Asymptotic Behavior of the Expected Optimal Value of the Multidimensional Assignment Problem

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Abstract

The Multidimensional Assignment Problem (MAP) is a higher-dimensional version of the Linear Assignment Problem that arises in the areas of data association, target tracking, resource allocation, etc. This paper elucidates the question of asymptotical behavior of the expected optimal value of the large-scale MAP whose assignment costs are independent identically distributed random variables with a prescribed probability distribution. We demonstrate that for a broad class of continuous distributions the limiting value of the expected optimal cost of the MAP is determined by the location of the left endpoint of the support set of the distribution, and construct asymptotical bounds for the expected optimal cost.

Keywords: multidimensional assignment problem, random assignment problem, expected optimal value, asymptotical analysis, asymptotical bounds

1 Introduction

The Multidimensional Assignment Problem (MAP) is a higher dimensional version of the two-dimensional, or Linear Assignment Problem (LAP) [32]. If a classical textbook formulation of the Linear Assignment Problem is to find an optimal assignment of "N jobs to M workers," then, for example, the 3-dimensional Assignment Problem can be interpreted as finding an optimal assignment of "N jobs to M workers in K time slots," etc. In general, the objective of the MAP is to find tuples of elements from given sets, such that the total cost of the tuples is minimized. The MAP was first introduced by Pierskalla [34], and since then has found numerous applications in the areas of data association [5, 25], image recognition [39], multisensor multitarget tracking [35, 36, 24], tracking of elementary particles [38], etc. For a discussion of the MAP and its applications see, for example, [7, 9, 8] and references therein.

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A d-dimensional "axial" MAP with equal number n of elements in each dimension has the form

$$\begin{aligned}
\min \quad & \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} c_{i_1\cdots i_d}\, x_{i_1\cdots i_d} && && (1)\\
\text{s.t.} \quad & \sum_{i_2=1}^{n} \cdots \sum_{i_d=1}^{n} x_{i_1\cdots i_d} = 1, && i_1 = 1, \ldots, n, \\
& \sum_{i_1=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} \sum_{i_{k+1}=1}^{n} \cdots \sum_{i_d=1}^{n} x_{i_1\cdots i_d} = 1, && k = 2, \ldots, d-1, \quad i_k = 1, \ldots, n, \\
& \sum_{i_1=1}^{n} \cdots \sum_{i_{d-1}=1}^{n} x_{i_1\cdots i_d} = 1, && i_d = 1, \ldots, n, \\
& x_{i_1\cdots i_d} \in \{0, 1\}^{n^d}.
\end{aligned}$$

An instance of the MAP with different numbers of elements in each dimension, $n_1 \geq n_2 \geq \ldots \geq n_d$, is reducible to form (1) by introduction of dummy variables.

Problem (1) admits the following geometric interpretation: given a d-dimensional cubic matrix, find such a permutation of its rows and columns that the sum of the diagonal elements is minimized (which explains the term "axial"). This rendition leads to an alternative formulation of the MAP (1) in terms of permutations $\pi_1, \ldots, \pi_{d-1}$ of numbers 1 to n, i.e., one-to-one mappings $\pi_i : \{1, \ldots, n\} \mapsto \{1, \ldots, n\}$,

$$\min_{\pi_1,\ldots,\pi_{d-1}\in\Pi^n} \sum_{i=1}^{n} c_{i,\pi_1(i),\ldots,\pi_{d-1}(i)},$$

where Π^n is the set of all permutations of the set $\{1, \ldots, n\}$. A feasible solution to the MAP (1) can be conveniently described by specifying its cost,

$$z = c_{i_1^{(1)}\cdots\, i_d^{(1)}} + c_{i_1^{(2)}\cdots\, i_d^{(2)}} + \ldots + c_{i_1^{(n)}\cdots\, i_d^{(n)}}, \tag{2}$$

where $\left(i_j^{(1)}, i_j^{(2)}, \ldots, i_j^{(n)}\right)$ is a permutation of the set $\{1, 2, \ldots, n\}$ for every $j = 1, \ldots, d$. In contrast to the LAP that represents a $d = 2$ special case of the MAP (1) and is polynomially solvable [7, 32], the MAP with $d \geq 3$ is generally NP-hard, a fact that follows from a reduction of the 3-dimensional matching problem (3DM) [14]. Although a number of exact and heuristic algorithms for the MAP [1, 6, 10, 33] have been developed in the literature, only small-to-moderate-sized instances of the MAP can be solved routinely as of today due to immense complexity of the problem.

If one denotes the optimal value of an instance of the MAP (1) with d dimensions and n elements per dimension as $z^*_{d,n}$, then it is of interest to determine the (asymptotical) behavior of $z^*_{d,n}$ when either d or n is large: $n \gg 1$ or $d \gg 1$. Evidently, such an analysis would require certain assumptions on the behavior of the assignment costs of the MAP. This is facilitated by assuming that the cost coefficients $c_{i_1\cdots i_d}$ in (1) are independent identically distributed (iid) random variables from some prescribed continuous distribution. This reduces the question at hand to asymptotical analysis of the expected optimal value $\mathsf{E}\big[z^*_{d,n}\big]$ of random instances of the MAP, which is denoted throughout the paper as $Z^*_{d,n}$.

During the last two decades, expected optimal values of random assignment problems have been studied intensively in the context of random LAP. Perhaps, the most widely known result in this area is the conjecture by Mézard and Parisi [22] that the expected optimal value $\mathsf{E}[L_n] = Z^*_{2,n}$ of an LAP of size

n with iid uniform or exponential with mean 1 cost coefficients satisfies $\lim_{n\to\infty} \mathsf{E}[L_n] = \zeta(2) = \frac{\pi^2}{6}$, where $\zeta(\cdot)$ is Riemann's zeta function. In fact, this conjecture was preceded by an upper bound on the expected optimal value of the LAP with uniform (0,1) costs: $\limsup_{n\to\infty} L_n \le 3$ due to Walkup [40], which was soon improved by Karp [18]: $\limsup_{n\to\infty} L_n \le 2$. A lower bound on the limiting value of L_n was first provided by Lazarus [19, 20]: $\liminf_{n\to\infty} L_n \ge 1 + e^{-1} \approx 1.37$, and then has been improved to 1.44 by Goemans and Kodilian [15] and 1.51 by Olin [27]. Experimental evidence in support of the Mézard-Parisi conjecture was provided by Pardalos and Ramakrishnan [30]. Recently, Aldous [3] has shown that indeed $\lim_{n\to\infty} \mathsf{E}[L_n] = \frac{\pi^2}{6}$, thereby proving the conjecture (in an earlier paper [2] the author has established the existence of the limit of L_n). Another conjecture due to Parisi [31] stating that the expected optimal value of a random LAP of finite size n with exponentially distributed iid costs is equal to $\mathsf{E}[L_n] = Z^*_{2,n} = \sum_{i=1}^{n} i^{-2}$ has been proven independently in [21] and [26].

This paper contributes to the existing literature on random assignment problems by establishing the limiting value and asymptotic behavior of the expected optimal cost $Z^*_{d,n}$ of random Multidimensional Assignment Problem with iid cost coefficients for a broad class of continuous distributions. We demonstrate that, unlike the 2-dimensional LAP, the MAP with $d \ge 3$ has an expected optimal value that depends on the location of the left endpoint of the support set of the costs' distribution. In particular, the expected optimal value of an MAP with exponentially distributed cost coefficients approaches zero when either d or n approaches infinity. The presented analysis is constructive in the sense that it allows for deriving asymptotical lower and upper bounds for $Z^*_{d,n}$ that are converging when the support of distribution is bounded from below, as well as to estimate the rate of convergence for $Z^*_{d,n}$ when the left endpoint of the support is located at the origin.

The rest of the paper is organized as follows. The next section introduces the concept of the index graph representation of the MAP that our approach is based upon. Using the structure of the index graph, we construct simple expressions for the upper and lower bounds of the expected optimal cost $Z^*_{d,n}$ of the MAP. In Section 3 we establish the main result of the present endeavor by determining the limiting values of large-scale random MAPs for a wide spectrum of continuous distributions. Section 4 discusses asymptotical bounds on the expected optimal cost of random MAPs and presents the corresponding numerical results.

2 Index graph representation of the MAP and related lemmata

Our approach to determining the asymptotic behavior of the expected optimal cost $Z^*_{d,n}$ of an MAP (1) with random cost coefficients is based on analysis of a directed graph that can be constructed to represent the set of feasible solutions of the MAP. It is a variation of the *index tree* introduced by Pierskalla [34], a tree structure in which each path from the root node to a leaf node corresponds to a feasible solution of the MAP. For the purposes of current development, it is convenient to consider a directed *index graph*, such that feasible solutions of the MAP are represented by directed paths in this graph.

The index graph $\mathcal{G} = (V, E)$ of the MAP (1) has a set of vertices V that is partitioned into n *levels*[1] and a distinct *root node.* A node at level j of the graph represents an assignment $(i_1, \ldots, i_d)$ with $i_1 = j$ and cost $c_{j\, i_2 \cdots i_d}$. Thus, each level contains $\kappa = n^{d-1}$ nodes. Root node disregarded, the index graph is n-partite, with directed arcs connecting nodes at level j to nodes at level $(j+1)$, $j = 1, \ldots, n-1$. The

[1]In general, MAP may have n_1 elements in dimension 1, n_2 elements in dimension 2, and so on, so that in assignment $(i_1, i_2, \ldots, i_n)$ one has $1 \le i_k \le n_k$, $k = 1, \ldots, d$. In this case, the index graph would contain n_1 levels.

set E of arcs in the index graph is constructed in such a way that any feasible solution of the MAP (1) can be represented as a directed path connecting the root node to a leaf node at level n (we call such a directed path a *feasible* path).

