Lectures on Semidefinite Programming

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Topics for this talk

- Convex conic optimization and semidefinite programming (SDP);

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- The conic duality theorem;
- Positive semidefinite matrices: a short review;
- Special cases/examples of SDP;
- Primal–dual interior point algorithms.

Conic Linear Optimization

A conic linear optimization problem (conic LP) takes the form:

$$\min \left\{ c^T x \;\middle|\; Ax = b,\; x \in \mathcal{K} \right\},$$

where the data is $c \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ and $\mathcal{K} \subset \mathbb{R}^n$ is a convex cone.

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Some choices for $\mathcal{K}$:

- nonnegative orthant in $\mathbb{R}^n$ — linear programming (LP);
- the cone of $m \times m$ positive semidefinite matrices such that $n = \binom{m+1}{2}$ — semidefinite programming (SDP).

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- Can be solved efficiently by interior point algorithms.

More on convex cones

A subset $\mathcal{K}$ of $\mathbb{R}^m$ is a *cone* if

$$a \in \mathcal{K}, \;\; \lambda \geq 0 \;\Rightarrow\; \lambda a \in \mathcal{K},$$

and the cone $\mathcal{K}$ is a *convex cone* if moreover

$$a, a' \in \mathcal{K} \Rightarrow a + a' \in \mathcal{K}.$$

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- $\mathcal{K}$ must be *pointed* (may not contain a ray);
- $\mathcal{K}$ must be closed;
- $\mathcal{K}$ must be solid (have a nonempty interior).

Dual cones

The *dual cone* of a convex cone $\mathcal{K}$ is:

$$\mathcal{K}_* = \left\{ \lambda \in \mathbb{R}^m \;:\; \lambda^T a \geq 0 \;\; \forall a \in \mathcal{K} \right\}.$$

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(iv) If $\mathcal{K}$ is a closed convex cone, then so is $\mathcal{K}_*$, and the cone dual to $\mathcal{K}_*$ is $\mathcal{K}$ itself.

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Corollary: If $\mathcal{K} \subset \mathbb{R}^m$ is a closed, pointed, solid, convex cone then so is $\mathcal{K}_*$, and vice versa.

Dual problem

Consider the conic optimization problem:

$$(COP): \quad \inf \left\{ c^T x \;\middle|\; Ax = b,\; x \in \mathcal{K} \right\},$$

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Associated *dual problem*

$$(COD): \quad \sup \left\{ b^T y \mid A^T y + s = c, \; s \in \mathcal{K}_* \right\}.$$

Feasible set $\mathcal{D}$, optimal set $\mathcal{D}^*$ (possibly empty).

Weak duality theorem

Let $x \in \mathcal{P}$ and $(y, s) \in \mathcal{D}$. Then

$$\begin{aligned} c^T x - b^T y &= (A^T y + s)^T x - b^T y \\ &= (Ax)^T y + s^T x - b^T y \\ &= s^T x \geq 0, \end{aligned}$$

because $x \in \mathcal{K}$ and $s \in \mathcal{K}_*$.

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because $x \in \mathcal{K}$ and $s \in \mathcal{K}_*$.

In words, the *duality gap* is *nonnegative* at feasible primal-dual solutions.

Strong duality theorem

Assume (COP) and (COD) are *strictly feasible*:

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We call (x^*, s^*) *complementary solutions*.

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The inner product induces the *Frobenius* (Euclidean) norm:

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which is sub-multiplicative: $\|AB\| \leq \|A\| \|B\|$.

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A nonsingular matrix $X \succeq 0$ is called positive definite ($X \succ 0$).

P.s.d. matrices (ctd.)

