Behavioral Measures and their Correlation with IPM Iteration Counts on Semi-Definite Programming Problems

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Abstract

We study four measures of problem instance behavior that might account for the observed differences in interior-point method (IPM) iterations when these methods are used to solve semidefinite programming (SDP) problem instances: (i) an aggregate geometry measure related to the primal and dual feasible regions (aspect ratios) and norms of the optimal solutions, (ii) the (Renegar-) condition measure $C(d)$ of the data instance, (iii) a measure of the near-absence of strict complementarity of the optimal solution, and (iv) the level of degeneracy of the optimal solution. We compute these measures for the SDPLIB suite problem instances and measure the sample correlation (CORR) between these measures and IPM iteration counts (solved using the software SDPT3) when these measures have finite values. Our conclusions are roughly as follows: the aggregate geometry measure is highly correlated with IPM iterations (CORR = 0.896), and provides a very good explanation of IPM iterations, particularly for problem instances with solutions of small norm and aspect ratio. The condition measure $C(d)$ is also correlated with IPM iterations, but less so than the aggregate geometry measure (CORR = 0.630). The near-absence of strict complementarity is weakly correlated with IPM iterations (CORR = 0.423). The level of degeneracy of the optimal solution is essentially uncorrelated with IPM iterations.

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1 Introduction

When applied to the solution of semidefinite programming (SDP) problems, modern interior-point methods (IPMs) enjoy both excellent theoretical complexity (see [24]) as well as practical performance. Computational experience has shown that state-of-the-art IPM software significantly outperforms the best theoretical worst-case complexity in terms of the number of Newton-type iterations for such algorithms (see [17]); indeed, the software SDPT3 solves 85 problems in the SDPLIB suite using between 10 and 60 IPM iterations. For these problems, the number of IPM iterations is essentially uncorellated with traditional measures of problem size such as the number of equality constraints m or the dimension of the space of variables $\bar{n}$ of a primal SDP in standard form. (For example, we computed sample correlations of IPM iterations and these dimension meausures for the 85 SDPLIB instances, and obtained $\text{CORR}(m, \text{IPM Iterations}) = 0.060$ and $\text{CORR}(\bar{n}, \text{IPM Iterations}) = -0.107$.)

Herein we study the extent to which certain measures of problem instance behavior might be correlated with the computational performance of IPMs on SDP problems. We examine four measures of problem instance behavior that might account for the observed differences in interior-point method (IPM) iterations. Two of these measures were previously studied in connection to the theoretical complexity of interior-point methods, namely an aggregate geometry measure related to the primal and dual feasible regions (aspect ratios) and norms of the optimal solutions developed in [7], and the (Renegar-) condition measure $C(d)$ of the data instance studied in [19]. In addition, we also develop and study a measure of the near-absence of strict complementarity of the optimal solution, as well as a measure of the level of degeneracy of the optimal solution. (These two measures have been shown to be related to the superlinear convergence of some variants of the interior point method, see [2].) We compute these measures for the SDPLIB suite problem instances and measure the correlation between these measures and IPM iteration counts when these instances are solved using the software SDPT3.

Our conclusions are roughly as follows: the aggregate geometry measure is highly correlated with IPM iterations ($\text{CORR} = 0.896$), and provides a very good explanation of IPM iterations, particularly for problem instances with solutions of small norm and aspect ratio. The condition measure $C(d)$ is also correlated with IPM iterations, but less so than the aggregate geometry measure ($\text{CORR} = 0.630$). The near-absence of strict complementarity is weakly correlated with IPM iterations ($\text{CORR} = 0.423$). The level of degeneracy of the optimal solution is essentially uncorrelated with IPM iterations.

The rest of the paper is organized as follows. We present the SDP problem format and notation used in this paper in the remainder of this introductory section. In Section 2 we present our aggregate geometry measure and computational results. We discuss the computation of the condition measure and present associated computational results in Section 3. We present a measure of non-strict complementarity and associated computational results in Section 4. We present our measure of degeneracy and associated computational results in Section 5. Summary conclusions and some further issues are discussed in Section 6.

1.1 SDP Problem Format and Notation

We consider the standard form primal convex conic optimization problem:

$$(CP) \qquad \begin{array}{rl} \min_x & \langle c, x \rangle \\ \text{s.t.} & A(x) = b \\ & x \in K \ , \end{array} \tag{1}$$

where $x, c \in \Re^n$, $b \in \Re^m$, $A(\cdot)$ is a linear operator from $\Re^n$ to $\Re^m$, $\langle \ , \ \rangle$ is a dot product on $\Re^n$, and K is a closed convex cone in $\Re^n$. The (Lagrange) conic dual problem of (CP) is:

$$(CD) \qquad \begin{array}{rl} \max_{y,z} & b^T y \\ \text{s.t.} & A^*(y) + z = c \\ & z \in K^* \ , \end{array} \tag{2}$$

where K^* is the (positive) dual cone, i.e., $K^* := \{z \in \Re^n : \langle z, x \rangle \geq 0 \text{ for all } x \in K\}$, and $A^*(\cdot)$ is the adjoint operator of $A(\cdot)$, namely $A^*(\cdot)$ satisfies $y^T A(x) = \langle A^*(y), x \rangle$ for all x, y, and for the space $\Re^m$ we consider the coordinate-wise dot-product $\langle y, b \rangle = y^T b$.

Let S^k denote the space of $k \times k$ symmetric matrices, and let $S^k_+, \ S^k_{++} \subset S^k$ denote the cones of positive semi-definite and positive definite symmetric matrices, respectively. Let "$\succeq$" and "$\succ$"denote the partial orderings induced by S^k_+ and S^k_{++}, respectively. Similarly, let $\Re^k_+, \Re^k_{++} \subset \Re^k$ denote the cones of nonnegative k-vectors (the nonnegative orthant) and positive k-vectors, respectively. The problem instances in the SDPLIB suite are conic optimization problems of the form (1) where K is the cartesian product of one or more semidefinite cones and the nonnegative orthant. A problem instance in the SDPLIB can therefore be characterized as follows: let S^s denote the space of symmetric block-diagonal matrices with n_s blocks of dimensions $s_1, \ldots, s_{n_s}$, and let $S^s_+, \ S^s_{++} \subset S^s$ denote the cones of positive semidefinite and positive definite matrices in S^s, respectively. We also consider that matrices in S^s are of size $|s| \times |s|$, with $|s| = \sum_{j=1}^{n_s} s_j$. Each problem instance in the SDPLIB suite can be written as:

$$(SDP) \qquad \begin{array}{rl} \min & c^s \bullet x^s + (c^l)^T x^l \\ \text{s.t.} & A^s_i \bullet x^s + (A^l_i)^T x^l = b_i, \quad i = 1, \ldots, m \\ & x^s \in S^s_+, \ x^l \in \Re^{n_l}_+ \end{array} \tag{3}$$

where the dot product in the product space is given by $\langle (c^s, c^l), (x^s, x^l) \rangle = \langle c^s, x^s \rangle + \langle c^l, x^l \rangle = c^s \bullet x^s + (c^l)^T x^l$ and "$\bullet$" denotes the trace inner product $\langle c^s, x^s \rangle := \text{trace}((c^s)^T x^s)$.