Namely, the root node is connected to all nodes at level 1. Node $\left(1, i_2^{(r)}, \ldots, i_n^{(r)}\right)$ at level 1 is connected to (or, is a *parent* to) node $\left(2, i_2^{(s)}, \ldots, i_n^{(s)}\right)$ at level 2 if and only if $i_l^{(r)} \neq i_l^{(s)},\ l = 2, \ldots, n$. Clearly, a node at level 1 is a parent to $(n-1)^{d-1}$ nodes at level 2, and any arc connecting a node at level 1 to a node at level 2 belongs to some feasible path. Similarly, each node at level $j = 1, \ldots, n-1$ is a parent to $(n-1)^{d-1}$ nodes at level $j+1$. However, only $(n-j)^{d-1}$ of these descendent nodes are feasible with respect some node at level 1, i.e., only $(n-j)^{d-1}$ of arcs connecting a node at level j to nodes at level $j+1$ belong to feasible paths. In this way, the index tree contains $(n!)^{d-1}$ feasible paths, by the number of feasible solutions of the MAP (1).

The index graph representation of MAP aids in constructing of lower and upper bounds for the expected optimal cost of MAP (1) with random iid costs via the following lemmas [16].

Lemma 1 *Given the index graph $\mathcal{G} = (V, E)$ of $d \geq 3$, $n \geq 3$ MAP whose assignment costs that are iid random variables from an absolutely continuous distribution, construct set $\mathcal{A} \subset V$ by randomly selecting α different nodes from each level of the index graph. Then, $\mathcal{A}$ is expected to contain a feasible solution of the MAP if*

$$\alpha = \left\lceil \frac{n^{d-1}}{n!^{\frac{d-1}{n}}} \right\rceil. \tag{3}$$

Lemma 2 *For a $d \geq 3$, $n \geq 3$ MAP whose cost coefficients are iid random variables from an absolutely continuous distribution F with existing first moment, define*

$$\underline{Z}^*_{d,n} := n\mathsf{E}_F\left[X_{(1|\kappa)}\right] \quad \textit{and} \quad \overline{Z}^*_{d,n} := n\mathsf{E}_F\left[X_{(\alpha|\kappa)}\right], \tag{4}$$

where $X_{(i|\kappa)}$ is the i-th order statistic of $\kappa = n^{d-1}$ iid random variables with distribution F, and α is determined as in (3)*. Then, $\underline{Z}^*_{d,n}$ and $\overline{Z}^*_{d,n}$ constitute lower and upper bounds for the expected optimal cost $Z^*_{d,n}$ of the MAP, respectively: $\underline{Z}^*_{d,n} \leq Z^*_{d,n} \leq \overline{Z}^*_{d,n}$.*

Above, by an absolutely continuous distribution we mean any continuous distribution whose c.d.f. $F(\cdot)$ is representable as $F(x) = \int_{-\infty}^{x} f(y)\,\mathrm{d}y$, where $f(\cdot)$ is the (continuous) density of the distribution.

Proofs of the lemmas are based on the probabilistic method [4] and can be found in [16]. In particular, the proof of Lemma 2 considers set $\mathcal{A}_{\min}$ that is constructed by selecting from each level of the index graph α nodes with the smallest costs among the κ nodes at that level. The continuity of distribution F ensures that assignment costs in the MAP (1) are all different almost surely, hence locations of the nodes that comprise the set $\mathcal{A}_{\min}$ are random with respect to the array of nodes in each level of $\mathcal{G}(V, E)$. In the remainder of the paper, we always refer to α and κ as defined above.

As the problem size of an instance of MAP (1) is governed by two parameters, the number d of "dimensions," and number n of elements in each dimension, asymptotic analysis of the expected optimal value $Z^*_{d,n}$ of large-scale instances of the MAP (1) involves two cases: $n \to \infty$ while d is fixed, and $d \to \infty$ for a fixed n. Throughout the paper, we use notation $\kappa \xrightarrow{n} \infty$ and $\kappa \xrightarrow{d} \infty$, respectively, to address these two situations. If a certain statement holds for both cases of $n \to \infty$ and $d \to \infty$, we indicate this by $\kappa \xrightarrow{n,d} \infty$. The rationale behind the proposed notation is that an increase in either n or

d entails polynomial or exponential growth in the number κ of nodes in each level of the index graph: $\kappa = n^{d-1} \to \infty$.

The behavior of quantity α (3) in the cases $\kappa \xrightarrow{n} \infty$ and $\kappa \xrightarrow{d} \infty$ is more contrasting. When the number of dimensions d stays constant, and the number n of elements in each dimension increases indefinitely, α approaches a finite limiting value,

$$\alpha \to \alpha^* := \lceil e^{d-1} \rceil, \quad \kappa \xrightarrow{n} \infty, \tag{5}$$

while in the case of fixed n and unbounded d it increases exponentially:

$$\alpha \sim \kappa^{\gamma_n}, \quad \kappa \xrightarrow{d} \infty, \quad \text{where} \quad \gamma_n = 1 - \frac{\ln n!}{n \, \ln n}. \tag{6}$$

The range of values that parameter γ_n takes for different n is specified by the following

Lemma 3 *For all $n \geq 3$, the coefficient γ_n in* (6) *satisfies* $0 < \gamma_n < \frac{1}{2}$.

Proof: We start with establishing a simple inequality

$$n + 1 \geq \left(1 + \tfrac{1}{n}\right)^n, \quad n \geq 2, \tag{7}$$

that is easily validated by the induction argument. Indeed, observe that (7) holds for $n = 2$. Then, assume that (7) holds for some n, which, due to the fact that $\frac{1}{n} \geq \frac{1}{n+1}$, further implies

$$n + 1 \geq \left(1 + \tfrac{1}{n+1}\right)^n. \tag{8}$$

Multiplying both sides of inequality (8) by $\left(1 + \frac{1}{n+1}\right)$ we obtain

$$n + 2 = (n+1)\left(1 + \tfrac{1}{n+1}\right) \geq \left(1 + \tfrac{1}{n+1}\right)^{n+1},$$

which verifies (7) for $n + 1$. Now, to obtain the statement of the Lemma, it is convenient to represent γ_n as

$$\gamma_n = 1 - \frac{1}{n} \sum_{r=1}^{n} \frac{\ln r}{\ln n}. \tag{9}$$

Evidently, since $0 < \frac{\ln r}{\ln n} < 1$ for $r = 2, \ldots, n-1$, the second term in (9) satisfies $0 < \frac{1}{n} \sum_{r=1}^{n} \frac{\ln r}{\ln n} < 1$. Hence, $0 < \gamma_n < 1$ for $n \geq 3$. The upper bound $\gamma_n < \frac{1}{2}$ for $n \geq 3$ is established by showing that

$$\sum_{r=1}^{n} \ln r > \frac{n}{2} \ln n, \quad n \geq 3. \tag{10}$$

Direct evaluation confirms that (10) holds for $n = 3$. Then, following the induction argument, we demonstrate that (9) holds for some fixed $n + 1$, provided that it holds for n. For $n + 1$ the right-hand side of (10) equals to

$$\frac{n+1}{2} \ln(n+1) = \frac{n}{2} \ln n + \frac{1}{2} \ln(n+1) + \frac{n}{2} \ln\left(1 + \tfrac{1}{n}\right),$$

hence, to prove that (10) holds for $n+1$ it suffices to show that

$$\ln(n+1) \geq \frac{1}{2}\ln(n+1) + \frac{n}{2}\ln\left(1+\tfrac{1}{n}\right).$$

The last inequality is equivalent to (7), which proves the Lemma. ■

The reader has already noticed that the presented lemmata addresses only MAPs with $d \geq 3,\ n \geq 3$. Before proceeding further, we comment shortly on two special cases of the MAP (1) when $d = 2$ or $n = 2$. The case $d = 2$ represents, as noted earlier, the Linear Assignment Problem, whose asymptotic behavior is distinctly different from that of MAPs with $d \geq 3$ (see Sections 3 and 4). It can be shown that in the case of $d = 2$ Lemmas 1 and 2 produce only trivial bounds that are rather inefficient in determining the asymptotic behavior of the expected optimal value of the LAP within the presented approach.

When $n = 2$, the costs of feasible solutions to the MAP (1) have the form

$$z = c_{i_1^{(1)}\cdots\, i_d^{(1)}} + c_{i_1^{(2)}\cdots\, i_d^{(2)}}, \quad \text{where} \quad i_j^{(1)}, i_j^{(2)} \in \{1,2\}, \quad i_j^{(1)} \neq i_j^{(2)},$$

and consequently are iid random variables with distribution F_2, which is the convolution of F with itself: $F_2 = F * F$ [17]. Then, obviously, the expected optimal value of a random MAP with $n = 2$ can be expressed as expectation of the minimum order statistic of $(2!)^{d-1}$ iid variables with distribution F_2:

$$Z^*_{d,2} = \mathsf{E}_{F*F}\big[X_{(1|2^{d-1})}\big], \tag{11}$$

which simplifies the analysis of the expected optimal cost as compared to the general case of $n \geq 3$ that necessitates the use of the lower and upper bounds (4).

3 Main result

In this section we derive the limiting values of the expected optimal cost $Z^*_{d,n}$ of a random MAP (1) with iid cost coefficients drawn from a broad class of continuous distributions, when one of the dimension parameters n or d of the problem increases indefinitely. First we lay out preliminary considerations that make a foundation for more intricate analysis presented below.

According to Lemma 2, the upper bound on the expected optimal cost $Z^*_{d,n}$ of a random MAP with iid cost coefficients from an absolutely continuous distribution F is given by $\overline{Z}^*_{d,n} = n\,\mathsf{E}_F\left[X_{(\alpha|\kappa)}\right]$, where $X_{(\alpha|\kappa)}$ is the α-th order statistic among κ independent F-distributed random variables. It is well-known (see, e.g., [11]) that order statistics have distributions of beta type; for instance, the distribution of $X_{(\alpha|\kappa)}$ is given by

$$\mathsf{P}[X_{(\alpha|\kappa)} \leq x\,] = \frac{\kappa!}{(\alpha-1)!\,(\kappa-\alpha)!}\int_0^{F(x)} t^{\alpha-1}\,(1-t)^{\kappa-\alpha}\,\mathrm{d}t.$$

Then, assuming that the interior of the support set of distribution F has the form (a,b), where

$$a = \inf\{\,x \mid F(x) > 0\,\}, \quad b = \sup\{\,x \mid F(x) < 1\,\},$$

the upper bound $\overline{Z}^*_{d,n}$ can be represented as

$$\overline{Z}^*_{d,n} = \frac{n\,\Gamma(\kappa+1)}{\Gamma(\alpha)\,\Gamma(\kappa-\alpha+1)}\,I_{\alpha,\kappa}, \tag{12}$$

with $I_{\alpha,\kappa}$ being the integral of the form

$$I_{\alpha,\kappa} = \int_a^b x\,[F(x)]^{\alpha-1}[1-F(x)]^{\kappa-\alpha}\;\mathrm{d}F(x) = \int_0^1 F^{-1}(u)\,u^{\alpha-1}(1-u)^{\kappa-\alpha}\;\mathrm{d}u. \tag{13}$$

In the last equality, $F^{-1}(\cdot)$ denotes the inverse of the c.d.f. $F(x)$ of distribution of assignment costs in the MAP (1). While it is practically impossible to evaluate integral (13) exactly in the general case, the asymptotic behavior of (13) for large n and d can be determined for a wide range of distributions F. In particular, it will be seen that the asymptotic value of $I_{\alpha,\kappa}$, and, consequently, that of $Z^*_{d,n}$ for $n, d \gg 1$ depends on the location of the left endpoint a of the support of F.

