_2 = \{x_2 = 0\}$. It is easy to see that $M_1 = M_2 = \{0\}$. By Theorem 4.4, all the trivial solutions in $\mathcal{T} \subset \mathcal{T}_{\text{weak}}$ for the resulting switched system are globally asymptotically stable. This example can be seen as a complement to the results in [9], as for now nonlinear vector fields are allowed and the same conclusion holds, i.e., global asymptotic stability of the resulting switched system under weak dwell-time switchings.

Example 5.4. [1, Example 2] Consider two subsystems given by

$$x' = f_1(x) = \begin{pmatrix} -x_2 \\ x_1 - x_2^k \end{pmatrix},$$

and

$$x' = f_2(x) = \begin{pmatrix} x_2 \\ -x_1 - x_2^k \end{pmatrix},$$

where $k \geq 1$ is any odd integer. This example is essentially similar to the previous one. By defining a common Lyapunov function $V_1 = V_2 = |x|^2$, Theorem 4.4 shows that all the trivial solutions in $\mathcal{T} \subset \mathcal{T}_{\text{weak}}$ for the resulting switched system are globally asymptotically stable, while in [1] similar results were obtained only for dwell-time switchings.

Remark 5.2. It can be shown that, for Example 5.3 and Example 5.4, there exists no common strict Lyapunov functions. Otherwise, the sum of the two right-hand sides would define an asymptotically stable system, which is not the case for both examples.

Example 5.5. [1, Example 5] Consider two subsystems given by

$$x' = f_1(x) = \begin{pmatrix} -x_1 - x_2 \\ x_1 \end{pmatrix},$$

and

$$x' = f_2(x) = \begin{cases} \begin{pmatrix} -x_1 - x_2 \\ x_1 \end{pmatrix}, & x_1 < 0, \\ \\ \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, & x_2 \geq 0, \end{cases}$$

where the second system appears to be with discontinuous right-hand side but indeed defines a continuous differential system. In [1], it is pointed out that if one uses the common Lyapunov function $V_1 = V_2 = |x|^2$, no useful information can be obtained by their results, since $Z = \{x_1 \geq 0\}$ itself becomes a weakly invariant set (by their notation). However, according to our results, we have $E_1 \subset \{x_1 = 0\}$ and $E_2 \subset \{x_1 \geq 0\}$. It follows that $M_1 = \{0\}$ and $M_2 = \{x_1 \geq 0\}$. Although $M_1 \bigcup M_2 \neq \{0\}$, we do have $M_1 = \{0\}$. We may still apply our Theorem 4.4. Since $f_1(x) = f_2(x)$ when $x_1 < 0$, an arbitrary switching can be equivalently modified so that $\sigma = 1$ on $\{x_1 < 0\}$, which is equivalent to restrict the switching to those switchings defined by a covering $\{\mathcal{X}_1, \mathcal{X}_2\}$ of $\mathbb{R}^2$ such that $\mathcal{X}_2 \subset \{x_1 \geq 0\}$. Since $\sigma = 1$ on $\{x_1 < 0\}$, it follows that the first mode is a persistent mode of σ (it actually takes at least a constant time for the system to travel out $\{x_1 < 0\}$). Now according to Theorem 4.4, all the trivial solutions in $\mathcal{T}_1 \subset \mathcal{T}_{\text{weak}}$ for the resulting switched system are globally asymptotically stable. By the fact that an arbitrary switching has an equivalent modification in $\mathcal{T}_1$, we actually obtain that the origin is globally asymptotically stable under arbitrary switching, which agrees with the results shown in [3] by the method of "worse case" argument (see also [6]).

Remark 5.3. Example 5.1 and Example 5.5 are originally presented in [1] to show the limitations of their results. As we can see from these examples, our invariance principles and stability theorems presented in this paper can effectively overcome these limitations and are applicable to an even larger class of switched systems.

Example 5.6. Consider a family of Liénard's equations

$$y'' + h_p(y)y' + g_p(y) = 0, \quad p \in \mathcal{P}.$$

Let $x_1 = y$, $x_2 = y'$. Then an equivalent family of systems is

$$x' = f_p(x) = \begin{pmatrix} x_2 \\ -g_p(x_1) - h_p(x_1)x_2 \end{pmatrix}, \quad p \in \mathcal{P}.$$

Let $\mathcal{X}_p$, $p \in \mathcal{P}$, be a covering of $\mathbb{R}^2$ that defines the state-dependent switchings for a switched system given by the above family of Liénard's equations, and let $\mathcal{X}_p^1$ denote the projection of $\mathcal{X}_p$ into its first component. Assume that, for all $p \in \mathcal{P}$,

(i) $G_p(x) = \int_0^x g_p(\xi)d\xi > 0$ for all $x \in \mathcal{X}_p^1$ such that $x \neq 0$,

(ii) $G_p(x) \to \infty$, as $x \in \mathcal{X}_p^1$ such that $|x| \to \infty$,

(iii) $g_p(x) \neq 0$, for all $x \in \mathcal{X}_p^1$ such that $x \neq 0$,

(iv) $h_p(x) > 0$, for all $x \in \mathcal{X}_p^1$ such that $x \neq 0$.