Here, $(A_i^l)^T$ denotes the ith row of the matrix $A^l \in \Re^{m \times n_l}$ and we let A^s denote the linear operator that maps S^s to $\Re^m$ by $A^s x^s = (A_1^s \bullet x^s, \dots, A_m^s \bullet x^s)^T$. Note that with

$$K := S_+^s \times \Re_+^{n_l} \tag{4}$$

we see that (SDP) is an instance of (CP). Note also that the linear operator A^s is indexed by a triplet: $A_{ijf}^s = (A_i^s)_{jf}$, so that A_{ijf}^s is the entry in row j and column f of the matrix A_i^s, and that $A_i^s \in S^s$, $i = 1, \dots, m$. We let $A_{\bullet jf}^s$ represent the m-dimensional vector of the j, f entries of the matrices $A_1^s, \dots, A_m^s$. The dual problem of (SDP) is given by

$$(SDD) \qquad \begin{array}{ll} \max & b^T y \\ \text{s.t.} & \sum_{i=1}^m A_i^s y_i + z^s = c^s \\ & (A^l)^T y + z^l = c^l \\ & z^s \in S_+^s, \ z^l \in \Re_+^{n_l} \ . \end{array} \tag{5}$$

We let $e = (1, \dots, 1) \in \Re^{n_l}$, let e_i denote the i-th canonical vector of appropriate dimension, and let I denote the identity matrix in appropriate spaces. If $x, z \in \Re^k$ are vectors, let $x \circ z := (x_1 z_1, \dots, x_k z_k)^T$, and for P, Q arbitrary matrices, let $P \otimes Q$ denote the Kronecker matrix product of P and Q. If $x \in \Re^k$ is a vector, let $\|x\|_p := \left(\sum_{j=1}^k |x_j|^p\right)^{1/p}$ denote the usual L_p-norm. If $x \in S^k$ is a matrix, let $\lambda(x)$ denote the k-vector of eigenvalues of x, and let $\|x\|_{Ep} := \left(\sum_{j=1}^k |\lambda_j(x)|^p\right)^{1/p}$ denote the L_p-norm of the eigenvalues of x ($p = 1$ is the Ky Fan norm, $p = 2$ is the Frobenius norm, see [3]). Given a norm $\|\cdot\|$ on a vector space, let $\|\cdot\|_*$ denote the (dual) norm on the dual vector space. Let $B(\bar{x}, r)$ denote the ball of radius r centered at the point $\bar{x}$, namely $B(\bar{x}, r) := \{x \mid \|x - \bar{x}\| \le r\}$, and let $\text{dist}(x, T)$ denote the distance from a point x to the set T. Given a linear operator A mapping $\Re^k$ to $\Re^l$ with norms on these spaces given by $\|\cdot\|^X$ and $\|\cdot\|^Y$, respectively, the operator norm of A is defined to be $\|A\| := \sup\{\|Ax\|^Y : \|x\|^X \le 1\}$.

2 Aggregate Geometry Measure

2.1 Motivation

In [7] two primal geometry measures were used to provide a theoretical complexity bound for a particular primal-based IPM for convex optimization problems in a format more general than the conic form (CP). The two primal geometry measures will be denoted here by D_p^ε and g_p and will be reviewed shortly; essentially D_p^ε measures the norm of the largest ε-optimal primal solution and g_p is an "aspect ratio" measure that is smaller to the extent that there is a primal feasible solution of relatively small norm whose distance from the boundary of the feasible region is relatively large. It is shown in [7] that a theoretical bound on IPM iterations of the primal-based algorithm involves the term $O(\sqrt{\vartheta}(\log(D_p^\varepsilon) + \log(g_p)))$ (ϑ is the complexity parameter of the barrier function used therein). Herein we test the practical relevance of these geometry measures as applied to SDP problems solved using a standard primal-dual IPM. However, because SDP problems are solved by primal-dual algorithms (where the roles of the primal and dual are interchangeable), we also consider dual versions D_d^ε and g_d of these geometry measures (defined on the dual feasible region) and test the correlation of IPM iterations with the logarithm of the simple aggregate measure:

$$g^m := \left(D_p^\varepsilon \times g_p \times D_d^\varepsilon \times g_d\right)^{\frac{1}{4}} \; ;$$

note that the geometric mean is appropriate since we are interested in studying the correlation between IPM iterations and $\log(g^m)$.

2.2 Primal Geometry Measures

For the primal conic problem (CP), the first primal geometry measure, originally introduced in [7], is the maximum norm over all ε-optimal primal solutions, which we denote by D_p^ε. Given a norm $\|\cdot\|$ specified for the space of variables x, D_p^ε is defined as:

$$(PM): \quad \begin{array}{lll} D_p^\varepsilon := & \text{maximum}_x & \|x\| \\ & \text{s.t.} & A(x) = b \\ & & x \in K \\ & & \langle c, x \rangle \le \text{VAL} + \varepsilon \ , \end{array} \tag{6}$$

where VAL is the optimal objective function value of (CP). At first glance it may seem odd to maximize rather than minimize in defining D_p^ε. However, consider the ill-posed case when VAL is finite but the set of optimal solutions is unbounded and hence $D_p^\varepsilon = +\infty$. Then the dual feasible region has no interior, and we would expect it to be more difficult for an interior-point method to compute an approximate solution of (CP). Also, in [6] it is shown that D_p^ε is inversely proportional to the size of the largest ball contained in the level sets of the dual problem, and so D_p^ε contains specific information about the interior of the dual feasible region in a neighborhood of the dual optimal solution(s).

