Referee Report: Theory of Semidefinite Localization for Sensor Network Localization

This manuscript contributes some interesting ideas, based on a fundamental connection between the Euclidean distance matrices (EDMs) and the symmetric positive semidefinite matrices (PSDMs). Regrettably, the authors convey the false impression that this connection was only recently discovered:

> "An alternative approach, called the semidefinite programming method, is recently developed in [13] and related earlier work [3,23]."

References [13,3,23] were published in 2004, 1999, and 2001. In fact, the connection between EDMs and PSDMs was discovered in the 1930s and a large body of relevant work has been based on it. It is incumbent on the authors to at least this cite this work, if not compare their method to other methods that solve what the authors call the sensor network localization (SNL) problem.

Let $\Delta = (\delta_{ij})$ denote an $r \times r$ *pre-distance* or *dissimilarity* matrix, i.e., a symmetric matrix with nonnegative entries and zero diagonal entries. Let $\Delta_2 = (\delta_{ij}^2)$ and let $e_r = (1, \ldots, 1)^t \in \Re^r$. Schoenberg (1935) and Young and Householder (1938) established (a slight variant of) the following result: Δ is an EDM with embedding dimension d (i.e., at least d dimensions are required to realize an embedding of Δ) if and only if

$$\tau(\Delta_2) = -\frac{1}{2}\left(I - \frac{e_r e_r^t}{r}\right)\Delta_2\left(I - \frac{e_r e_r^t}{r}\right)$$

is a PSDM of rank d. Furthermore, any $r \times d$ matrix X such that $XX^t = \tau(\Delta_2)$ provides a realization: the r rows of X specify r points in $\Re^d$ whose interpoint distances are the given δ_{ij}. This embedding theorem is the basis for what is now called classical multidimensional scaling (CMDS), originally developed by Torgerson (1952) and by Gower (1966).

Given a dissimilarity matrx, not necessarily an EDM, and a dimension d, CMDS produces r points in $\Re^d$ whose interpoint distances approximate the dissimilarities. It does so by replacing $\tau(\Delta_2)$ with the (not necessarily unique) nearest (in the sense of Frobenius norm) PSDM of rank $\leq d$. Despite the fact that the closed cone of PSDMs of rank $\leq d$ is not convex, this matrix can be computed explicitly. Let $\lambda_1 \geq \cdots \geq \lambda_r$ denote the eigenvalues of $\tau(\Delta_2)$ and let $q_1, \ldots, q_r$ denote the corresponding eigenvectors. For $i = 1, \ldots, d$, let $\bar{\lambda}_i = \max(\lambda_i, 0)$. Then the points produced by CMDS are the rows of

$$X = \left[\begin{array}{ccc} \bar{\lambda}_1^{1/2} q_1 & \cdots & \bar{\lambda}_d^{1/2} q_d \end{array}\right].$$

Of course, if Δ is already an EDM with embedding dimension $\leq d$, then $\bar{\lambda}_i = \lambda_i$ and $XX^t = \tau(\Delta_2)$.

Now consider the SNL problem. Because the anchor locations are knnown, so are the anchor-anchor distances. Suppose that *all* anchor-sensor distances $(\bar{d}_{kj})$ and sensor-sensor distances (d_{ij}) are also known. We store these distances in an $r \times r$ EDM, Δ, where $r = m+n$. We then use CMDS to compute X, a matrix of locations with the correct interpoint distances. The anchor locations in X will not be $a_1, \ldots, a_m$; however, we can use Procrustes analysis (Mardia et al., 1979, Section 14.7) to recover the desired locations.

Let

$$A = \begin{pmatrix} a_1^t \\ \vdots \\ a_m^t \end{pmatrix}$$

denote the specified anchor locations and let

$$Y = \begin{pmatrix} x_1^t \\ \vdots \\ x_m^t \end{pmatrix}$$

denote the corresponding locations produced by CMDS. Because these configurations have the same interpoint distances, there is an affine transformation that maps Y to A. Let $\bar{a}$ denote the centroid of $a_1, \ldots, a_m$, let $\bar{A} = A - e_m \bar{a}^t$, and compute the singular value decomposition $Y^t \bar{A} = V \Gamma U^t$. Then $A = YVU^t + e_m \bar{a}^t$, so $XVU^t + e_r \bar{a}^t$ solves the SNL problem.[1]

The SNL problem is difficult because some of the anchor-sensor and/or sensor-sensor distances may be unknown. For such pairs, we replace the corresponding δ_{ij} with an interval of possible values, $[\ell_{ij}, u_{ij}]$. Let $[L, U]$ denote the rectangle of dissimilarity matrices that satisfy these bounds. Then the EDM completion problem is the problem of finding an EDM that lies in $[L, U]$, whereas the SNL problem is the problem of finding an EDM of embedding dimension ≤ 2 that lies in $[L, U]$. (Once the EDM has been found, the solution of the SNL problem is obtained by the methods described above.)

Because of the relation between EDMs and PSDMs, both the EDM completion problem and the SNL problem can be posed as problems about PSDMs rather than problems about EDMs. The former problem admits any matrix in the closed convex cone of PSDMs and can be formulated as a semidefinite programming problem, for which Alfakih et al. (1998) proposed an algorithm. The latter problem, however, imposes an additional rank restriction that destroys the convexity of the feasible set, so that it is not clear how to apply semidefinite programming methodology. The exciting contribution of the present manuscript is that the authors have found a way to do so.

The authors correctly note that, because the algorithm proposed by Alfakih et al. does not enforce rank restrictions, it does not guarantee a solution in the prescribed number of dimensions and therefore is "not quite suitable to our application." However, they have overlooked a closely related formulation of the EDM completion problem that can accommodate rank restrictions.

Trosset (2000, 2002) suggested solving

$$\begin{array}{ll} \min\limits_{\Delta_2} & F_d \circ \tau(\Delta_2) \\ \text{s.t.} & \Delta_2 \in [L_2, U_2], \end{array} \tag{1}$$

where $F_d(B)$ is the squared distance (in Frobenius norm) between B and the closed cone of PSDMs of rank $\leq d$. This quantity can be computed using the methods described above and turns out to be a function of the eigenvalues of $\tau(\Delta_2)$. Thus, Trosset circumvented the difficulty of managing rank restrictions by encoding the restrictions in the objective function.

How should the present manuscript be modified in light of the above? Clearly, the ideas detailed in this report should be summarized and the relevant papers cited. The issue of which approach is superior will not be resolved easily and it would be unrealistic to ask the authors to do so. They might observe that their semidefinite programming algorithm has faster local convergence than the limited memory algorithm (L-BFGS-B) that Trosset (2002) used to solve (1). On the other hand, what is interesting about (1) is the formulation itself, which undoubtedly can be solved by algorithms more efficient than L-BFGS-B.

[1] This construction works when $d = 2$, presumably the case of interest. It might fail when $d = 3$ if A and Y have chiral fragments, but this possiblity seems unlikely in practice. I believe that it is precluded by the authors' uniqueness assumption.

References

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