The exposition of the results in the sequel assumes familiarity with the foundations of asymptotical analysis and theory of special functions. For example, the asymptotic properties of integrals of type (13) for large values of n and d can be quantified in terms of the Beta and Gamma functions,

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \quad \Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,\mathrm{d}t. \tag{14}$$

Although we make extensive use of these and other related special functions, we do not discuss herein their numerous properties that are used to derive the majority of the results of this paper. Instead, we refer the reader to an excellent treatise on special functions edited by Erdélyi [13]; also, a comprehensive collection of facts and formulae on special functions is available at [41]. Text by Olver [28] is another excellent reference to both asymptotical analysis and the theory of special functions.

3.1 Distributions with a finite left endpoint of the support

The asymptotic analysis of the expected optimal value of the random MAPs with continuous distributions is facilitated by means of the following lemma, which provides asymptotic series representation on a scale of Beta functions for integrals of type (13).

Lemma 4 *Let function $h(u)$ have the following asymptotic expansion at $0+$,*

$$h(u) \sim \sum_{s=0}^{\infty} a_s u^{(s+\lambda-\mu)/\mu}, \quad u \to 0+, \tag{15}$$

where $\lambda, \mu > 0$. Then for any positive integer m one has

$$\int_0^1 h(u)\,u^{\alpha-1}(1-u)^{\kappa-\alpha}\,\mathrm{d}u = \sum_{s=0}^{m-1} a_s\,\phi_s(\kappa) + \mathcal{O}\big(\phi_m(\kappa)\big), \quad \kappa \xrightarrow{n,\,d} \infty, \tag{16}$$

where $\phi_s(\kappa) = \mathrm{B}\big(\frac{s+\lambda}{\mu} + \alpha - 1,\ \kappa - \alpha + 1\big)$, $s = 0, 1, \ldots$, provided that the integral is absolutely convergent for $\kappa = \alpha = 1$.

Proof: First we demonstrate that the sequence of functions $\{\phi_s(\kappa)\}_{s=0}^{\infty}$ represents a *scale* [28] in both cases of large n and d, i.e., $\phi_{s+1}(\kappa) = o\big(\phi_s(\kappa)\big)$, $\kappa \xrightarrow{n,\,d} \infty$. The ratio of $\phi_{s+1}(\kappa)$ and $\phi_s(\kappa)$ has the form

$$\frac{\phi_{s+1}(\kappa)}{\phi_s(\kappa)} = \frac{\Gamma\big(\frac{s+1+\lambda}{\mu} + \alpha - 1\big)}{\Gamma\big(\frac{s+\lambda}{\mu} + \alpha - 1\big)} \cdot \frac{\Gamma\big(\kappa + \frac{s+\lambda}{\mu}\big)}{\Gamma\big(\kappa + \frac{s+1+\lambda}{\mu}\big)}.$$

Using a well-known expansion for the quotient of the Gamma functions [13]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\Big(1+\mathcal{O}\big(z^{-1}\big)\Big), \quad z\to\infty, \tag{17}$$

we obtain that for a fixed d and $n \to \infty$,

$$\frac{\phi_{s+1}(\kappa)}{\phi_s(\kappa)} \sim \frac{\Gamma\big(\frac{s+1+\lambda}{\mu}+\alpha^*-1\big)}{\Gamma\big(\frac{s+\lambda}{\mu}+\alpha^*-1\big)}\,\kappa^{-1/\mu} \;\to\; 0, \quad \kappa \xrightarrow{n} \infty, \quad \mu > 0.$$

Similarly, for $d \to \infty$ and n fixed, by virtue of Lemma 3 one has

$$\frac{\phi_{s+1}(\kappa)}{\phi_s(\kappa)} \sim \frac{\kappa^{\gamma_n/\mu}}{\kappa^{1/\mu}} = \kappa^{\frac{\gamma_n-1}{\mu}} \to 0, \quad \kappa \xrightarrow{d} \infty, \quad \mu > 0.$$

Now we show that expansion (16) holds. For any non-negative integer m, let

$$h(u) = \sum_{s=0}^{m-1} a_s u^{(s+\lambda-\mu)/\mu} + \varphi_m(u), \quad 0 < u < 1. \tag{18}$$

Substituting the above equality in the left-hand side of (16), one obtains

$$\int_0^1 h(u)\, u^{\alpha-1}(1-u)^{\kappa-\alpha}\,\mathrm{d}u$$
$$= \sum_{s=0}^{m-1} a_s \mathrm{B}\big(\tfrac{s+\lambda}{\mu}+\alpha-1,\ \kappa-\alpha+1\big) + \int_0^1 \varphi_m(u)\, u^{\alpha-1}(1-u)^{\kappa-\alpha}\,\mathrm{d}u. \tag{19}$$

To estimate the remainder term in (19), observe that $\varphi_m(u) = \mathcal{O}\big(u^{(m+\lambda-\mu)/\mu}\big)$ for $u \to 0+$, in accordance with the definition of asymptotic expansion (15). Consequently, there exist $u_m \in (0,1)$ such that $C_m = \sup\limits_{u\in(0,\,u_m]} \big|\varphi_m(u)\, u^{-(m+\lambda-\mu)/\mu}\big|$ is finite, whence

$$\left|\int_0^{u_m} \varphi_m(u)\, u^{\alpha-1}(1-u)^{\kappa-\alpha}\,\mathrm{d}u\right| \le C_m\, \mathrm{B}\big(\tfrac{m+\lambda}{\mu}+\alpha-1,\ \kappa-\alpha+1\big). \tag{20}$$

Since integral (16) is absolutely convergent for $\kappa = \alpha = 1$, the integral $\int_{u_m}^1 \varphi_m(u)\,\mathrm{d}u$ is absolutely convergent too, which implies absolute convergence of $\int_{u_m}^1 \varphi_m(u)\, u^{-M}\,\mathrm{d}u$ for any $M > 0$:

$$\int_{u_m}^1 \big|\varphi_m(u)\big|\,\mathrm{d}u \le \int_{u_m}^1 \big|\varphi_m(u)\, u^{-M}\big|\,\mathrm{d}u \le u_m^{-M}\int_{u_m}^1 \big|\varphi_m(u)\big|\,\mathrm{d}u.$$

For $M = \frac{m+\lambda}{\mu} + 1$ denote $\Xi_m = \sup\limits_{\tau\in[u_m,\,1]} \big|\xi_m(\tau)\big|$, where $\xi_m(\tau) = \int_{u_m}^{\tau} \varphi_m(u)\, u^{-M}\,\mathrm{d}u$. Integrating the remainder term in (19) by parts, we have

$$\int_{u_m}^1 \varphi_m(u)\, u^{\alpha-1}(1-u)^{\kappa-\alpha}\,\mathrm{d}u = -(\alpha+M-1)\int_{u_m}^1 \xi_m(u)\, u^{\alpha+M-2}(1-u)^{\kappa-\alpha}\,\mathrm{d}u$$
$$+ (\kappa-\alpha)\int_{u_m}^1 \xi_m(u)\, u^{\alpha+M-1}(1-u)^{\kappa-\alpha-1}\,\mathrm{d}u, \tag{21}$$

which implies that the first integral in (21) is bounded as

$$\left| \int_{u_m}^{1} \varphi_m(u)\, u^{\alpha-1}(1-u)^{\kappa-\alpha}\, \mathrm{d}u \right| < 2\, \Xi_m \frac{\Gamma(\alpha+M)\,\Gamma(\kappa-\alpha+1)}{\Gamma(\kappa+M)}. \tag{22}$$

Considering the behavior of the right-hand side of (22) with respect to $\phi_m(\kappa)$ for large κ,

$$\frac{\Gamma(\alpha+M)\,\Gamma(\kappa-\alpha+1)}{\Gamma(\kappa+M)} \cdot \frac{\Gamma\big(\kappa+\frac{m+\lambda}{\mu}\big)}{\Gamma\big(\alpha+\frac{m+\lambda}{\mu}-1\big)\,\Gamma(\kappa-\alpha+1)},$$

we see that their ratio is of order of κ^{-1} when $\kappa \xrightarrow{n} \infty$, and is $\mathcal{O}\big(\kappa^{2\gamma_n-1}\big)$ if $\kappa \xrightarrow{d} \infty$. Hence, by Lemma 3 we have that

$$\left| \int_{u_m}^{1} \varphi_m(u)\, u^{\alpha-1}(1-u)^{\kappa-\alpha}\, \mathrm{d}u \right| = o\left(\mathrm{B}\big(\tfrac{m+\lambda}{\mu}+\alpha-1,\, \kappa-\alpha+1\big)\right), \quad \kappa \xrightarrow{n,d} \infty. \tag{23}$$

Combining (19), (20), and (23) we arrive at the sought equality (16). ∎

With the help of Lemma 4 it is now straightforward to derive the limiting value of the expected optimal cost of MAP with random iid cost coefficients that have a continuous p.d.f. with a finite left endpoint of the support.