Note that (PM) is in general a non-convex optimization problem, which is disconcerting. However, (PM) is a convex optimization problem if the norm $\|\cdot\|$ has the property that it is a linear function on K. Specifying (CP) to (SDP) where K is given by (4), we define the following norm on the vector space of variables:

$$\|x\| = \|(x^s, x^l)\| := \|x^s\|_{E1} + \|x^l\|_1 \ . \tag{7}$$

Proposition 1 *Suppose that K is given by (4), and that $\|x\| = \|(x^s, x^l)\|$ is defined using (7). Then $\|x\| = I \bullet x^s + e^T x^l$ for all $x \in K$, and*

$$PM: \quad \begin{array}{rl} D_p^\varepsilon = \text{maximum}_x & I \bullet x^s + e^T x^l \\ s.t. & A^s x^s + A^l x^l = b \\ & x^s \in S_+^s, \ x^l \in \Re_+^{n_l} \\ & c^s \bullet x^s + (c^l)^T x^l \leq VAL + \varepsilon \ . \end{array} \tag{8}$$

Proof: From (4) and (7) we have that $\|x\| = \sum_{j=1}^s |\lambda_j(x^s)| + \sum_{i=1}^{n_l} |x_i^l| = e^T \lambda(x^s) + e^T x^l$, which implies the equivalent objective function since $e^T \lambda(x^s) = \text{trace}(x^s) = I \bullet x^s$. We complete the proof by replacing the definitions in (6). ■

The second primal geometry measure we consider (also originally introduced in [7]) is defined for problem (CP) to be the optimal objective function value of the optimization problem:

$$\mathrm{P}_{g_p}: \quad \begin{array}{rl} g_p := \text{minimum}_x & \max\left\{ \|x\|, \dfrac{\|x\|}{\text{dist}(x, \partial K)}, \dfrac{1}{\text{dist}(x, \partial K)} \right\} \\ \text{s.t.} & A(x) = b \\ & x \in K \ . \end{array} \tag{9}$$

Note that g_p will be smaller to the extent that the feasible region of (CP) contains a point x whose norm is not too large and whose distance from ∂K is not too small. (For a further discussion of g_p see [7].) We can compute the value of g_p by instead working with the following convex problem:

$$\mathrm{P}_{t_p^*}: \quad \begin{array}{rl} t_p^* := \text{maximum}_{w,\theta,t} & t \\ \text{s.t.} & A(w) - b\theta = 0 \\ & B(w,t) \subset K \\ & \|w\| \leq 1 \\ & t \leq \theta \leq 1 \ . \end{array} \tag{10}$$

(Recall that $B(w,t)$ denotes the ball centered at w with radius t.) It is easy to show that $g_p = \frac{1}{t_p^*}$ under the transformations:

$$x \leftarrow \frac{w}{\theta} \quad \text{and} \quad (w, \theta, t) \leftarrow \left(\frac{x}{\max\{\|x\|, 1\}}, \frac{1}{\max\{\|x\|, 1\}}, \frac{\min\{\text{dist}(x, \partial K), 1\}}{\max\{\|x\|, 1\}} \right) \ .$$

While $\mathrm{P}_{t_p^*}$ is a convex program, it is not clear if the constraint "$B(w,t) \subset K$" can be conveyed efficiently. Specifying (CP) to (SDP) where K is given by (4), we see that this can be done for our particular choice of norm:

Proposition 2 *Suppose that K is given by (4), and that $\|x\| = \|(x^s, x^l)\|$ is defined using (7), and let $r \geq 0$ be given. Then*

$$B(x, r) = B((x^s, x^l), r) \subset K \quad \text{if and only if} \quad x^s - rI \in S^s_+,\ x^l - re \geq 0\ ,$$

whereby

$$P_{t_p^*}: \quad \begin{array}{lll} t_p^* := \text{maximum}_{w,\theta,t} & t \\ \quad s.t. & A^s w^s + A^l w^l - b\theta = 0 \\ & w^s - tI \in S^s_+ \\ & w^l - te \geq 0 \\ & I \bullet w^s + e^T w^l \leq 1 \\ & t \leq \theta \leq 1\ . \end{array} \tag{11}$$

Proof: By Proposition 1, we have $\|w\| = I \bullet w^s + e^T w^l$ for (w, θ, t) feasible for (11) with $t \geq 0$. We only need to prove the characterization of the inclusion constraint, since Problem (11) follows immediately from (10) with that characterization. Assume first that $B(x, r) \subset K$. Let v be a unit eigenvector corresponding to the smallest eigenvalue $\lambda_{\min}(x^s)$ of x^s. Then $y := (x^s - rvv^T, x^l) \in B(x, r)$, which implies that $\lambda_{\min}(x^s) - r \geq \lambda_{\min}(x^s - rvv^T) \geq 0$, and therefore $x^s - rI \in S^s_+$. Likewise, for any $j \in \{1, \ldots, n_l\}$, $y_j = (x^s, x^l - re_j) \in B(x, r)$, which means that $x^l_j - r \geq 0$ and therefore $x^l - re \geq 0$. For the converse, assume that $x^l - re \geq 0$ and $x^s - rI \in S^s_+$, which is equivalent to $\lambda_i(x^s) \geq r$. Let $y \in B(x, r)$, this implies that $|y^l_j - x^l_j| \leq r$ for all $j \in \{1, \ldots, n_l\}$, which gives $y^l_j \geq x^l_j - r \geq 0$. Since $y \in B(x, r)$ we also have that $|\lambda_i(y^s - x^s)| \leq r$ for all $i \in \{1, \ldots, |s|\}$. Letting $w_1, \ldots, w_s$ and $z_1, \ldots, z_s$ be the orthonormal bases of eigenvectors for x^s and $y^s - x^s$, we have for any vector v, $v^T y^s v = v^T x^s v + v^T (y^s - x^s) v = \sum_{i=1}^s \lambda_i(x^s) \left(v^T w_i\right)^2 + \sum_{i=1}^s \lambda_i(y^s - x^s) \left(v^T z_i\right)^2 \geq r\|v\|_2^2 - r\|v\|_2^2 = 0$. Thus we have that $y \in K$. ■

Taken together, Propositions 1 and 2 demonstrate that if we use the specified norm (7), then D_p^ε and g_p can each be computed by solving an associated convex optimization problem whose size and structure is compatible with the original problem instance (SDP).

