Boundries of von Neumann Algebras

Jesse Peterson

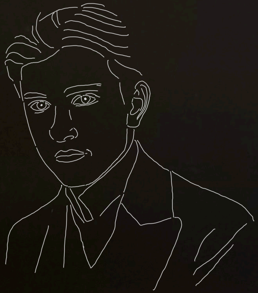
University of Waterloo

ICM 2026

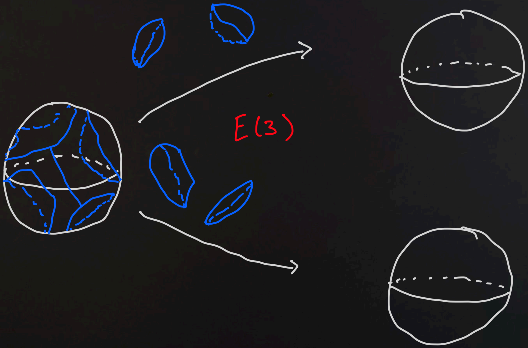
Philadelphia

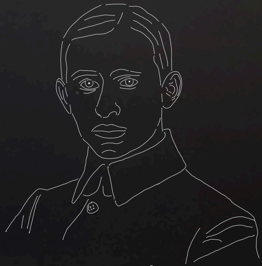
Slides

Banach - Tarski Paradox (1924)



S. Banach





A. Tarski

Amenable Groups (von Neumann 1929)

Γ is amenable if there is an invariant mean: A state $\varphi : \ell^\infty G \to \mathbb{C}$ s.t. $\varphi(L_t(f)) = \varphi(f)$ $t \in \Gamma$.

$$\left(L_t(f)(x) = f(t^{-1}x) \right)$$

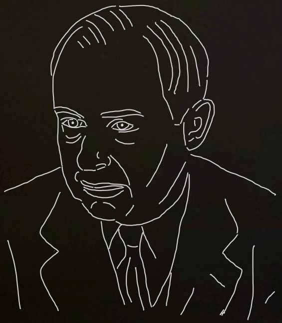
Ex: $\mathbb{Z}$ is amenable.

Set $\varphi_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} f(k)$.

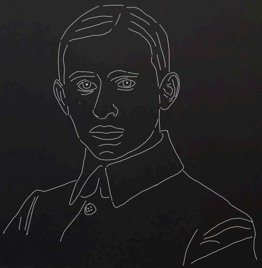
$$\| \varphi_n - \varphi_n \circ L_1 \|_\infty = \frac{2}{n}$$

Take φ a wk*-cluster point of $\{\varphi_n\}_n$.

Tarski 1938: Γ is amenable iff there is no paradoxical decomposition.

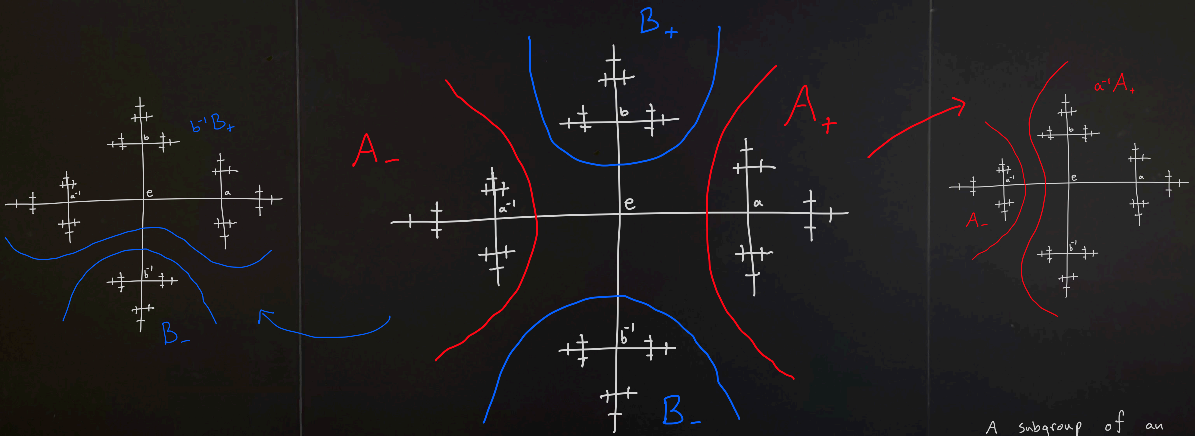


J. von Neumann



A. Tarski

Paradoxical Decomposition of $\mathbb{F}_2$



A subgroup of an amenable group is amenable.