Any $A \in \mathcal{S}_n$ has a *spectral decomposition*:

$$A = \sum_{i=1}^{n} \lambda_i q_i q_i^T := Q \Lambda Q^T$$

where

$$Aq_i = \lambda_i q_i, \quad Q = [q_1 \ \ldots \ q_n], \quad Q^T Q = I,$$

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If $A \in \mathcal{S}_n^+$ then $\lambda_i \geq 0$. Square root factorization:

$$A^{\frac{1}{2}} := \sum_{i=1}^{n} \sqrt{\lambda_i} q_i q_i^T.$$

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Proof: Let $A \in \mathcal{S}_n^+$ and $B \in \mathcal{S}_n^+$; then

$$\begin{aligned} \langle A, B \rangle &= \operatorname{Tr}\left(A^{\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}} \right) \\ &= \operatorname{Tr}\left(A^{\frac{1}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}} \right) \\ &= \left\| A^{\frac{1}{2}} B^{\frac{1}{2}} \right\|^2 \geq 0. \end{aligned}$$

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Conversely, if $A \in \mathcal{S}_n$ and $\langle A, B\rangle \geq 0$ for all $B \in \mathcal{S}_n^+$, then let $x \in \mathbb{R}^n$ be given and set $B = xx^T \in \mathcal{S}_n^+$. The result follows (why?)

SDP: The standard form

Let $\mathcal{K} = \mathcal{S}_n^+$. We get the conic optimization problems

$$(P): \quad p^* = \inf_{X \in \mathcal{S}_n^+} \mathrm{Tr}(CX)$$

$$\text{subject to } \mathrm{Tr}(A_i X) = b_i, \quad \forall\, i.$$

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Dual problem:

$$(D): \quad d^* = \sup_{y \in \mathbb{R}^m, S \in \mathcal{S}_n^+} b^T y$$

$$\text{subject to } \sum_{i=1}^{m} y_i A_i + S = C.$$

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\begin{aligned}
&\mathrm{Tr}(CX) - b^T y \\
&= \mathrm{Tr}\left(\left(S + \sum_{i=1}^{m} y_i A_i\right) X\right) - \sum_{i=1}^{m} y_i \mathrm{Tr}(A_i X) \\
&= \mathrm{Tr}(SX) \geq 0 \text{ (since } X, S \succeq 0\text{)}
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(Duality gap always nonnegative.)

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If there exist *strictly* feasible $X \succ 0$ and $S \succ 0$, then there exist *complementary solutions* $X^* \in \mathcal{P}$ and $S^* \in \mathcal{D}$, i.e.

$$\mathrm{Tr}(X^* S^*) = 0 \Leftrightarrow X^* S^* = 0.$$

Bad duality example

$$(D): \quad \sup_{y \in \mathbb{R}^2} \; y_2$$

subject to

$$S = -y_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - y_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \succeq 0.$$

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(D) is *not solvable* but $\sup_{y \in \mathcal{D}} y_2 = 1$ (why?).

Bad duality example (ctd)

The dual problem of the example is:

$$(P): \quad \min_X \operatorname{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} X\right)$$

$$\text{subject to } X = \begin{bmatrix} 0 & x_{12} \\ x_{12} & 1 \end{bmatrix} \succeq 0.$$

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Note that $X \succeq 0$ implies $x_{12} = 0$ so that

$$X^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0,$$

is the unique optimal solution with optimal value 1.

Optimality conditions

From the strong duality theorem we get *sufficient* optimality conditions for (P) and (D):

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\begin{aligned}
\mathrm{Tr}(A_i X) &= b_i, \qquad\qquad i = 1, \ldots, m \\
\sum_{i=1}^m y_i A_i + S &= C \\
XS &= 0 \\
X \succeq 0, &\quad y \in \mathbb{R}^m, S \succeq 0
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NB: Also *necessary* if (P) and (D) are *strictly feasible*. If (X, S) are complementary solutions $XS = 0$, and $X + S \succ 0$ then we call (X, S) a *strictly complementary solution pair*.

Special cases of SDP

Tools and examples

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- The Schur complement theorem: a useful tool;
- Convex quadratic problems;
- Eigenvalue and matrix norm optimization;
- Logarithmic Chebychev approximation;
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- The Lovász theta function.