2.3 Dual Geometry Measures and Aggregate Measure g^m

Given a norm $\|\cdot\|$ on the space of dual cone variables z, we define D_d^ε and g_d as the following obvious analogs of D_p^ε and g_p for the dual problem (CD):

$$(DM): \quad \begin{array}{ll} D_d^\varepsilon := \text{maximum}_{y,z} & \|z\| \\ \quad \text{s.t.} & A^*(y) + z = c \\ & z \in K^* \\ & b^T y \geq \text{VAL} - \varepsilon \end{array} \tag{12}$$

and

$$
\mathrm{P}_{g_d}: \quad \begin{array}{lll} g_d := \text{minimum}_{y,z} & \max\left\{\|z\|, \dfrac{\|z\|}{\text{dist}(z, \partial K^*)}, \dfrac{1}{\text{dist}(z, \partial K^*)}\right\} \\ \quad\quad \text{s.t.} & A^*(y) + z = c \\ & z \in K^* \ . \end{array} \tag{13}
$$

Here selecting the proper norm is again crucial to obtain tractable formulations of these problems. Similar to the results in Propositions 1 and 2, if we use the specified norm (7) for the dual cone variables $z = (z^s, z^l)$, we can compute D_d^ε and g_d by solving the problems:

$$
(DM): \quad \begin{array}{ll} D_d^\varepsilon := \text{maximum}_{y,z} & I \bullet z^s + e^T z^l \\ \quad\quad \text{s.t.} & \sum_{i=1}^m A_i^s y_i + z^s = c^s \\ & (A^l)^T y + z^l = c^l \\ & z^s \in S_+^s, \ z^l \in \Re_+^{n_l} \\ & b^T y \geq \text{VAL} - \varepsilon \end{array} \tag{14}
$$

and

$$
\mathrm{P}_{t_d^*}: \quad \begin{array}{ll} t_d^* := \text{maximum}_{v,u,\theta,t} & t \\ \quad\quad \text{s.t.} & \sum_{i=1}^m A_i^s v_i + u^s - c^s\theta = 0 \\ & (A^l)^T v + u^l - c^l\theta = 0 \\ & u^s - tI \in S_+^s \\ & u^l - te \geq 0 \\ & I \bullet u^s + e^T u^l \leq 1 \\ & t \leq \theta \leq 1 \ , \end{array} \tag{15}
$$

and setting $g_d = \frac{1}{t_d^*}$.

We aggregate the four geometry measures D_p^ε, g_p, D_d^ε, and g_d into the following single aggregate measure using their geometric mean:

$$
g^m := \left(D_p^\varepsilon \times g_p \times D_d^\varepsilon \times g_d\right)^{\frac{1}{4}} \ .
$$

Roughly speaking, g^m is smaller to the extent that the primal and dual problems have near-optimal solutions with small norm, and whose feasible regions have points of small norm that are far from the boundary of the respective cones.

2.4 Computation of Geometry Measures for the SDPLIB Suite

We computed the aggregate geometry measure g^m (by computing D_p, g_p, D_d, and g_d) for the SDPLIB suite of 92 semidefinite optimization problems, which are available on the worldwide web at `http://www.nmt.edu/~sdplib/`. Of the 92 problems that make up the SDPLIB suite, we removed four instances that are infeasible (`infd1, infd2,` `infp1, infp2`) and three large problems for which even computing a solution was excessively difficult (`maxG55` (5000×5000), `maxG60` (7000×7000), `thetaG51` (6910×1001)), yielding a working set of 85 problem instances that formed the basis of our computational experiments. All computation was performed using the software SDPT3-3.1, see [23, 22].

We used the following methodology to specify the value of ε for the formulation and computation of D_p^ε and D_d^ε. Let x_k and (y_k, z_k) be the approximate optimal solutions returned by a solver to the original conic problem (SDP). These are approximate optimal solutions: they satisfy feasibility and complementary slackness within a given tolerance. Because setting very small values of ε can result in formulating a nearly-infeasible problem to determine either D_p^ε or D_d^ε, we used the following rule for assigning the value of ε for each problem instance:

$$\varepsilon = \frac{1}{2} \max \left\{ z_k^T x_k, (c^T x_k - b^T y_k), 10^{-3} \right\} \ ,$$

which was designed to set ε to be one half of the computed duality gap. To ensure that (PM) and (DM) are feasible we replace the objective function constraints by $\langle c, x \rangle \le \langle c, x_k \rangle + \varepsilon$ for (PM) and by $b^T y \ge b^T y_k - \varepsilon$ for (DM). We denote the values obtained from these modified versions of (PM) and (DM) as D_p and D_d, respectively.

Table 5 in the Appendix contains the resulting values of g^m as well as D_p, g_p, D_d, and g_d for the 85 SDPLIB problems under consideration. Notice from Table 5 that $g_p = \infty \iff D_d = \infty$ and $g_d = \infty \iff D_p = \infty$. This follows since for a primal and dual feasible conic problem, the objective function level sets of the primal problem are unbounded ($D_p = \infty$) if and only if the dual problem contains no slack vector in the interior of the dual cone ($g_d = \infty$), and similarly for the dual. Table 1 presents summary statistics for the four geometry measures: 32 of the 85 SDPLIB problem instances have no primal interior solution within software tolerance; however, all 85 instances have dual interior solutions.

Table 1: Summary Statistics of Geometry Measures for 85 Problems in the SDPLIB Suite

Status	D_p	D_d	g_p	g_d
Finite	85	53	53	85
Infinite	-	32	32	-
Total	85	85	85	85

In order to assess any relationship between the aggregate geometry measure g^m and IPM iterations for the SDPLIB suite, we first solved and recorded the IPM iterations taken by SDPT3 version 3.1 with default settings for the 85 SDPLIB suite problems considered herein. Algorithm SDPT3-3.1 exits with an approximate solution if either (i) it achieves a small relative error "err" (defined below), (ii) it identifies the problem as primal or dual infeasible, or (iii) it perceives slow progress or encounters numerical difficulties. Regardless of the exiting condition, we recorded the iteration count as a measure of the difficulty faced by the solver to reach termination on each problem under the same default settings. Table 5 in the Appendix presents the IPM iterations obtained by SDPT3-3.1 with default settings as well as the relative error, which SDPT3-3.1 defines as:

$$\text{err} := \max\left\{\frac{\langle x, z\rangle}{\max\{1, (|\langle c, x\rangle| + |b^T y|)/2\}}, \frac{\|A(x) - b\|}{\max\{1, \|b\|\}}, \frac{\|A^*(y) + z - c\|}{\max\{1, \|c\|\}}\right\}, \qquad (16)$$

for instances in which the final iterate has err $> 10^{-6}$. Figure 1 shows a histogram of IPM iterations for SDPT3-3.1 for the 85 problems in the SDPLIB suite.