Von Neumann Algebras (1936)

(F. J. Murray – J. von Neumann)

A $*$-algebra $M \subset \mathbb{B}(\mathcal{H})$ is a **von Neumann algebra** if it contains 1 and is closed in the Strong operator topology (pointwise convergence)

Bicommutant Theorem (von Neumann 1929):

A $*$-algebra $M \subset \mathbb{B}(\mathcal{H})$ containing 1 is SOT-closed iff $M = (M')'$

$$\left(S' := \{ T \in \mathbb{B}(\mathcal{H}) \mid Tx = xT \ \ \forall \ x \in S \} \right)$$

Ex: (X, μ) a probability space, then

$$L^\infty(X,\mu) \subset \mathbb{B}(L^2(X,\mu))$$

$$L^\infty(X,\mu)' = L^\infty(X,\mu).$$

A C^*-algebra is only closed in the norm topology.

Ex: Γ a group

$$\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$$

$$\lambda_t \cdot \delta_x = \delta_{tx}.$$

$$L\Gamma = \overline{\mathbb{C}\lambda(\Gamma)}^{SOT}$$

II_1 Factors

A vN algebra M is a factor if $Z(M)=\mathbb{C}$.

A vN algebra M is finite if it has a trace: A state $\tau: M \to \mathbb{C}$ s.t.

$$\tau(xy)=\tau(yx).$$

A II_1 factor is an infinite dimensional finite factor.

$$\boxed{\text{Murray - von Neumann: } L\Gamma \text{ is a factor iff } \Gamma \text{ has infinite conjugacy classes (ICC).}}$$

Ex: S_∞, $\mathbb{F}_2$, $\mathbb{Z} \wr \mathbb{Z}$, $PSL_n\mathbb{Z}$, $\mathbb{F}_2 \times \mathbb{F}_2$, ...

Trace: $\tau(f)=\int f \, d\mu$
$= \langle f \cdot \hat{1}, \hat{1} \rangle$

Ex: (X,μ) a probability space, then

$$L^\infty(X,\mu) \subset \mathbb{B}(L^2(X,\mu))$$

$$L^\infty(X,\mu)' = L^\infty(X,\mu).$$

Trace: $\tau(\lambda_t) = \langle \lambda_t \cdot \delta_e, \delta_e \rangle = \begin{cases} 1 & \text{if } t=e, \\ 0 & \text{if } t \neq e. \end{cases}$

Ex: Γ a group

$\lambda: \Gamma \to \mathcal{U}(\ell^2\Gamma)$

$\lambda_t \cdot \delta_x = \delta_{tx}.$

$$L\Gamma = \overline{\mathbb{C}\lambda(\Gamma)}^{SOT}$$

Amenability and von Neumann algebras

M a II_1 factor.

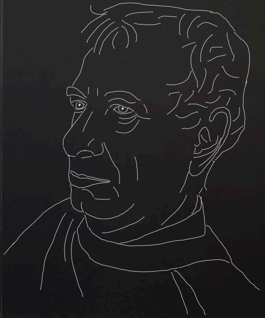
M is <u>hyperfinite</u> if $M = R := \overline{\bigcup_n M_{2^n}(\mathbb{C})}^{SOT}$.

M is <u>injective</u> if $\exists\ \mathbb{E} : \mathbb{B}(\mathcal{H}) \to M$ a conditional expectation (projection of norm 1).

J. Schwartz (1963):

$L\Gamma$ hyperfinite $\Longrightarrow$ $L\Gamma$ injective $\Longrightarrow$ Γ amenable.

$$R \not\cong L(S_\infty \times \mathbb{F}_2) \not\cong L\mathbb{F}_2$$



J. Schwartz

Classification of Injective Factors

A. Connes (1976): M a separable II_1 factor
M is injective iff $M \simeq R$

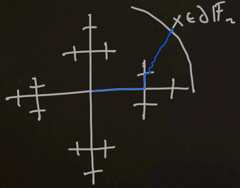
If Γ is countably infinite ICC, then
$L\Gamma \simeq R$ iff Γ is amenable.