Theorem 1 *Consider a $d \geq 3$, $n \geq 3$ MAP (1) with cost coefficients that are iid random variables from an absolutely continuous distribution with existing first moment. Let $(a,\ b)$, where $a \in \mathbb{R}$, $b \in (a, +\infty]$, be the interior of the support set of this distribution. Further, assume that the inverse $F^{-1}(x)$ of the c.d.f. $F(x)$ of the distribution is such that*

$$F^{-1}(x) = a + \mathcal{O}\big(x^{\beta}\big), \quad x \to 0+, \quad \beta > 0. \tag{24}$$

Then, for a fixed n and $d \to \infty$, or a fixed d and $n \to \infty$, the expected optimal value $Z^*_{d,n}$ satisfies

$$\lim_{d\to\infty} Z^*_{d,n} = na, \qquad \lim_{n\to\infty} Z^*_{d,n} = \begin{cases} +\infty, & a > 0, \\ 0, & a = 0, \\ -\infty, & a < 0. \end{cases} \tag{25}$$

Proof: According to the aforesaid, an upper bound on the expected optimal value of a random MAP whose assignment costs are iid random variables with distribution F is obtained via (12)–(13). Existence of the first moment of F ensures the existence of its order statistics, as well as absolute integrability of $F^{-1}(\cdot)$ on the interval $(0, 1)$. Given the asymptotic behavior (24) of function $F^{-1}(u)$ at $u = 0+$, application of Lemma 4 to integral $I_{\alpha,\kappa}$ (13) produces

$$I_{\alpha,\kappa} = a\frac{\Gamma(\alpha)\,\Gamma(\kappa-\alpha+1)}{\Gamma(\kappa+1)} + \mathcal{O}\left(\frac{\Gamma(\alpha+\beta)\,\Gamma(\kappa-\alpha+1)}{\Gamma(\kappa+\beta+1)}\right), \quad \kappa \xrightarrow{n,d} \infty,$$

which, upon substitution into (12), leads to

$$\overline{Z}^*_{d,n} = na + \mathcal{O}\left(n\frac{\Gamma(\alpha+\beta)\,\Gamma(\kappa+1)}{\Gamma(\alpha)\,\Gamma(\kappa+\beta+1)}\right), \quad \kappa \xrightarrow{n,d} \infty. \tag{26}$$

By asymptotic expansion (17) the second term in (26) vanishes as $\kappa \xrightarrow{n,d} \infty$, yielding that the upper bound $\overline{Z}^*_{d,n}$ satisfies the relations (25). On the other hand, since a is the left endpoint of the support set of distribution F, the cost z of a feasible solution (2) of the MAP is bounded from below by $\inf z = na$. It is easy to see that $\inf z$ does also satisfy equalities (25), which, by virtue of inequality

$$\inf z \ \le \ Z^*_{d,n} \ \le \ \overline{Z}^*_{d,n},$$

proves the statement (25) of the Theorem. ■

The conditions of Theorem 1 are quite general: it is only required that function $F^{-1}(\cdot)$ is absolutely integrable on (0, 1) and there exists $\beta > 0$ such that

$$\lim_{x \to 0+} \left| x^{-\beta} \big(F^{-1}(x) - a \big) \right| < \infty.$$

The case when distribution F is such that the rate of convergence of $F^{-1}(x) - a$ as $x \to 0+$ is slower than that of x^β, for some $\beta > 0$, can be considered similarly, based on the particular form of $F^{-1}(\cdot)$.

The proved theorem evidences that the asymptotic behavior of the expected optimal value of a random MAP whose distribution has a bounded from below support is quite dissimilar in cases of $d \gg 1$ and $n \gg 1$. Indeed, for a fixed d and $n \to \infty$, the expected optimal value $Z^*_{d,n}$ of a random MAP with corresponding distribution F is finite only when the left endpoint of the support of F is located at the origin: $a = 0$. On the contrary, $\lim_{d \to \infty} Z^*_{d,n}$ for a fixed n is finite as long as a is finite.

Trivially, finiteness of the left endpoint a of the support of the distribution implies finiteness of the function $F^{-1}(\cdot)$ at the origin: $F^{-1}(0) = a \in \mathbb{R}$. If the support of distribution F has infinite left endpoint, $a = -\infty$, the inverse $F^{-1}(x)$ of the distribution's c.d.f. necessarily has a singularity at $x = 0+$:

$$\lim_{x \to 0+} F^{-1}(x) = -\infty.$$

The next subsection contains an analysis of the expected optimal value of MAP with iid cost coefficients drawn from a continuous distribution whose support has infinite left endpoint.

3.2 Distributions with infinite left endpoint of the support

The case of a continuous distribution with infinite left endpoint of the support set is more cumbersome as different singularities of $F^{-1}(x)$ at $x = 0+$ require differing approaches to their treatment. To this end, we consider two most common singularities of $F^{-1}(0+)$ encountered in applications: a power singularity and a logarithmic singularity.

We say that $F^{-1}(x)$ has a power singularity at $x = 0+$ if $F^{-1}(x) \sim -\nu x^{-\beta}$, $x \to 0+$, where ν and β are positive constants. It is easy to see that for function $F^{-1}(x)$ to be integrable over the interval (0, 1), the range of values for parameter β must be limited to $0 < \beta < 1$. The next theorem addresses this case.

Theorem 2 *Consider a $d \ge 3$, $n \ge 3$ MAP* (1) *with cost coefficients that are iid random variables from an absolutely continuous distribution whose expectation exits. Let $(-\infty, b)$, where $b \in (-\infty, +\infty]$, be the support set of this distribution. Further, assume that the inverse $F^{-1}(x)$ of the c.d.f. $F(x)$ of the distribution has an asymptotic expansion at $x \to 0+$*

$$F^{-1}(x) \sim -\nu x^{-\beta}, \quad x \to 0+, \quad 0 < \beta < 1, \quad \nu > 0. \tag{27}$$

Then, for a fixed n and $d \to \infty$, or a fixed d and $n \to \infty$, the expected optimal value $Z^*_{d,n}$ of the MAP is unbounded from below:

$$\lim_{d\to\infty} Z^*_{d,n} = \lim_{n\to\infty} Z^*_{d,n} = -\infty. \tag{28}$$

Proof: Essentially the same as that of Theorem 1. ■

Now we examine the case of the inverse $F^{-1}(x)$ of distribution $F(\cdot)$ having a (weaker) logarithmic singularity at zero, $F^{-1}(x) \sim -\nu\big(\ln \frac{1}{x}\big)^{\beta},\ x \to 0+$, where ν and β are again positive numbers. Similarly, in this case the expected optimal value of MAP is also unbounded from below. A key element of the subsequent analysis is the asymptotical valuation of the integral

$$\int_0^1 u^{\alpha+\delta-1}(1-u)^{\kappa-\alpha} \ln\tfrac{1}{u}\, \mathrm{d}u, \tag{29}$$

where $\delta > 0$, for large values of n and d. By differentiating the identity for the Beta function (see (14))

$$\int_0^1 u^{\alpha+\delta-1}(1-u)^{\kappa-\alpha}\, \mathrm{d}u = \frac{\Gamma(\alpha+\delta)\,\Gamma(\kappa-\alpha+1)}{\Gamma(\kappa+\delta+1)}$$

with respect to α, one has

$$\int_0^1 u^{\alpha+\delta-1}(1-u)^{\kappa-\alpha} \ln\tfrac{1}{u}\, \mathrm{d}u = -\frac{1}{\Gamma(\kappa+\delta+1)}\frac{\partial}{\partial\alpha}\big[\Gamma(\alpha+\delta)\,\Gamma(\kappa-\alpha+1)\big] - \int_0^1 u^{\alpha+\delta}(1-u)^{\kappa-\alpha}\ln(1-u)\ \mathrm{d}u. \tag{30}$$

Given that $\ln(1-u) = -u + \mathcal{O}\big(u^2\big),\ u \to 0+$, the asymptotic value of the last integral in (30) is obtained by application of Lemma 4 as

$$\int_0^1 u^{\alpha+\delta}(1-u)^{\kappa-\alpha}\ln(1-u)\, \mathrm{d}u = \mathcal{O}\left(\frac{\Gamma(\alpha+\delta+1)\,\Gamma(\kappa-\alpha+1)}{\Gamma(\kappa+\delta+2)}\right).$$

Next, the first term in the right-hand side of (30) can be rewritten using the logarithmic derivative of the Gamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ in the form

$$\frac{\Gamma(\alpha+\delta)\,\Gamma(\kappa-\alpha+1)}{\Gamma(\kappa+\delta+1)}\big[\psi(\kappa-\alpha+1) - \psi(\alpha+\delta)\big].$$

Employing the asymptotic expansion of $\psi(z)$ for large values of argument [13],

$$\psi(z) = \ln z - \frac{1}{2z} + \mathcal{O}\big(z^{-2}\big), \quad z \to \infty, \tag{31}$$

we arrive at the sought asymptotical expressions for integral (29). In the case $n \to \infty$, we have

$$\int_0^1 u^{\alpha+\delta-1}(1-u)^{\kappa-\alpha} \ln\tfrac{1}{u}\, \mathrm{d}u = \frac{\Gamma(\alpha^*+\delta)}{\kappa^{\alpha^*+\delta}}\ \ln\kappa\,\left(1 + \mathcal{O}\left(\frac{1}{\ln\kappa}\right)\right), \quad \kappa \xrightarrow{n} \infty. \tag{32}$$

In case $d \to \infty$ integral (29) equals to

$$\int_0^1 u^{\alpha+\delta-1}(1-u)^{\kappa-\alpha} \ln\tfrac{1}{u}\,\mathrm{d}u = \frac{\Gamma(\kappa^{\gamma_n}+\delta)\,\Gamma(\kappa-\kappa^{\gamma_n}+1)}{\Gamma(\kappa+\delta+1)}\Big\{(1-\gamma_n)\ln\kappa + \mathcal{O}\big(\kappa^{-\gamma_n}\big)\Big\}, \quad \kappa \xrightarrow{d} \infty. \tag{33}$$

By use of the expansion

$$\frac{\Gamma(\kappa^{\gamma_n}+\delta)}{\Gamma(\kappa+\delta+1)} = \frac{e^{\kappa-\kappa^{\gamma_n}}\kappa^{(1-3\gamma_n)/2}}{\kappa^{\kappa-\gamma_n\kappa^{\gamma_n}}}\kappa^{-(1-\gamma_n)(\delta+1)}\Big(1+\mathcal{O}\big(\kappa^{-\gamma_n}\big)\Big), \quad \kappa \xrightarrow{d} \infty, \tag{34}$$

expression (33) can be transformed to

$$\int_0^1 u^{\alpha+\delta-1}(1-u)^{\kappa-\alpha} \ln\tfrac{1}{u}\,\mathrm{d}u = R_{\gamma_n}^{\kappa}\,\kappa^{-(1-\gamma_n)(\delta+1)}\,(1-\gamma_n)\ln\kappa\,\Big(1+\mathcal{O}\left(\kappa^{-\gamma_n}\right)\Big), \quad \kappa \xrightarrow{d} \infty, \tag{35}$$

where $R_{\gamma_n}^{\kappa}$ stands for

$$R_{\gamma_n}^{\kappa} = \Gamma(\kappa-\kappa^{\gamma_n}+1)\,e^{\kappa-\kappa^{\gamma_n}}\,\kappa^{-\kappa+\gamma_n\kappa^{\gamma_n}+(1-3\gamma_n)/2}. \tag{36}$$

Now we are ready to prove the main result of this subsection.