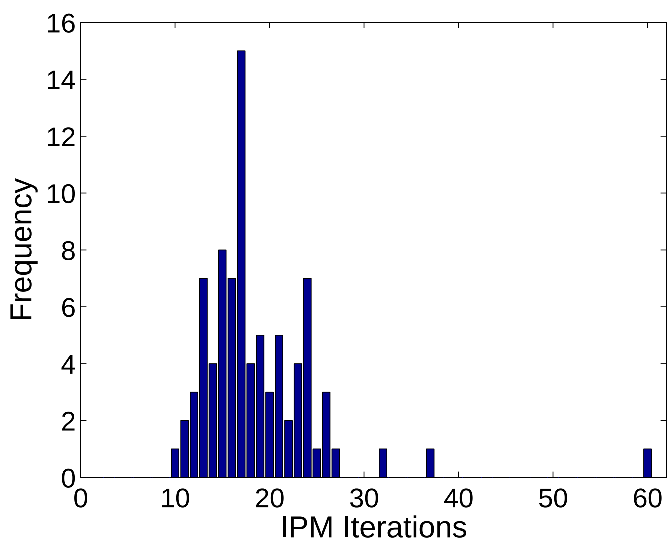

Figure 1: Histogram of IPM Iterations taken by SDPT3-3.1 for 85 Problems in the SDPLIB Suite

Figure 2 shows a scatter plot of the number of IPM iterations taken by SDPT3 and $\log(g^m)$ (all logarithms herein are base 10). In this and other relevant figures, non-finite values of the measure are shown on the far right. Figure 2 indicates that finite values of g^m are highly linearly related to IPM iterations. We also computed the sample correlation of $\log(g^m)$ versus IPM iterations for the 53 finitely-valued instances, obtaining $\text{CORR}\left(\log(g^m),\ \text{IPM Iterations}\right) = 0.896$. These results indicate a significant linear relationship between IPM iterations and $\log(g^m)$. In particular, note from Figure 2 and Table 5 that $\log(g^m)$ explains particularly well the number of IPM iterations for problem instances where g^m is relatively small (say, ≤ 5000).

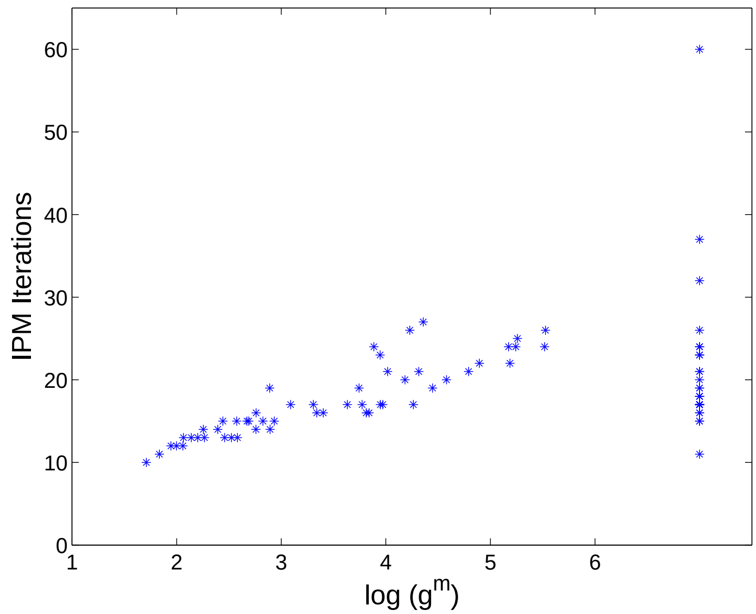

Figure 2: Scatter Plot of IPM iterations and $\log(g^m)$ for 85 problems in the SDPLIB Suite.

We also analyzed some different aggregate geometry measures based on the four individual measures $D_p,\ g_p,\ D_d,\ g_d$, obtaining similar results. For example, the aggregate measure $G^M := \max\{D_p,\ g_p,\ D_d,\ g_d\}$ yields CORR $\left(\log(G^M),\ \text{IPM Iterations}\right) = 0.883$, which is not appreciably different from the corresponding value for g^m.

3 Condition Number

Considering the cone K in the problem (CP) to be fixed, a problem instance is characterized by its data $d = (A(\cdot), b, c)$. Given a norm $\|\cdot\|^X$ for the x variables and a norm $\|\cdot\|^Y$ for the space $\Re^m$ of the image of $A(\cdot)$, we define the norm on the space of data d by $\|d\| := \max\{\|A(\cdot)\|, \|b\|^Y, \|c\|_*^X\}$ where $\|A(\cdot)\|$ is the operator norm. Renegar's theory of condition numbers for (CP) focuses on three quantities – $\rho_P(d), \rho_D(d)$, and $C(d)$, to bound various behavioral and computational quantities pertaining to (CP). The quantity $\rho_P(d)$ is called the "distance to primal infeasibility" and is defined as:

$$\rho_P(d) := \inf\{\|\Delta d\| \mid X_{d+\Delta d} = \emptyset\} ,$$

where X_d denotes the feasible region of (CP):

$$X_d := \{x \in \Re^n \mid A(x) = b, x \in K\} .$$

The quantity $\rho_D(d)$ is called the "distance to dual infeasibility" for the dual problem (CD) and is defined similarly to $\rho_P(d)$ but using the dual problem instead. The quantity $C(d)$ is called the "condition number" of the problem instance d and is a (positively) scale-invariant reciprocal of the smallest data perturbation Δd that will render the perturbed data instance either primal or dual infeasible:

$$C(d) := \frac{\|d\|}{\min\{\rho_P(d), \rho_D(d)\}} \ . \tag{17}$$

A problem is called "ill-posed" if $\min\{\rho_P(d), \rho_D(d)\} = 0$, equivalently $C(d) = \infty$. These three condition measure quantities have been shown in theory to be connected to a wide variety of bounds on behavioral characteristics of (CP) as well as the complexity of interior-point algorithms for (CP), see the literature review in [14].