Schur complement theorem

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$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

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is called the *Schur complement of A in M*. The following are equivalent:

- M is positive (semi)definite;
- $C - B^T A^{-1} B$ is positive (semi)definite.

Convex quadratic problems

$$\min \left\{ c^T x \;:\; f_i(x) \leq 0,\; i = 1, \ldots, m \right\},$$

where

$$f_i(x) = (B_i x + b_i)^T (B_i x + b_i) - c_i^T x - d_i, \quad \forall i.$$

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Via Schur complement theorem equivalent to:

$$\begin{bmatrix} I & Bx + b \\ (Bx + b)^T & c^T x + d \end{bmatrix} \succeq 0.$$

Smallest eigenvalue problem

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Both problems are strictly feasible: In (D), take $\lambda < \lambda_1$ and in (P): take $X = \frac{1}{n} I$.

Eigenvalue optimization

Notation: $\lambda_{\max}(A)$ denotes the *largest eigenvalue* of $A \in \mathcal{S}_n$. Consider

$$\min_y \lambda_{\max}(A(y))$$

$$A(y) := A_0 + y_1 A_1 + \cdots + y_m A_m,$$

for given $A_i \in \mathcal{S}_n$ ($i = 0, \ldots, m$).

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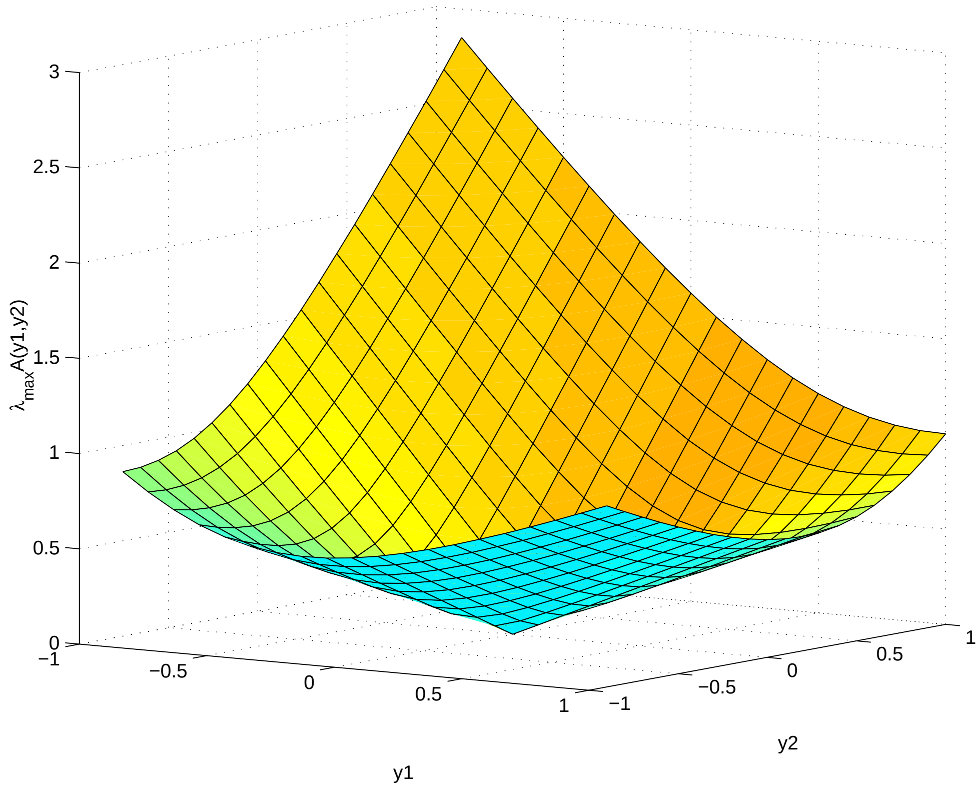
NB: the function $f(y) = \lambda_{\max}(A(y))$ is convex but not *differentiable*.