M-D. Choi – E. Effros (1977): A C^*-algebra A
is nuclear iff A^{**} is injective.

A. Connes

Amenable Actions

$\mathbb{F}_2 \curvearrowright \partial \mathbb{F}_2$



$\mu_n(x)$: Uniform measure on first n steps toward x.

$$\boxed{\Gamma \text{ is amenable iff } \exists\ \mu_n \in \mathrm{Prob}(\Gamma) \text{ st } \| t\cdot\mu_n - \mu_n \| \longrightarrow 0 \quad t \in \Gamma.}$$

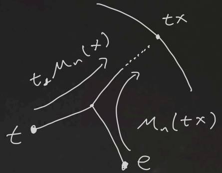
C. Anantharaman-Delaroche 1987:

An action on a compact Hausdorff space $\Gamma \curvearrowright K$ is <u>amenable</u> if $\exists$ continuous maps

$$\mu_n : K \longrightarrow \mathrm{Prob}(\Gamma) \quad \text{st}$$

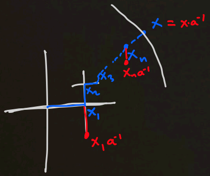
$$\sup_{x \in K} \| t\cdot\mu_n(x) - \mu_n(tx) \| \xrightarrow[n\to\infty]{} 0 \quad t \in \Gamma.$$

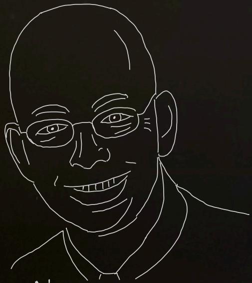
$$\boxed{\Gamma \curvearrowright K \text{ is amenable iff } C(K) \rtimes_r \Gamma \text{ is nuclear}}$$



C. Anantharaman-Delaroche

$\mathbb{F}_2 \curvearrowright \partial\mathbb{F}_2$





N. Higson

Biexact Groups

Higson compactification:

$$S(\Gamma) := \{ f \in \ell^\infty\Gamma \mid R_t(f) - f \in c_0(\Gamma) \ t \in \Gamma \}$$

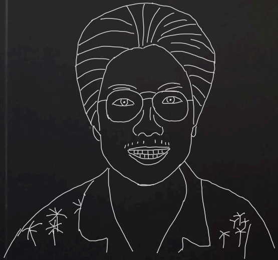
$$S(\Gamma) \simeq C(\overline{\Gamma}^S)$$

N. Ozawa 2004: Γ is biexact if the action $\Gamma \curvearrowright \overline{\Gamma}^S$ is amenable

> If Γ is biexact, then $L\Gamma$ is solid
> (If $B \subset L\Gamma$ has no minimal projections, then $B' \cap L\Gamma$ is injective)

Ex: $\mathbb{F}_2$, $\mathbb{Z}^2 \rtimes SL_2\mathbb{Z}$, $\mathbb{Z} \wr \mathbb{F}_2$, Hyperbolic groups, ...

$$C(\overline{\mathbb{F}_2}) \subset C(\overline{\mathbb{F}_2}^S)$$



N. Ozawa

Properly Proximal Groups

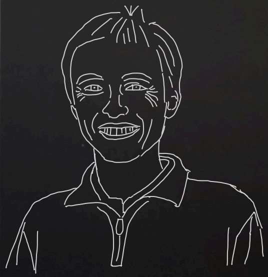
R. Boutonnet – A. Ioana – P (2021):

Γ is <u>properly proximal</u> if there is no invariant state on $S(\Gamma)$

R. Bountonnet

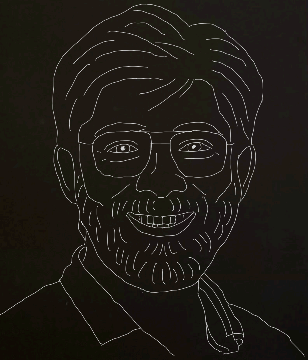
$\boxed{\text{If } \Gamma \text{ is properly proximal, then } L\Gamma \text{ has no weakly compact Cartan subalgebra in the sense of Ozawa and Popa.}}$

Ex: Nonamenable Biexact, free products, relatively hyperbolic, lattices in Semi-simple Lie groups.