Theorem 3 *Consider a $d \geq 3$, $n \geq 3$ MAP (1) with cost coefficients that are iid random variables from an absolutely continuous distribution F whose expectation exits. Let $(-\infty, b)$, where $b \in (-\infty, +\infty]$, be the interior of the support set of F, and assume that the inverse $F^{-1}(x)$ of the c.d.f. $F(x)$ of the distribution has a logarithmic singularity at $x \to 0+$,*

$$F^{-1}(x) \sim -\nu\left(\ln\tfrac{1}{x}\right)^{\beta}, \quad x \to 0+, \quad \textit{where} \quad \beta > 0, \quad \nu > 0. \tag{37}$$

Then, for a fixed n and $d \to \infty$, or a fixed d and $n \to \infty$, the expected optimal value $Z_{d,n}^*$ of the MAP is unbounded from below:

$$\lim_{d\to\infty} Z_{d,n}^* = \lim_{n\to\infty} Z_{d,n}^* = -\infty. \tag{38}$$

Proof: As before, the proof is based on examining the asymptotical behavior of the upper bound $\overline{Z}_{d,n}^*$, which in the case of a logarithmic singularity (37) of $F^{-1}(x)$ at $x = 0+$ comes down to asymptotic evaluation of the integral

$$\int_0^1 u^{\alpha-1}(1-u)^{\kappa-\alpha}\left(\ln\tfrac{1}{u}\right)^{\beta}\,\mathrm{d}u \tag{39}$$

for large values of n and d. The asymptotical properties of this integral as n or d approach infinity are determined by the behavior of the integrand in the vicinity of 0; in particular, for any $\beta > 0$ one has

$$\int_0^1 u^{\alpha-1}(1-u)^{\kappa-\alpha}\left(\ln\tfrac{1}{u}\right)^{\beta}\mathrm{d}u = \int_0^{1/e} u^{\alpha-1}(1-u)^{\kappa-\alpha}\left(\ln\tfrac{1}{u}\right)^{\beta}\mathrm{d}u + \mathcal{O}\left(\frac{\Gamma(\alpha)\,\Gamma(\kappa-\alpha+1)}{\Gamma(\kappa+1)}\right), \tag{40}$$

for $\kappa \xrightarrow{n,d} \infty$. Indeed, observe that $\frac{1}{e} \leq u \leq 1$ implies that $0 \leq \left(\ln\frac{1}{u}\right)^{\beta} \leq 1$ for $\beta > 0$, therefore the following integral is bounded as

$$\int_{1/e}^1 u^{\alpha-1}(1-u)^{\kappa-\alpha}\left(\ln\tfrac{1}{u}\right)^{\beta}\mathrm{d}u \leq \int_{1/e}^1 u^{\alpha-1}(1-u)^{\kappa-\alpha}\,\mathrm{d}u \leq \int_0^1 u^{\alpha-1}(1-u)^{\kappa-\alpha}\,\mathrm{d}u = \mathrm{B}(\alpha,\ \kappa-\alpha+1),$$

which entails the following asymptotic expression that verifies equality (40):

$$\int_{1/e}^{1} u^{\alpha-1}(1-u)^{\kappa-\alpha} \left(\ln \tfrac{1}{u}\right)^{\beta} \, \mathrm{d}u = \mathcal{O}\left(\frac{\Gamma(\alpha)\,\Gamma(\kappa-\alpha+1)}{\Gamma(\kappa+1)}\right), \quad \kappa \xrightarrow{n,d} \infty. \tag{41}$$

The choice of constant $1/e$ that splits the integration interval in (40) is merely due to our convenience; a similar decomposition can be obtained for any small enough positive number between 0 and 1. Following [12], to estimate the asymptotical behavior of integral

$$\int_{0}^{1/e} u^{\alpha-1}(1-u)^{\kappa-\alpha} \left(\ln \tfrac{1}{u}\right)^{\beta} \, du = J(\alpha,\, \kappa-\alpha+1,\, \beta), \tag{42}$$

that enters the right-hand side of decomposition (40), we employ the functional equation

$$\frac{1}{\Gamma(\varrho)} \int_{0}^{\infty} z^{\varrho-1}\, J(\alpha+z,\, \varkappa,\, \beta)\, \mathrm{d}z = J(\alpha,\, \varkappa,\, \beta-\varrho), \quad \alpha, \beta, \varkappa, \varrho > 0, \tag{43}$$

which is easily derivable by plugging (42) into the left-hand side of (43) and interchanging the order of integration. First we consider the case of $0 < \beta < 1$. For $\varrho \in (0,1)$ let $\beta = 1-\varrho$, then integral $J(\alpha, \kappa-\alpha+1, \beta)$ (42) can be written as a sum of two other integrals, J_1 and J_2,

$$J(\alpha,\ \kappa-\alpha+1,\ 1-\varrho) = \frac{1}{\Gamma(\varrho)} \left(\int_{0}^{\zeta} + \int_{\zeta}^{\infty}\right) z^{\varrho-1} J(\alpha+z,\ \kappa-\alpha+1,\ 1)\, \mathrm{d}z = J_1 + J_2, \tag{44}$$

where $\zeta > 0$ is a positive number. With interchange of the order of integration, J_2 reduces to

$$J_2 = \frac{1}{\Gamma(\varrho)} \int_{0}^{1/e} u^{\alpha+\zeta}(1-u)^{\kappa-\alpha} \ln\tfrac{1}{u}\, \mathrm{d}u \int_{\zeta}^{\infty} z^{\varrho-1} u^{z-\zeta}\, \mathrm{d}z. \tag{45}$$

Since $0 \le u \le \frac{1}{e}$, the inner integral in (45) is a bounded function of u,

$$\int_{\zeta}^{\infty} z^{\varrho-1} u^{z-\zeta}\, \mathrm{d}z = \int_{\zeta}^{\infty} z^{\varrho-1} e^{-(z-\zeta)\ln u}\, \mathrm{d}z \le \int_{\zeta}^{\infty} z^{\varrho-1} e^{-(z-\zeta)}\, \mathrm{d}z,$$

hence, J_2 can be estimated as

$$J_2 = \mathcal{O}\left(\int_{0}^{1/e} u^{\alpha+\zeta-1}(1-u)^{\kappa-\alpha} \ln\tfrac{1}{u}\, \mathrm{d}u\right) \quad \kappa \xrightarrow{n,d} \infty. \tag{46}$$

Next, consider integral J_1 in (44):

$$J_1 = \frac{1}{\Gamma(\varrho)} \int_{0}^{\zeta} z^{\varrho-1}\, \mathrm{d}z \int_{0}^{1/e} u^{\alpha+z-1}(1-u)^{\kappa-\alpha} \ln\tfrac{1}{u}\, \mathrm{d}u. \tag{47}$$

By means of decomposition (40), the inner integral in (47) can be represented as

$$\begin{aligned} &\int_{0}^{1/e} u^{\alpha+z-1}(1-u)^{\kappa-\alpha} \ln\tfrac{1}{u}\, \mathrm{d}u \\ &\qquad = \int_{0}^{1} u^{\alpha+z-1}(1-u)^{\kappa-\alpha} \ln\tfrac{1}{u}\, \mathrm{d}u + \mathcal{O}\left(\frac{\Gamma(\alpha+z)\,\Gamma(\kappa-\alpha+1)}{\Gamma(\kappa+z+1)}\right), \quad \kappa \xrightarrow{n,d} \infty. \end{aligned} \tag{48}$$

Further treatment of integrals J_1 and J_2 depends on the rate with which κ approaches infinity, i.e., which of the parameters n and d is fixed and which is infinitely increasing. We structure the remainder of the proof of the theorem in the following steps.

Step 1: $\kappa \xrightarrow{n} \infty$. If n goes to infinity and d is fixed, we have by (5) that $\alpha \to \alpha^* \in \mathbb{R}$. In accordance to the derived asymptotic expansion (32), the integral in the right-hand side of (48) equals to

$$\int_0^1 u^{\alpha+z-1}(1-u)^{\kappa-\alpha} \ln\tfrac{1}{u}\, \mathrm{d}u = \frac{\Gamma(\alpha^*+z)}{\kappa^{\alpha^*+z}} \ln\kappa \, \left(1+\mathcal{O}\left(\frac{1}{\ln\kappa}\right)\right), \quad \kappa \xrightarrow{n} \infty.$$

Similarly, the last term in (48) admits representation

$$\frac{\Gamma(\alpha+z)\,\Gamma(\kappa-\alpha+1)}{\Gamma(\kappa+z+1)} = \mathcal{O}\left(\kappa^{-\alpha^*-z}\right), \quad \kappa \xrightarrow{n} \infty.$$

On inserting the last two equalities in (48) we obtain

$$J_1 = \frac{1}{\Gamma(\varrho)\,\kappa^{\alpha^*}} \left\{\Big(\ln\kappa + \mathcal{O}(1)\Big)\int_0^\zeta z^{\varrho-1}\Gamma(\alpha^*+z)\,\kappa^{-z}\,\mathrm{d}z + \int_0^\zeta z^{\varrho-1}\mathcal{O}\big(\kappa^{-z}\big)\,\mathrm{d}z\right\}. \tag{49}$$