Eig. optimization: example

$$\min_{y_1,y_2}\left\{\lambda_{\max}\left(y_1\begin{bmatrix}1 & -1\\ -1 & 0\end{bmatrix}+y_2\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}\right)\right\}$$



Optimal solution $y_1^* = y_2^* = 0$.

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is equivalent to:

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SDP formulation:

$$\min \left\{ t \ : \ \begin{bmatrix} t - a_i^T x / b_i & 0 & 0 \\ 0 & a_i^T x / b_i & 1 \\ 0 & 1 & t \end{bmatrix} \succeq 0 \ \ \forall i \right\}.$$

Minimization of polynomials

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Theorem:

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We call p a *sum of squares* (SOS) in this case.

Minimization of polynomials (ctd.)

Now we have

$$\min_{x\in\mathbb{R}} p(x) = \max_{t,x} \{t \ : \ p(x) - t \geq 0 \ \forall x \in \mathbb{R}\}$$

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SDP can be used to determine if a polynomial is an SOS (*Gram matrix method*).

The Gram matrix method

Theorem: A polynomial $p : \mathbb{R}^n \mapsto \mathbb{R}$ of degree $2m$ is an SOS iff

$$p(x) = \tilde{x}^T M \tilde{x}, \text{ for some } M \succeq 0, \tag{1}$$

where $\tilde{x} = [1\ x_1\ x_2\ \ldots\ x_n\ x_1^2\ x_1x_2\ \ldots\ x_n^m]^T$ is a vector of all possible monomials of degree at most m.

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where $\tilde{x} = [1 \; x_1 \; x_2 \; \ldots \; x_n \; x_1^2 \; x_1 x_2 \; \ldots \; x_n^m]^T$ is a vector of all possible monomials of degree at most m.

- Dimension of $\tilde{x}$ is $\binom{n+m}{m}$: polynomial in n if m is fixed.
- The right-hand-side in (3) is linear in the entries of $M \Rightarrow$ (3) is a linear matrix inequality (LMI) (semidefinite feasibility problem).

Example (Parrilo)

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$$P(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -\lambda & 1 \\ -\lambda & 5 & 0 \\ 1 & 0 & -1+2\lambda \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}.$$

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If we call the 3×3 matrix in the last expression $M(\lambda)$, then $M(\lambda)$ defines an *affine space* parametrized by λ.

SDP problem: is there a λ such that $M(\lambda) \succeq 0$ (positive semidefinite)?

Example (ctd.)

for $\lambda = 3$, $M(\lambda)$ is positive semidefinite, and

$$M(3) = L^T L, \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

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$$M(3) = L^T L, \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

and consequently

$$P(x) = \tilde{x}^T M(3) \tilde{x} = \tilde{x}^T L^T L \tilde{x} = \|L\tilde{x}\|^2,$$

where $\tilde{x} = [x_1^2 \;\; x_2^2 \;\; x_1 x_2]^T$.

Thus P can be written as a sum of squares.

Nesterov's aproach

We have obtained:

$$\min_{x \in \mathbb{R}} p(x) = \max_{t,x} \left\{ t \ : \ p(x) - t = \sum_i p_i(x)^2 \right\}$$
$$= \max_{t,x} \{ t \ : \ p(x) - t = \tilde{x}^T M \tilde{x} \}$$

for some $M \succeq 0$, where $\tilde{x}^T = [1 \ x \ x^2 \ldots x^{\frac{1}{2}\deg(p)}]$.