We take the Lyapunov function candidates to be the total energy of each subsystem

$$V_p(x) = G_p(x_1) + \frac{x_2^2}{2}. \tag{5.1}$$

Then

$$V_p'(x) = -h_p(x_1)x_2^2 \leq 0. \tag{5.2}$$

Hence, by the assumptions above, we can see that V_p defines a family of positive definite and radially unbounded multiple Lyapunov functions for the switched system. According to (5.2), we have $E_p \subset \{x_1 = 0\} \cup \{x_2 = 0\}$. Now it is easy to check that $M_p = \{0\}$ for all $p \in \mathcal{P}$ by the assumptions above. Therefore, we can conclude, by Theorem 4.4, that all the trivial solutions in $\mathcal{T} \subset \mathcal{T}_{\text{weak}} \cap \mathcal{T}_{\text{cover}}[\mathcal{X}]$ for the resulting switched system are globally asymptotically stable, provided that Assumption 2.1 is satisfied by $\mathcal{T}$. The assumptions (iii) and (iv) on g_p and h_p can be replaced by the following weaker ones, for some particular $p \in \mathcal{P}$,

(iii') $g_p(x) \neq 0$, for all $x \in \mathcal{X}_p^1$ such that $x \neq 0$,

(iv') $h_p(x) > 0$, for all $x \in \mathcal{X}_p^1$ such that $x \neq 0$,

and

(iv") $h_p(x) \geq 0$, for all $x \in \mathcal{X}_p^1$ such that $x \neq 0$, for all $p \in \mathcal{P}$.

Now, by the same argument, we are only able to conclude that $M_p = \{0\}$ for this particular p. Nevertheless, Theorem 4.4 guarantees that all the trivial solutions in $\mathcal{T}_p \subset \mathcal{T} \subset \mathcal{T}_{\text{weak}} \cap \mathcal{T}_{\text{cover}}[\mathcal{X}]$ for the resulting switched system are globally asymptotically stable, provided that Assumption 2.1 is satisfied by $\mathcal{T}_p$. It is worth noting that none of the results in [1, 9, 10, 21] can give any useful information in this case.

Remark 5.4. Example 5.6 also presents an example showing that, by defining the multiple Lyapunov function conditions based on a covering $\mathcal{X}$, we actually relax the assumptions on each function V_p (see Remark 2.4 after Definition 2.4). For example, if $\mathcal{P} = \{1, 2\}$ and we choose $\mathcal{X}_1 = \{x_1 \geq 0\}$ and $\mathcal{X}_2 = \{x_1 < 0\}$, then $\mathcal{X}_1^1 = [0, \infty)$ and $\mathcal{X}_1^1 = (-\infty, 0)$. Now we consider Example 5.6 with

$$g_1(y) = \begin{cases} y, & y \geq 0, \\ 0, & y < 0, \end{cases} \qquad g_2(y) = \begin{cases} 0, & y \geq 0, \\ y, & y < 0, \end{cases}$$

and

$$h_1(y) = y, \quad h_2(y) = -y.$$

Then neither the Lyapunov functions V_1 and V_2 defined by (5.1) are positive definite, nor the derivatives V_1' and V_2' given by (5.2) are semi-negative definite on the entire plane.

5.2 Switched systems with impulse effects

Example 5.7. Revisit Example 5.6 by introducing an impulse function

$$\delta(x) = -\begin{pmatrix} 0 \\ (1+r)x_2 \end{pmatrix},$$

where $r \in (0,1)$ is a constant. Now associate this impulse function to all switching signals in $\mathcal{S}$ and activate it at each switching instant, i.e., let $\mathbb{I} = \{\delta\}$. Consider the case that all g_p are equal. Then according to Example 5.6, the impulsive switched system enjoys a common Lyapunov function given by

$$V(x) = G(x_1) + \frac{x_2^2}{2}.$$

It is easy to check that Condition 2.1 is satisfied and so is Assumption 2.1. Therefore, if g_p and h_p satisfy the assumptions (i)–(iv) in Example 5.6, then, according to Theorem 4.4, all the trivial solutions in $\mathcal{T} \subset \mathcal{T}_{\text{weak}} \cap \mathcal{T}_{\text{cover}}[\mathcal{X}]$ for the resulting impulsive switched system are globally asymptotically stable. Moreover, if g_p and h_p satisfy the assumptions (i)–(ii) and (iii')–(iv") in Example 5.6, then, according to Theorem 4.4, all the trivial solutions in $\mathcal{T}_p \subset \mathcal{T} \subset \mathcal{T}_{\text{weak}} \cap \mathcal{T}_{\text{cover}}[\mathcal{X}]$ for the resulting impulsive switched system are globally asymptotically stable.

6 Conclusions

Invariance principles are established for a general class of impulsive switched systems, which generalize the classical LaSalle's principle and the results in [1,21] to the general setting of impulsive switched systems under weak dwell-time conditions. Asymptotic stability results for impulsive switched systems under weak dwell-time switchings are derived as important applications of these invariance principles. In summary, the paper contributes to the theory of impulsive switched hybrid systems in the following aspects:

(i) invariance principles for systems with impulse effects have not yet been well explored in the literature;

(ii) specialized to the switched system framework, the results here are sharper than the known results in the literature as shown by several examples; and

(iii) assuming only weak dwell-time conditions makes the results applicable to a larger class of impulsive switched systems.

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