In particular, it is shown in [19] that a theoretical bound on the number of iterations of a suitable IPM involves the term $O(\sqrt{\vartheta}\log(C(d)))$ (ϑ is the complexity parameter of the barrier function used therein). Furthermore, $\log(C(d))$ is shown to have some explanatory value for IPM iteration counts for the NETLIB suite of linear programming problems, see [14] as well. Herein, just as we did with the aggregate geometry measure g^m, we test the correlation between $\log(C(d))$ and IPM iteration counts for SDP problems in the SDPLIB suite.

3.1 Distances to infeasibility and norm of data

In order to estimate $C(d)$ efficiently we need to compute and/or estimate the three quantities $\rho_P(d), \rho_D(d)$, and $\|d\|$. The computation of these quantities is hard or easy depending on the choice of norms, see [9]. Specifying to the case of (SDP) where K is defined by (4), we use the following choice of norms:

$$\|x\|^X = \|(x^s, x^l)\|^X := \|x^s\|_{E1} + \|x^l\|_1 \quad \text{and} \quad \|v\|^Y := \|v\|_1 \ . \tag{18}$$

We discuss the computation of $\rho_P(d), \rho_D(d)$, and $\|d\|$ below.

3.1.1 Computation of $\rho_P(d)$

With the choice of norms (18), it follows directly from Remark 6 of [8] that

$$\rho_P(d) = \min_{k=1,\ldots,2m} \rho_P^k(d)$$

where

$$\begin{array}{rll} \rho_P^k(d) = & \min_{y,z,u} & \max\{\|A^*(y) + z\|_*^X, \ |-b^T y + u|\} \\ & \text{s.t.} & y_{\lceil \frac{k}{2} \rceil} = (-1)^k \\ & & z \in K^*, \ y \in \Re^m, \ u \geq 0 \ , \end{array} \tag{19}$$

for $k = 1, \ldots, 2m$. However, noting from (18) that

$$\|z\|_*^X = \|(z^s, z^l)\|_*^X = \max\{\|z^s\|_{E\infty}, \|z^l\|_\infty\} \ ,$$

problem (19) is equivalently:

$$\begin{array}{rll} \rho_P^k(d) = & \min_{y,z,u,\gamma} & \gamma \\ & \text{s.t.} & \sum_{i=1}^n A_i^s y_i + z^s = \gamma I \\ & & (A^l)^T y + z^l = \gamma e \\ & & -b^T y + u \le \gamma \\ & & y_{\lceil \frac{k}{2} \rceil} = (-1)^k \\ & & z^s \in S_+^s,\ z^l \in \Re^{n_l},\ y \in \Re^m,\ \gamma \in \Re,\ u \ge 0 \ , \end{array} \tag{20}$$

for $k = 1, \ldots, 2m$, which is a conic convex problem of size and structure compatible with the original dual problem instance (SDD). Therefore $\rho_P(d)$ can be computed by solving $2m$ SDP instances of compatible size and structure as the original dual problem instance.

3.1.2 Computation of $\rho_D(d)$

Using Theorem 2 of [8] and exchanging the roles of the primal and dual, it follows that

$$\begin{array}{rll} \rho_D(d) = & \min_{x,g} & \max\left\{\|A(x)\|^Y, |\langle c, x\rangle + g|\right\} \\ & \|x\|^X = 1 & \\ & x \in K & \\ & g \ge 0 \ , & \end{array} \tag{21}$$

which is generally a non-convex problem due to the norm constraint "$\|x\|^X = 1$." However, under the choice of norms (18) $\|x\|^X$ is a linear function on K from Proposition 1, whereby (21) is equivalently:

$$\begin{array}{rll} \rho_D(d) = & \min & \gamma \\ & \text{s.t.} & \|A^s x^s + A^l x^l\|_1 \le \gamma \\ & & |c^s \bullet x^s + (c^l)^T x^l + g| \le \gamma \\ & & I \bullet x^s + e^T x^l = 1 \\ & & x^s \in S_+^s,\ x^l \in \Re_+^{n_l},\ g \ge 0 \ , \end{array} \tag{22}$$

which can easily be converted to a conic convex optimization problem whose size and structure is compatible with the original problem instance (SDP).

3.1.3 Estimation of $\|d\|$

Recalling that $\|d\| := \max\{\|A(\cdot)\|, \|b\|^Y, \|c\|_*^X\}$, with the choice of norms given by (18) we have $\|b\|^Y = \|b\|_1$ and $\|c\|_*^X = \max\{\|c^s\|_{E\infty}, \|c^l\|_\infty\}$, whose computation are straightforward. However, under this choice of norms we have

$$\begin{aligned} \|A(\cdot)\| &= \max \ \{\|A^s x^s + A^l x^l\|_1 : \|x^s\|_{E1} + \|x^l\|_1 \le 1\} \\ &= \max \ \left\{\max_{\|x^s\|_{E1}\le 1} \|A^s x^s\|_1, \max_{\|x^l\|_1 \le 1} \|A^l x^l\|_1\right\} \\ &= \max \ \left\{\|A^s\|_{E1,1}, \|A^l\|_{1,1}\right\} \\ &= \max \ \left\{\|A^s\|_{E1,1}, \|A^l_{\bullet 1}\|_1, \ldots, \|A^l_{\bullet n_l}\|_1\right\} \ , \end{aligned}$$

where $A^l_{\bullet j}$ denotes the jth column of A^l, and so the only difficulty in estimating $\|d\|$ lies in estimating $\|A^s\|_{E1,1}$. We use standard norm inequalities to bound this quantity as follows:

Proposition 3 *Let $\lambda_i^{\max}$ denote the eigenvalue of A_i^s of largest absolute value, let v^i be the corresponding eigenvector normalized to $\|v^i\|_2 = 1$, and define $\hat{x}^i := v^i(v^i)^T$. Then*

$$L \le \|A^s\|_{E1,1} \le U \ ,$$

where

$$U = \min\left\{\sum_{j=1}^{s}\sum_{f=1}^{s} \|A^s_{\bullet jf}\|_1, \sqrt{m}\|A^s\|_{E2,2}, \|(\lambda_1^{\max}, \ldots, \lambda_m^{\max})\|_1\right\} \ ,$$

and

$$L = \max\left\{\frac{\|A^s\|_{E2,2}}{\sqrt{|s|}}, \ \|A^s\hat{x}^1\|_1, \ldots, \|A^s\hat{x}^m\|_1\right\} \ .$$