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for some $M \succeq 0$, where $\tilde{x}^T=[1\ x\ x^2 \ldots x^{\frac{1}{2}\deg(p)}]$.
Let $p(x)=\sum_\alpha a_\alpha x^\alpha$. Then the optimization problem becomes: maximize t such that

$$a_0 - t = M_{00}, \quad a_\alpha = \sum_{i+j=\alpha} M_{ij}, \quad M\succeq 0.$$

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Equivalent problem: $\max t$ such that

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for some $M \succeq 0$.
From (6):

$$M_{00} = -t, \quad M_{01} = M_{10} = -1, \quad M_{11} = 1.$$

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We therefore get

$$\min_{x \in \mathbb{R}} p(x) = \max_{t,M} t$$

such that

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such that

$$M = \begin{bmatrix} -t & -1 \\ -1 & 1 \end{bmatrix} \succeq 0.$$

Note that the optimal value is -1, as it should be.

Co-cliques

A *co-clique* of $G = (V, E)$ is a subset $V' \subset V$ such that the *induced subgraph* on V' has no edges.

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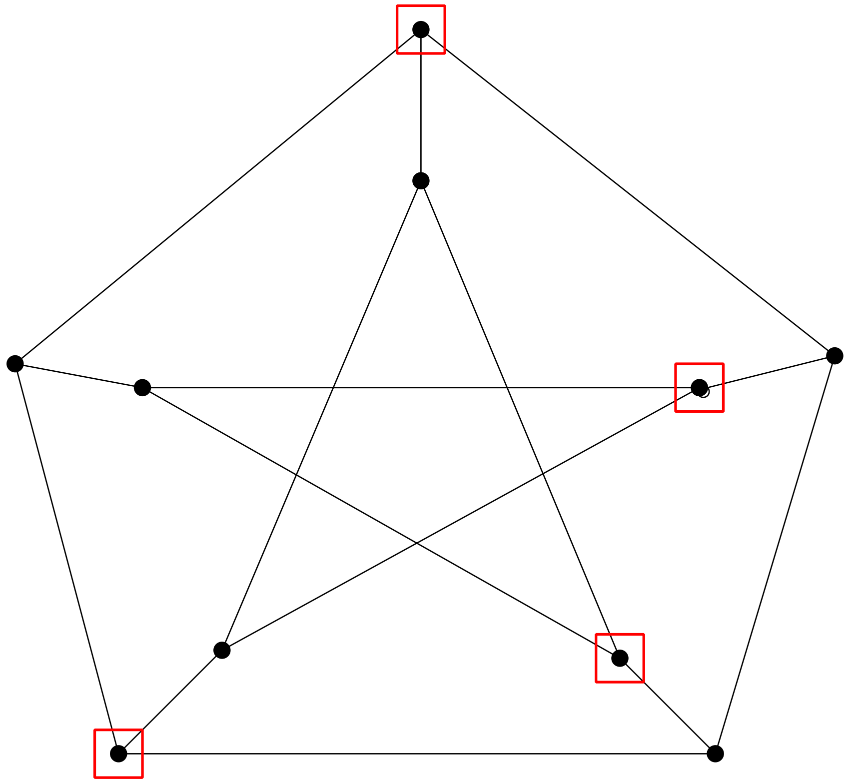