Now, consider the first integral in the right-hand side of (49). Since $\Gamma(\alpha^*+z) = \Gamma(\alpha^*)+\mathcal{O}(z),\ z \to 0+$, this integral can be restated as

$$\int_0^\zeta z^{\varrho-1}\Gamma(\alpha^*+z)\,\kappa^{-z}\,\mathrm{d}z = \Gamma(\alpha^*)\int_0^\zeta z^{\varrho-1}\,\kappa^{-z}\,\mathrm{d}z + \mathcal{O}\left(\int_0^\zeta z^{\varrho}\,\kappa^{-z}\,\mathrm{d}z\right). \tag{50}$$

The first integral in the right-hand side of (50) can be expressed in terms of the incomplete Gamma function $\Gamma(a,\ \zeta) = \int_\zeta^\infty e^{-t}t^{a-1}\,\mathrm{d}t$ [13]:

$$\int_0^\zeta z^{\varrho-1}\,\kappa^{-z}\,\mathrm{d}z = \frac{\Gamma(\varrho)}{(\ln\kappa)^\varrho} - \frac{\Gamma(\varrho,\ \zeta\ln\kappa)}{(\ln\kappa)^\varrho}. \tag{51}$$

Using the following asymptotic relation for the incomplete Gamma function [13]

$$\frac{\Gamma(a,\ \zeta x)}{x^a} = \mathcal{O}\left(\frac{e^{-\zeta x}}{x}\right), \quad x \to \infty,\ \ a > 0, \tag{52}$$

the integrals in (50) can be evaluated as

$$\int_0^\zeta \kappa^{-z}z^{\varrho-1}\,\mathrm{d}z = \frac{\Gamma(\varrho)}{(\ln\kappa)^\varrho} + \mathcal{O}\left(\frac{1}{\kappa^\zeta\,\ln\kappa}\right),\ \ \mathcal{O}\left(\int_0^\zeta z^\varrho\,\kappa^{-z}\,\mathrm{d}z\right) = \mathcal{O}\left((\ln\kappa)^{-\varrho-1}\right), \quad \kappa \xrightarrow{n} \infty,$$

which yields the asymptotic value of (50) in the form

$$\int_0^\zeta z^{\varrho-1}\Gamma(\alpha^*+z)\,\kappa^{-z}\,\mathrm{d}z = \frac{\Gamma(\alpha^*)\,\Gamma(\varrho)}{(\ln\kappa)^\varrho}\left(1+\mathcal{O}\left(\frac{1}{\ln\kappa}\right)\right), \quad \kappa \xrightarrow{n} \infty.$$

On substitution of the last expression for integral (50) into representation (49) for J_1 we have

$$J_1 = \frac{\Gamma(\alpha^*)}{\kappa^{\alpha^*}}\,(\ln\kappa)^\beta\left(1+\mathcal{O}\left(\frac{1}{\ln\kappa}\right)\right), \quad \kappa \xrightarrow{n} \infty,$$

and, invoking expansions (46) and (32) with ζ large enough, we obtain

$$\int_0^1 u^{\alpha-1}(1-u)^{\kappa-\alpha}\left(\ln\tfrac{1}{u}\right)^\beta\,\mathrm{d}u=\frac{\Gamma(\alpha^*)}{\kappa^{\alpha^*}}(\ln\kappa)^\beta\left(1+\mathcal{O}\left(\frac{1}{(\ln\kappa)^\beta}\right)\right),\quad \kappa\xrightarrow{n}\infty. \tag{53}$$

To derive the counterpart of equality (53) for the case $\kappa\xrightarrow{d}\infty$, one has to take a slightly different route.

<u>Step 2: $\kappa\xrightarrow{d}\infty$.</u> In the case of d increasing and n fixed, we have that both α and κ are increasing, such that $\alpha\sim\kappa^{\gamma_n}$, $\kappa\to\infty$, where $0<\gamma_n<\frac{1}{2}$, according to (6) and Lemma 3. By virtue of representation (35), the integral in the right-hand side of (48) has the form

$$\int_0^1 u^{\kappa^{\gamma_n}+z-1}(1-u)^{\kappa-\kappa^{\gamma_n}}\ln\tfrac{1}{u}\,\mathrm{d}u=R^\kappa_{\gamma_n}\,\kappa^{-(1-\gamma_n)(z+1)}(1-\gamma_n)\ln\kappa\,\left(1+\mathcal{O}\big(\kappa^{-\gamma_n}\big)\right),\quad \kappa\xrightarrow{d}\infty.$$

Similarly, (34) yields

$$\mathcal{O}\left(\frac{\Gamma(\kappa^{\gamma_n}+z)\,\Gamma(\kappa-\kappa^{\gamma_n}+1)}{\Gamma(\kappa+z+1)}\right)=R^\kappa_{\gamma_n}\,\kappa^{-(1-\gamma_n)}\,\mathcal{O}\Big(\kappa^{-(1-\gamma_n)z}\Big),\quad \kappa\xrightarrow{d}\infty.$$

Combining the above two equalities, we have

$$\begin{aligned}\int_0^{1/e} & u^{\kappa^{\gamma_n}+z-1}(1-u)^{\kappa-\kappa^{\gamma_n}}\ln\tfrac{1}{u}\,\mathrm{d}u\\ &=R^\kappa_{\gamma_n}\,\kappa^{-(1-\gamma_n)}\left\{(1-\gamma_n)\ln\kappa\,\left(1+\mathcal{O}\big(\kappa^{-\gamma_n}\big)\right)\kappa^{-(1-\gamma_n)z}+\mathcal{O}\Big(\kappa^{-(1-\gamma_n)z}\Big)\right\},\quad \kappa\xrightarrow{d}\infty.\end{aligned}$$

By substituting the last expression in (48) we reduce integral J_1 to the form

$$J_1=\frac{R^\kappa_{\gamma_n}}{\Gamma(\varrho)}\,\kappa^{-(1-\gamma_n)}\left\{(1-\gamma_n)\ln\kappa\,\left(1+\mathcal{O}\big(\kappa^{-\gamma_n}\big)\right)\int_0^\zeta\kappa^{-(1-\gamma_n)\,z}z^{\varrho-1}\,\mathrm{d}z+\int_0^\zeta z^{\varrho-1}\mathcal{O}\left(\kappa^{-(1-\gamma_n)\,z}\right)\,\mathrm{d}z\right\}. \tag{54}$$

Similarly to the above, we represent the integrals in (54) in terms of the incomplete Gamma function as

$$\int_0^\zeta\kappa^{-(1-\gamma_n)\,z}z^{\varrho-1}\,\mathrm{d}z=\frac{\Gamma(\varrho)}{\big[(1-\gamma_n)\ln\kappa\big]^\varrho}-\frac{\Gamma\Big(\varrho,\ \zeta(1-\gamma_n)\ln\kappa\Big)}{\big[(1-\gamma_n)\ln\kappa\big]^\varrho},$$

and use asymptotic expansion (52) to obtain

$$\int_0^\zeta\kappa^{-(1-\gamma_n)\,z}z^{\varrho-1}\,\mathrm{d}z=\frac{\Gamma(\varrho)}{\big[(1-\gamma_n)\ln\kappa\big]^\varrho}+\mathcal{O}\left(\frac{1}{\kappa^{\zeta(1-\gamma_n)}\ln\kappa}\right),\quad \kappa\xrightarrow{d}\infty.$$

Substituting this result in (54) and choosing ζ large enough, we find the asymptotic value of J_1 for $d\gg 1$:

$$J_1=\frac{\Gamma(\kappa^{\gamma_n})\,\Gamma(\kappa-\kappa^{\gamma_n}+1)}{\Gamma(\kappa+1)}\,(1-\gamma_n)^\beta\,(\ln\kappa)^\beta\left(1+\mathcal{O}\Big(\frac{1}{\ln\kappa}\Big)\right),\quad \kappa\xrightarrow{d}\infty.$$

Using (35), J_2 can be evaluated as

$$J_2=\mathcal{O}\left(\frac{\Gamma(\kappa^{\gamma_n}+\zeta)\,\Gamma(\kappa-\kappa^{\gamma_n}+1)}{\Gamma(\kappa+\zeta+1)}\,\ln\kappa\right),$$

which finally yields the sought asymptotic representation of integral (39) for $d \to \infty$ and $\beta \in (0,1)$:

$$\int_0^1 u^{\alpha-1}(1-u)^{\kappa-\alpha}\left(\ln\tfrac{1}{u}\right)^{\beta}\mathrm{d}u \\ = \frac{\Gamma(\kappa^{\gamma_n})\,\Gamma(\kappa-\kappa^{\gamma_n}+1)}{\Gamma(\kappa+1)}\,(1-\gamma_n)^{\beta}\,(\ln\kappa)^{\beta}\left(1+\mathcal{O}\Big((\ln\kappa)^{-\beta}\Big)\right),\quad \kappa \xrightarrow{d} \infty. \tag{55}$$

<u>Step 3.</u> Now we use the developed expansions (53) and (55) to prove the unboundedness of the expected optimal value z^* of MAP when $F^{-1}(\cdot)$ has a logarithmic singularity (37) with $0 < \beta < 1$. Upon substitution of the asymptotic relation (37) into integral $I_{\alpha,\kappa}$ (13) one has

$$I_{\alpha,\kappa} \sim -\nu\int_0^1 u^{\alpha-1}(1-u)^{\kappa-\alpha}\left(\ln\tfrac{1}{u}\right)^{\beta}\mathrm{d}u,\quad \kappa \xrightarrow{n,d} \infty.$$

Using the derived expressions (53) and (55) for the above integral, the upper bound $\bar{z}^*$ of the expected optimal value z^* of the MAP can be asymptotically evaluated as

$$\overline{Z}^*_{d,n} = -\nu\varpi_n n(\ln\kappa)^{\beta} + \mathcal{O}(n),\quad \kappa \xrightarrow{n,d} \infty,$$

where $\varpi_n = 1$ for $\kappa \xrightarrow{n} \infty$, and $\varpi_n = (1-\gamma_n)^{\beta}$ if $\kappa \xrightarrow{d} \infty$. From the last equality it immediately follows that $\overline{Z}^*_{d,n}$ is unbounded from below for large n and d, which validates the assertion (38) of the theorem: $\lim_{d\to\infty} Z^*_{d,n} = \lim_{n\to\infty} Z^*_{d,n} = -\infty$ for $0 < \beta < 1$.