A. Ioana

W*E of Proper Proximal Groups



I. Ishan

$\boxed{\begin{array}{l}\text{I. Ishan – P – L. Ruth (2024):}\\ \text{If } L\Gamma \simeq L\Lambda, \text{ then } \Gamma \text{ is properly}\\ \text{proximal iff } \Lambda \text{ is.}\end{array}}$

Proof uses ideas of J. Schwartz.

L. Ruth

Small-at-infinity Boundary of II_1 Factors

M a II_1 factor.

say $T \in \mathbb{B}(\mathcal{H})$ is nearly compact if

there exist projections $p_i \in \mathcal{P}(M)$ $q_i \in \mathcal{P}(M')$

$p_i \longrightarrow 1$ $q_i \longrightarrow 1$ st

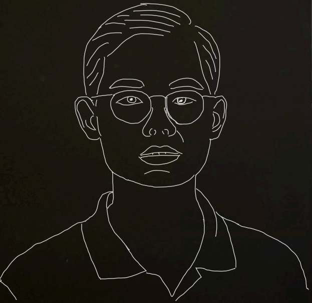
$$p_i q_i T q_i p_i \in \mathbb{K}(\mathcal{H})$$

Think Littlewoods Principles

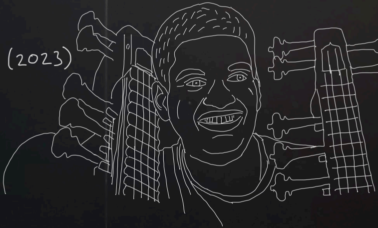
C. Ding – S. Kunnawalkam Elayavalli – P (2023)

$$\mathbb{S}(M) = \{ T \in \mathbb{B}(\mathcal{H}) \mid Tx - xT \text{ is nearly compact for all } x \in M' \}$$

$$\boxed{S(\Gamma) = \mathbb{S}(L\Gamma) \cap \ell^\infty\Gamma \subset \mathbb{B}(\ell^2\Gamma)}$$



C. Ding



S. Kunnawalkam Elayavalli

C. Ding – S. Kunnawalkam Elayavalli – P (2023)

M is properly proximal if there is no conditional expectation from $\mathbb{S}(M)$ to M

$$\boxed{\Gamma \text{ is properly proximal iff } L\Gamma \text{ is.}}$$

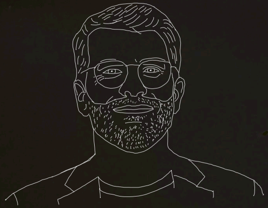
C. Ding – P (2026+)

M is biexact if the inclusion $M \subset \mathbb{S}(M)$ is "nearly nuclear".

$$\boxed{\Gamma \text{ is biexact iff } L\Gamma \text{ is.}}$$

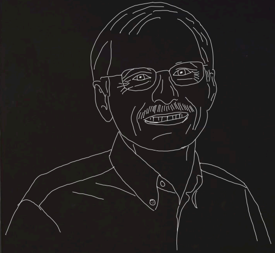
$$\boxed{\text{If } L\Gamma \simeq L\Lambda, \text{ then } \Gamma \text{ is biexact iff } \Lambda \text{ is.}}$$

Relative Boundaries and Applications Fit into Deformation / Rigidity Theory

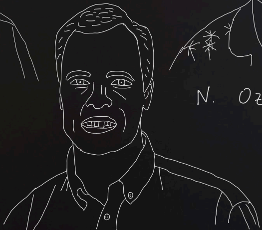


C. Houdayer

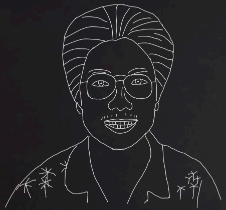
Slides



S. Popa

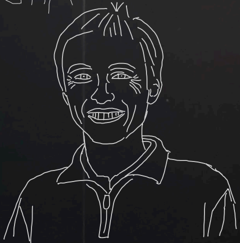


S. Vaes



N. Ozawa

I. Chifan



A. Ioana

Thank you!