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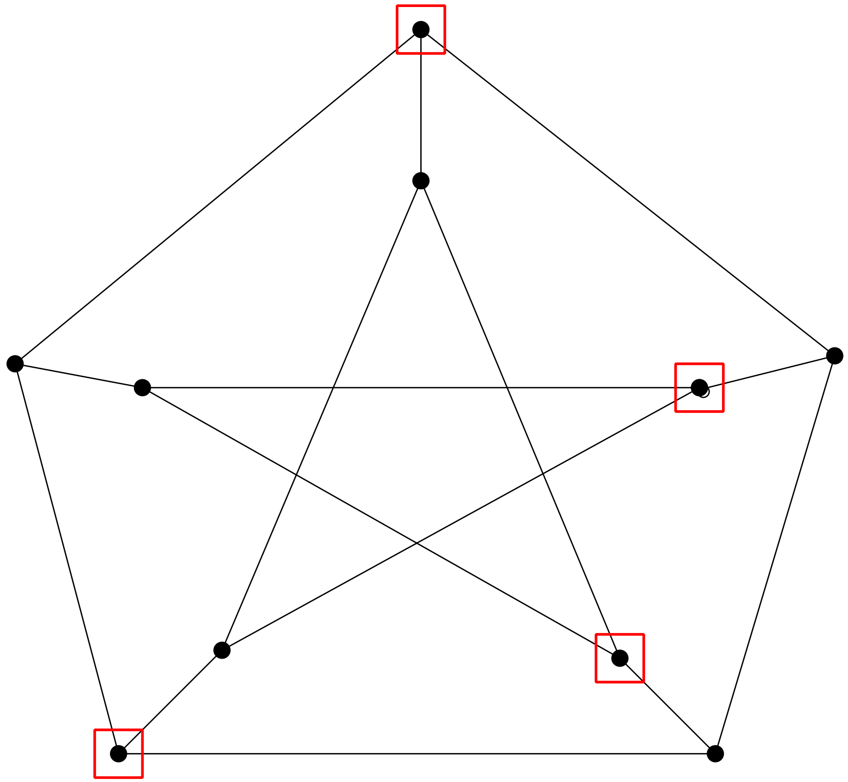

The *co-clique number* $\alpha(G)$ is the cardinality of the largest co-clique of G.

Vertex colourings

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- Chromatic number $\gamma(G)$: smallest number of colours needed to colour V;
- It is NP hard to compute $\gamma(G)$ (or $\alpha(G)$), or even to give a non-trivial polynomial time approximation.

Lovász ϑ-function

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A graph $G = (V, E)$ is given. Define:

$$\vartheta(G) := \max \operatorname{Tr}\left(ee^T X\right) = e^T X e$$

subject to

$$\begin{aligned} X_{ij} &= 0, \ \{i, j\} \in E \ (i \neq j) \\ \operatorname{Tr}(X) &= 1 \\ X &\in \mathcal{S}_n^+, \end{aligned}$$

where e denotes the all-one vector.

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First equality is easy. Second inequality via strong duality theorem.

Implication: we can compute $\alpha(G)$ and $\gamma(\bar{G})$ in pol. time for *perfect graphs*.

Algorithms: The central path

We change the system of optimality conditions to:

$$\begin{aligned}\mathrm{Tr}(A_iX) &= b_i,\ i = 1,\ldots,m,\\ \sum_{i=1}^{m} y_iA_i + S &= C,\\ XS &= \mu I,\end{aligned}$$

$$X \succeq 0,\ y \in \mathbb{R}^m,\ S \succeq 0,$$

for some parameter $\mu > 0$.

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for some parameter $\mu > 0$. These *centrality conditions* have a *unique* solution denoted by $(X(\mu), y(\mu), S(\mu))$.

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ii) the central path converges as $\mu \downarrow 0$;

iii) the limit point (X^*, y^*, S^*) (say) of the central path is in the relative interior of $\mathcal{P}^* \times \mathcal{D}^*$.

iv) Rate of convergence in case of *strict complementarity*:

$$\|X(\mu) - X^*\| = O(\mu), \ \|S(\mu) - S^*\| = O(\mu).$$

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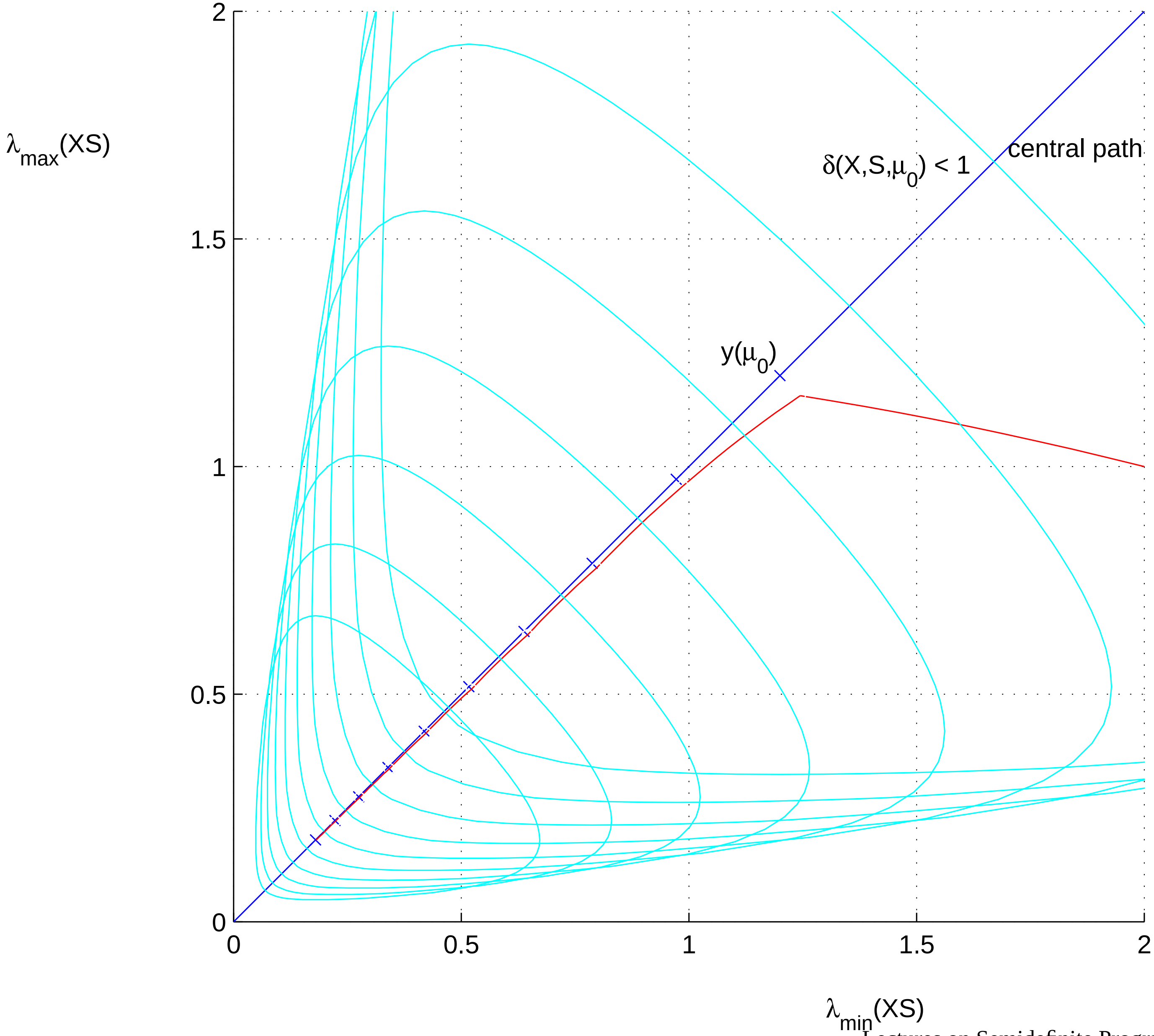
Geometrically, we view $(X(\mu), S(\mu))$ as a '*target point*' on the central path, and the parameter μ therefore determines the '*target duality gap*'.

Example



The central path: different interpretation

The point $y(\mu)$ on the dual central path minimizes

$$f_d^{\mu}(y) := -\frac{1}{\mu} b^T y - \ln\det\left(C - \sum_{i=1}^{m} y_i A_i\right).$$

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Also, f_d^μ is a *self-concordant* function — use Newton's method to minimize it efficiently.

More info

Christoph Helmberg's SDP page with links to papers and software downloads:

http://www-user.tu-chemnitz.de/∼helmberg/semidef.html

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Excellent introduction to SDP: L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review* 38, 49–95, 1996.

More info

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Solving optimization problems via internet (NEOS server):

http://www-neos.mcs.anl.gov/