<u>Step 4.</u> To extend the last result to values of $\beta \geq 1$ in (37), we recall that by virtue of decomposition (40) the main contribution to the asymptotic value of integral (39) is made by the first integral in the right-hand side of (40). For any $\beta \geq 1$, this integral is majorized as

$$-\nu\int_0^{1/e} u^{\alpha-1}(1-u)^{\kappa-\alpha}\left(\ln\tfrac{1}{u}\right)^{\beta}\mathrm{d}u \leq -\nu\int_0^{1/e} u^{\alpha-1}(1-u)^{\kappa-\alpha}\left(\ln\tfrac{1}{u}\right)^{\beta_0}\mathrm{d}u, \tag{56}$$

where $0 < \beta_0 < 1$. From the above results it follows that the right-hand side of (56) approaches $-\infty$ when $\kappa \xrightarrow{n,d} \infty$. Thus, we conclude that $\overline{Z}^*_{d,n} \to -\infty$, and, consequently, $Z^*_{d,n} \to -\infty$ when $\kappa \xrightarrow{n,d} \infty$ and $\beta \geq 1$ in (37). ■

It is easy to see that Theorems 2 and 3 can be applied to deal with situations when $F^{-1}(x)$ has a singularity of the form $F^{-1}(x) \sim -\nu x^{-\beta_1}\left(\ln\frac{1}{x}\right)^{\beta_2},\ x \to 0+$, where $\nu,\ \beta_2 > 0$, and $0 < \beta_1 < 1$. In such a case, again, one has $\lim_{d\to\infty} Z^*_{d,n} = \lim_{n\to\infty} Z^*_{d,n} = -\infty$.

To finalize the presented analysis of the expected optimal value of the MAP, we consider the special case of a random MAP (1) with $n = 2$, which has been omitted from the preceding discussion. Recall that the expected optimal cost $Z^*_{d,2}$ of a random $d \geq 3,\ n = 2$ MAP can be calculated explicitly by (11), which, in turn, may be rewritten in the form

$$Z^*_{d,2} = \kappa\int_0^1 F_2^{-1}(u)\,(1-u)^{\kappa-1}\,\mathrm{d}u = \kappa\int_0^{\infty} F_2^{-1}\left(1-e^{-\upsilon}\right)e^{-\kappa\upsilon}\,\mathrm{d}\upsilon. \tag{57}$$

Here $\kappa = 2^{d-1}$ and $F_2^{-1}(\cdot)$ is the inverse of the c.d.f. of distribution $F_2 = F * F$. Since $F * F$ represents the distribution of a sum of two independent F-distributed random variables, the interior of the support of F_2 has the form $(2a, 2b)$, where we adopt the usual convention of

$$n(\pm\infty) = \pm\infty,\quad n \in \mathbb{N}\backslash\{0\}. \tag{58}$$

Using techniques similar to those employed in Lemma 4, it can be shown that the optimal expected value of a random MAP with $n = 2$ satisfies $\lim_{d\to\infty} Z^*_{d,2} = 2a$ if $a \in \mathbb{R}$ and $F_2^{-1}(\cdot)$ has an asymptotic expansion of the form (24), i.e., $F_2^{-1}(u) = 2a + \mathcal{O}(u^\beta),\ u \to 0+,$ for some $\beta > 0$ (see Olver [28]). In the same way, the arguments of Theorem 3 in application to (57) yield that $\lim_{d\to\infty} Z^*_{d,2} = -\infty$ when $a = -\infty$ and $F_2^{-1}(\cdot)$ conforms to the conditions of Theorems 2 or 3. Next we illustrate the correspondence between the asymptotical behavior of $F_2^{-1}(x)$ at $x = 0+$ and that of $F^{-1}(x)$.

In the case of a finite a, the convolution representation of F_2 produces for $x \ll 1$

$$F_2(2a + x) = \int_a^{a+x} F\big(a + x - (y - a)\big)\, \mathrm{d}F(y) \le F(a + x) \int_a^{a+x} \mathrm{d}F(y) = F^2(a + x),$$

and

$$F_2(2a + x) \ge \int_a^{a+\frac{x}{2}} F\big(a + x - (y - a)\big)\, \mathrm{d}F(y) \ge F\big(a + \tfrac{x}{2}\big) \int_a^{a+\frac{x}{2}} \mathrm{d}F(y) = F^2\big(a + \tfrac{x}{2}\big),$$

which by inversion implies that $F^{-1}(\sqrt{x}) - a \le F_2^{-1}(x) - 2a \le 2\big(F^{-1}(\sqrt{x}) - a\big)$ for a sufficiently small positive x. Dividing the last inequalities by $x^{\beta/2}$ and taking into account (24) we obtain that the ratio $\big(F_2^{-1}(x) - 2a\big)\big/x^{\beta/2}$ is bounded for $x \to 0+$, which means that $F_2^{-1}(x)$ is equal to $2a + \mathcal{O}(x^{\beta/2})$ in the vicinity of $x = 0+$, and therefore complies with the conditions of Theorem 1.

In the case of $a = -\infty$, the function $F_2(x)$ for $-x \gg 1$ can be bounded from above as

$$F_2(2x) = \left(\int_{-\infty}^{x} + \int_x^b\right) F\big(x - (y - x)\big)\, \mathrm{d}F(y) \le \int_{-\infty}^{x} 1 \cdot \mathrm{d}F(y) + F(x) \int_x^b \mathrm{d}F(y) \le 2F(x),$$

whereas a lower bound is computable in the form

$$F_2(x) \ge \int_{-\infty}^{x} F(x - y)\, \mathrm{d}F(y) \ge F(0)F(x).$$

Inverting the derived bounds, we have $2F^{-1}\big(\frac{x}{2}\big) \le F_2^{-1}(x) \le F^{-1}\big(\frac{x}{F(0)}\big),\ x \to 0+$. Applying the same reasoning as above, we obtain that $F_2^{-1}(x)$ has an asymptotical expansion of the form (27) for $x \to 0+$, provided that $F^{-1}(\cdot)$ satisfies (27). If $F^{-1}(\cdot)$ has asymptotic representation (37) in the vicinity of 0, the derived bounds on $F_2^{-1}(\cdot)$ can be used to show that $\lim_{d\to\infty} Z^*_{d,2} = -\infty$ in this case as well.

Using the convention (58), we can summarize the developed results in the following theorem.

Theorem 4 (Expected Optimal Value of the MAP) *Consider a $d \ge 3, n \ge 2$ MAP (1) with cost coefficients that are iid random variables from an absolutely continuous distribution F with existing first moment. Let $(a,\ b)$, where a and b are finite or infinite, be the interior of the support set of this distribution. If the distribution F satisfies conditions of either of Theorems 1, 2, or 3, the expected optimal value of the MAP satisfies*

$$Z^*_{d,n} \to na, \quad n \to \infty \ \text{or} \ d \to \infty.$$

The foregoing analysis is constructive in the sense that it allows for deriving practically useful bounds for the expected optimal value of random MAP. The next section discusses this issue in detail and presents experimental evidence on the goodness of the developed bounds.

4 Asymptotical bounds and rate of convergence for the expected optimal value of MAP

Although the primary focus of the preceding sections has been on determining the limiting value of the expected optimal cost of "infinitely large" random MAPs, the obtained results can be readily employed to construct upper and lower asymptotical bounds for the expected optimal value of MAP when one of the parameters n or d is large but finite. The lower and upper bounds $\underline{Z}^*_{d,n}$ and $\overline{Z}^*_{d,n}$ introduced in Lemma 2 are especially useful when the support of distribution F of the MAP's cost coefficients has a finite left endpoint. The following statement is a special case of Lemma 4 and Theorem 1.

Lemma 5 *Consider a $d \geq 3$, $n \geq 3$ MAP* (1) *with cost coefficients that are iid random variables from an absolutely continuous distribution with existing first moment. Let $(a,\, b)$, where $a \in \mathbb{R}$, $b \in (a, +\infty]$, be the interior of the support set of this distribution, and assume that the inverse $F^{-1}(u)$ of the c.d.f. $F(u)$ of the distribution is such that*

$$F^{-1}(u) \sim a + \sum_{s=1}^{\infty} a_s u^{s/\mu}, \quad u \to 0+, \quad \mu > 0. \tag{59}$$

Then, for any integer $m \geq 1$, lower and upper bounds $\underline{Z}^*_{d,n}$, $\overline{Z}^*_{d,n}$ (4) *on the expected optimal cost $Z^*_{d,n}$ of the MAP can be asymptotically evaluated as*

$$\underline{Z}^*_{d,n} = an + \sum_{s=1}^{m-1} a_s \frac{n\,\Gamma(\kappa+1)\,\Gamma\big(\frac{s}{\mu}+1\big)}{\Gamma\big(\kappa+\frac{s}{\mu}+1\big)} + \mathcal{O}\left(n \frac{\Gamma(\kappa+1)\,\Gamma\big(\frac{m}{\mu}+1\big)}{\Gamma\big(\kappa+\frac{m}{\mu}+1\big)}\right), \quad \kappa \xrightarrow{n,d} \infty, \tag{60a}$$

$$\overline{Z}^*_{d,n} = an + \sum_{s=1}^{m-1} a_s \frac{n\,\Gamma(\kappa+1)\,\Gamma\big(\frac{s}{\mu}+\alpha\big)}{\Gamma(\alpha)\,\Gamma\big(\kappa+\frac{s}{\mu}+1\big)} + \mathcal{O}\left(n \frac{\Gamma(\kappa+1)\,\Gamma\big(\frac{m}{\mu}+\alpha\big)}{\Gamma(\alpha)\,\Gamma\big(\kappa+\frac{m}{\mu}+1\big)}\right), \quad \kappa \xrightarrow{n,d} \infty. \tag{60b}$$

It can be shown that the lower and upper bounds defined by (60) are convergent, i.e.,

$$|\overline{Z}^*_{d,n} - \underline{Z}^*_{d,n}| \to 0, \quad \kappa \xrightarrow{n,d} \infty,$$

whereas the corresponding asymptotical bounds for the case of distributions with support unbounded from below may be divergent in the sense that $|\overline{Z}^*_{d,n} - \underline{Z}^*_{d,n}| \nrightarrow 0$ when $\kappa \xrightarrow{n,d} \infty$.