Proof: From their definition we have that $\|\hat{x}^i\|_{E1} = 1$, therefore $\|A^s\hat{x}^i\|_1 \le \|A^s\|_{E1,1}$ for all $i = 1, \ldots, m$. Using the well known relationship between norms $\|y\|_2 \le \|y\|_1 \le \sqrt{k}\|y\|_2$ for $y \in \Re^k$, we can show that $\frac{1}{\sqrt{|s|}}\|A^s\|_{E2,2} \le \|A^s\|_{E1,1} \le \sqrt{m}\|A^s\|_{E2,2}$. Now we have that $\|A^s x^s\|_1 = \sum_{i=1}^m |A_i^s \bullet x^s| \le \sum_{i=1}^m \sum_{j,f=1}^s |A^s_{ijf}||x^s_{jf}| \le \sum_{j,f=1}^s \|A^s_{\bullet,jf}\|_1$, where we used the fact that $|x^s_{jf}| \le \|x^s\|_{E1} \le 1$. Finally, we use the eigenvalue-eigenvector decomposition of $A_i^s = \sum_{j=1}^n (\lambda_i)_j v_j^i (v_j^i)^T$ to show that $|A_i^s \bullet x^s| \le |\lambda_i^{\max}| \sum_{j=1}^s |(v_j^i)^T x^s v_j^i| \le |\lambda_i^{\max}|$. The last inequality is because $\sum_{j=1}^s |(v_j^i)^T x^s v_j^i| \le 1$ for any $\|x^s\|_{E1} \le 1$. Therefore $\|A^s\|_{E1,1} \le \|(\lambda_1^{\max}, \ldots, \lambda_m^{\max})\|_1$. ∎

Note that it can be readily shown that $\|A^s\|_{E2,2} = \|[\text{svec}(A_1^s), \ldots, \text{svec}(A_m^s)]\|_{2,2}$, where $\text{svec}(\cdot)$ denotes the standard linear isometry between S^s and the vector space $\Re^{\bar{n}}$, where $\bar{n} = \sum_{j=1}^{n_s} s_j(s_j+1)/2$. Thus the quantity $\|A^s\|_{E2,2}$ in Proposition 3 can be computed easily by using variants of the Lanczos method to compute the spectral-norm of the matrix $[\mathrm{svec}(A_1^s), \ldots, \mathrm{svec}(A_m^s)]$.

Therefore using Proposition 3 we can bound $\|d\|$ as follows:

$$\max\left\{L, \|A^l_{\bullet 1}\|_1, \ldots, \|A^l_{\bullet n_l}\|_1, \|b\|_1, \|c^s\|_{E\infty}, \|c^l\|_\infty\right\} \leq \|d\|$$

and

$$\|d\| \leq \max\left\{U, \|A^l_{\bullet 1}\|_1, \ldots, \|A^l_{\bullet n_l}\|_1, \|b\|_1, \|c^s\|_{E\infty}, \|c^l\|_\infty\right\} \ .$$

3.2 Computation of Condition Number Estimates for the SDPLIB Suite

The previous subsection shows that $\rho_P(d)$ can be computed by solving $2m$ SDP problems, $\rho_D(d)$ can be computed by solving a single SDP problem, and that lower and upper bounds on $\|d\|$ can be computed using straightforward matrix norms and maximum eigenvalue computations. This enables us to then compute lower and upper bounds on $C(d)$ using (17).

Table 6 in the Appendix contains the resulting computation of upper and lower bounds on $C(d)$ for the 85 problems in the SDPLIB suite. Blank entries in the table indicate that we were unable to compute the corresponding measure. We were able to estimate $\|d\|$ and to compute $\rho_D(d)$ for all 85 SDPLIB problems under consideration. However, we were not able to compute $\rho_P(d)$ for five problems, namely `control11, equalG51, maxG32, theta6`, and `thetaG11`. These five problems have large values of m (1596, 1001, 2000, 4375, and 2401, respectively), rendering the computation of the $2m$ problems needed for determining $\rho_P(d)$ excessive.

Some summary statistics from Table 6 are presented in Table 2. These statistics show that 48 out of 80 problems are well-posed, and the 32 problems that are ill-posed have primal distance to infeasibility equal to zero and positive dual distance to infeasibility. For the 48 problems with finite $C(d)$, the ratios between the upper and lower bounds on $C(d)$ are less than 20.4, see in Table 2. Therefore the logarithm of the geometric mean of the upper and lower bounds on $C(d)$ can differ from the true value of $\log(C(d))$ by no more $0.65 = \log_{10}(\sqrt{20.4})$, and so these estimates are fairly good.

Figure 3 shows a scatter plot of the number of IPM iterations taken by SDPT3 and $\log(C(d))$, using the average of logarithm of the lower and upper bounds on $C(d)$ from Table 6. Similar to the aggregate geometric measures, non-finite values of $C(d)$ are shown on the far right. The figure indicates that finite values are related to IPM iterations. We computed the sample correlation of $\log(C(d))$ versus IPM iterations for the 48 problems with finite $C(d)$. The sample correlation for these problems is

Table 2: Summary Statistics of Distances to ill-posedness for SDPLIB Suite

Status		$\rho_D(d)$ 0	$\rho_D(d)$ Positive	Total
$\rho_P(d)$	0	0	32	32
	Positive	0	48	48
	Total	0	80	80

CORR(log($C(d)$), IPM Iterations) = 0.630. These results indicate a somewhat linear relationship between IPM iterations and $\log(C(d))$ that is not much different from that found on the NETLIB suite of linear programming problems [14], but that is less significant than for the aggregate geometry measure g^m. We also considered other simple expressions involving $C(d)$ which are suggested by theory, such as $\sqrt{n}\log(C(d))$ where $n := |s|+n_l$, but none showed a stronger correlation with IPM iterations than $\log(C(d))$.