Asymptotical representations (60) for the bounds $\underline{Z}^*_{d,n}$ and $\overline{Z}^*_{d,n}$ are simplified when the inverse $F^{-1}(\cdot)$ of the c.d.f. of the distribution has a regular power series expansion in the vicinity of zero. Assume, for example, that function $F^{-1}(\cdot)$ can be written as

$$F^{-1}(u) = a + a_1 u + \mathcal{O}(u^2), \quad u \to 0+. \tag{61}$$

The asymptotic expressions (60) for the lower and upper bounds $\underline{Z}^*_{d,n}$ and $\overline{Z}^*_{d,n}$ for the expected optimal cost $Z^*_{d,n}$ then take the form

$$\underline{Z}^*_{d,n} \sim na + \frac{a_1}{n^{d-2}}, \quad \kappa \xrightarrow{n,d} \infty, \tag{62a}$$

$$\overline{Z}^*_{d,n} \sim na + a_1 \frac{\lceil e^{d-1} \rceil}{n^{d-2}}, \quad k \xrightarrow{n} \infty, \tag{62b}$$

$$\overline{Z}^*_{d,n} \sim na + a_1 n^{\gamma_n(d-1)-(d-2)}, \quad \kappa \xrightarrow{d} \infty. \tag{62c}$$

It is of interest to note that if $a = 0$ in (61) then for $n \gg 1$ and d fixed the expected optimal value of the MAP is asymptotically bounded as

$$\frac{a_1}{n^{d-2}} + \mathcal{O}\left(\frac{1}{n^{d-1}}\right) \;\le\; Z^*_{d,n} \;\le\; \frac{a_1\lceil e^{d-1}\rceil}{n^{d-2}} + \mathcal{O}\left(\frac{1}{n^{d-1}}\right), \quad k \xrightarrow{n} \infty, \tag{63}$$

which immediately yields the rate of convergence to zero for $Z^*_{d,n}$ as n approaches infinity:

Lemma 6 *Consider a $d \ge 3$, $n \ge 3$ MAP (1) with cost coefficients that are iid random variables from an absolutely continuous distribution with existing first moment. Let $(0, b)$, where $b \in (0, +\infty]$, be the interior of the support set of this distribution, and assume that the inverse $F^{-1}(u)$ of the c.d.f. $F(u)$ of the distribution satisfies (61) with $a = 0$. Then, for a fixed d and $n \to \infty$ the expected optimal value $Z^*_{d,n}$ of the MAP converges to zero as $\mathcal{O}\big(n^{-(d-2)}\big)$.*

Furthermore, inequalities (63) allow one to conjecture that the expected optimal value of a random MAP with an absolutely continuous distribution F, which satisfies (61) with $a = 0$, has the form

$$Z^*_{d,n} = \frac{C}{n^{d-2}} + \mathcal{O}\left(\frac{1}{n^{d-1}}\right), \quad n \gg 1,$$

where constant $C > 0$ is such that $a_1 \le C \le a_1\lceil e^{d-1}\rceil$.

We illustrate the tightness of the developed bounds (60) by comparing them to the computed expected optimal values of MAPs with coefficients $c_{i_1,\ldots,i_d}$ drawn from the uniform $U(0,1)$ distribution and exponential distribution with mean 1. The p.d.f. of the uniform distribution is given by $f_{U(0,1)}(x) = \mathbf{1}_{[0,1]}(x)$, and that of exponential with mean 1 distribution is $f_{\text{expo}(1)}(x) = e^{-x}\mathbf{1}_{[0,+\infty)}(x)$, where $\mathbf{1}_A(x)$ is the indicator function of a set $A \subseteq \mathbb{R}$. It is elementary that the inverse functions $F^{-1}(\cdot)$ of the c.d.f.'s for both these distributions are representable in the form (61) with $a = 0$, $a_1 = 1$, thereby reducing the general expressions (60) for the lower and upper bounds $\underline{Z}^*_{d,n}$ and $\overline{Z}^*_{d,n}$ to the form (62).

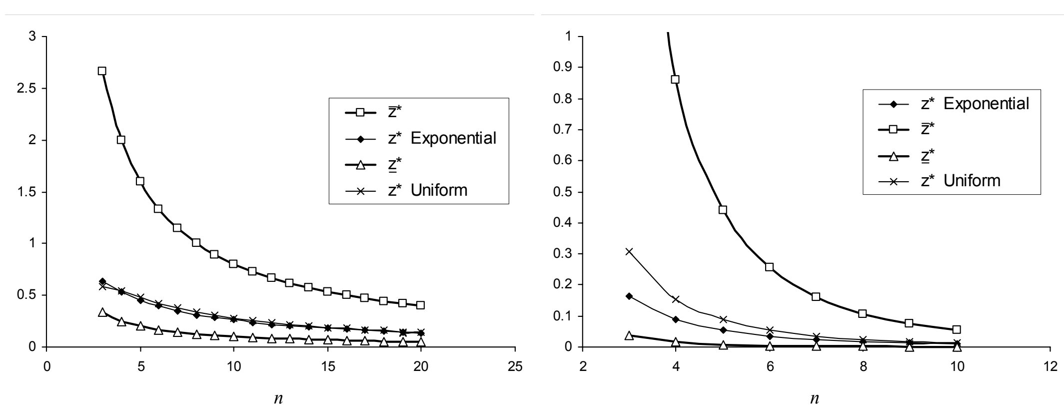

Figure 1: Expected optimal value $Z^*_{d,n}$, lower and upper bounds $\underline{Z}^*_{d,n}$, $\overline{Z}^*_{d,n}$ of an MAP with fixed $d = 3$ (left) and $d = 5$ (right) for uniform $U(0,1)$ and *exponential*(1) distributions.

The numerical experiments involved solving multiple instances of randomly generated MAPs with the number of dimensions d ranging from 3 to 10, and the number n of elements in each dimension running from 3 to 20. To solve the problems to optimality, we used a branch-and-bound algorithm that navigated

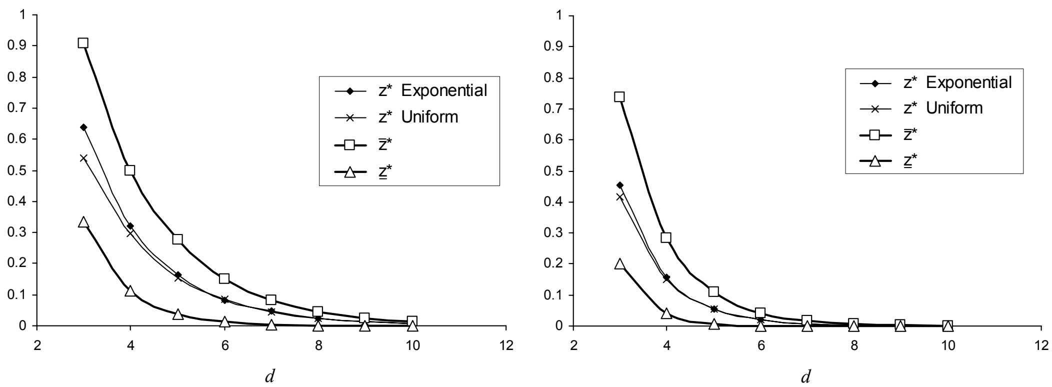

Figure 2: Expected optimal value $Z^*_{d,n}$, lower and upper bounds $\underline{Z}^*_{d,n}$, $\overline{Z}^*_{d,n}$ of an MAP with fixed $n = 3$ (left) and $n = 5$ (right) for uniform $U(0, 1)$ and *exponential*(1) distributions.

through the index tree representation of the MAP.[2] Figures 1 and 2 display the obtained expected optimal values of MAP with uniform and exponential iid cost coefficients when d is fixed at $d = 3$ or 5 and $n = 3, \ldots, 20$, and when $n = 3$ or 5 and d runs from 3 to 10. This "asymmetry" in reporting of the results is explained by the fact that the implemented branch-and-bound algorithm based on index tree is more efficient in solving "shallow" MAPs, i.e., instances that have larger n and smaller d. The solution times varied from several seconds to 20 hours on a 2GHz PC.

Figures 1 and 2 suggest that MAPs with exponentially and uniformly distributed costs and the same dimensionality possess very close expected optimal values. This observation is in complete agreement with the presented above theoretical findings: indeed, we argue that the asymptotical behavior of large-scale random MAPs is determined by the properties of the inverse $F^{-1}(x)$ of the c.d.f. $F(\cdot)$ at $x = 0$. In the considered case of uniformly and exponentially distributed random MAPs, the inverse functions $F^{-1}(\cdot)$ of c.d.f.'s of these two distributions share the first two terms of their asymptotical expansions at zero, which leads to similar numerical values of the expected optimal cost.

The conducted numerical experiments also suggest that the constructed lower and upper bounds for the expected optimal cost of random MAPs are quite tight, with the upper bound $\overline{Z}^*_{d,n}$ being tighter for the case of fixed n and large d (see Figs. 1–2).

Conclusions

In this paper we have conducted asymptotical analysis of the expected optimal value of the Multidimensional Assignment Problem where the assignment costs are iid random variables drawn from a continuous distribution. It has been demonstrated that asymptotical behavior of the expected optimal cost of a random MAP in the case when one of the problem's dimension parameters approaches infinity is determined by the location of the left endpoint a of the support $[a, b]$ of the distribution. Namely, the expected optimal cost of a d-dimensional random MAP with n elements in each dimension approaches the limiting value of na when one of the parameters n or d is fixed and the other increases infinitely.

[2] The index tree representation of MAP [34] differs from the introduced in Section 2 index graph in that *every* path from the root node to a leaf node of the tree represents a feasible solution to the MAP. As a result, each level of the index tree contains more nodes than the corresponding level of the index graph.

This statement has been proven to hold true for a broad class of continuous distributions with a finite or infinite left endpoint of the support. The presented analysis is constructive in the sense that it allows for derivation of lower and upper asymptotical bounds for the expected optimal value of the problem for a prescribed probability distribution. If the distribution's support set has a finite left endpoint, these bounds are converging.

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