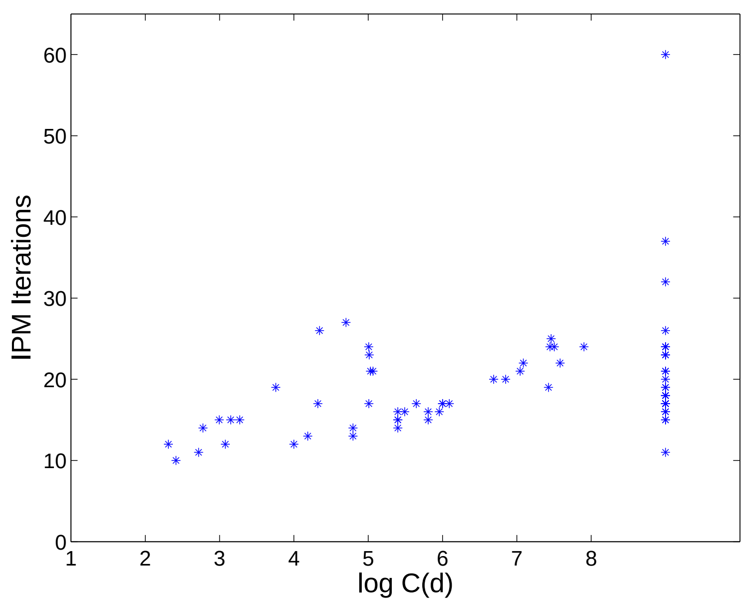

Figure 3: IPM iterations versus $\log(C(d))$.

Comparing Tables 5 and 6, one observes connections between values of the geometry measures and values of $C(d)$, for example, $C(d) = \infty$ precisely for those problem instances when $g_p = \infty$, etc. This is of course not a coincidence. The literature on condition numbers and related problems contains implicit connections between these measures, which we summarize as follows:

Proposition 4 *If $A(\cdot)$ has full row rank (which is the case by design for all problems* *in the SDPLIB suite), then:*

1. $g_p = \infty \iff \rho_P(d) = 0$
2. $g_d = \infty \iff \rho_D(d) = 0$
3. $g_p \leq 3(|s| + n_l)C(d)$
4. $g_d \leq 3(|s| + n_l)C(d)$
5. $D_p^\varepsilon \leq C(d)^2 + C(d)\frac{\varepsilon}{\|c\|_*}$
6. $D_d^\varepsilon \leq C(d)^2 + C(d)\frac{\varepsilon}{\|b\|}$

Proof: Item (1.) follows from Theorem 17 of [8] and Robinson [20], (2.) follows from Theorem 19 of [8] and Robinson [20], (3.) and (4.) follow from Theorems 17 and 19 of [8], and (5.) and (6.) follow from Theorem 1.1 and Lemma 3.2 of [18]. ■

4 Non-Strict Complementarity

Consider the following definition of strict complementarity, which is adapted from [1] and which considers an SDP instance with $A(\cdot)$ having full row rank (as is the case for SDPLIB instances).

Definition 1 *Let* $x = (x^s, x^l)$ *be primal feasible, and* y *and* $z = (z^s, z^l)$ *be a dual feasible pair such that* $\langle x, z\rangle = 0$*. Then strict complementarity is said to hold for* x *and* z *if* $x^s + z^s \succ 0$ *and* $x^l + z^l > 0$*.*

Strict complementarity is a desirable property of an SDP instance; in fact the strict complementarity of an optimal solution is a necessary condition for superlinear convergence of interior-point methods that take Newton-like steps, see [17], and much recent research has explored what conditions in addition to strict complementarity are needed to guarantee superlinear convergence for different interior-point algorithms [10, 11, 12, 13, 16]. However, even for linear programming (which must have a strictly complementary solution), there are instances for which the optimal solutions are nearly non-strictly complementary, and can be made arbitrarily badly so. Furthermore, in interior-point methods for either linear or semidefinite programming, we terminate the algorithm with a primal-dual solution that is almost optimal but not actually optimal. Hence there are genuine conceptual difficulties in trying to quantify and compute the extent of near-non-strict-complementarity for an SDP instance (and for LP instances as well).

Consider a point $(x, y, z) = (x(\mu), y(\mu), z(\mu))$ on the primal-dual central path of (SDP), then there exists an orthonormal matrix Q and diagonal matrices Λ_x, Λ_z corresponding to the vectors of eigenvalues λ_x, λ_z of x^s, z^s satisfying $x^s = Q\Lambda_x Q^T$, $z^s = Q\Lambda_z Q^T$, and

$$\Lambda_x \Lambda_z = \mu I \qquad \text{and} \qquad x^l(\mu) \circ z^l(\mu) = \mu e \ . \tag{23}$$

The duality gap of this solution is $\varepsilon := n\mu$ where $n = |s| + n_l$. Considering the j^{th} matrix equation of (23), we know that the two scalar quantities $\pi := (\lambda_x)_j$ and $\gamma := (\lambda_z)_j$ must satisfy $\pi, \gamma \geq 0$ and $\pi \cdot \gamma = \mu$. However, noticing that $\pi + \gamma \geq \min_{\pi,\gamma \geq 0}\{\pi + \gamma : \pi \cdot \gamma = \mu\} = 2\sqrt{\mu}$, this implies more generally that:

$$\Lambda_x + \Lambda_z \succeq 2\sqrt{\mu} I \qquad \text{and} \qquad x^l + z^l \geq 2\sqrt{\mu} e \ . \tag{24}$$

When a problem instance is non-strictly-complementary (NSC), then at least one index j must satisfy $(\lambda_x)_j \to 0$ and $(\lambda_z)_j \to 0$ (or $x^l_j \to 0$ and $z^l_j \to 0$) as $\mu \to 0$. In addition, (24) shows that $(\lambda_x)_j + (\lambda_z)_j$ (or $x^l_j + z^l_j$) must remain greater than $2\sqrt{\mu}$, which indicates that its slope must become unboundedly large near $\mu = 0$.

The above analysis, which is based on points being on the central path, suggests the following more general approach to measure the extent to which a computed approximate solution (x, y, z) is non-strictly complementary. Let $w = (x + z)/(2\sqrt{\mu})$, where $\mu = \langle x, z\rangle/n$. We partition w into $w = (w^s, w^l)$ and let λ_w denote the vector of eigenvalues of w^s. We consider (x, y, z) to be nearly non-strictly complementary if λ_w and/or w^l have small positive components. This is quantified by choosing a tolerance value T and defining the following index sets:

$$T^s := \{j : (\lambda_w)_j \leq T\} \quad \text{and} \quad T^l := \{j : w^l_j \leq T\} \ ,$$

and then using the following measure:

$$\kappa := \frac{-\left(\sum_{j \in T^s} \ln((\lambda_w)_j) + \sum_{j \in T^l} \ln(w_j)\right)}{|T^s| + |T^l|} \ .$$

Note that a larger value of κ indicates that the problem is closer to having a non-strict complementary optimal solution.

4.1 Computation of the non-strict complementarity measure κ for the SDPLIB Suite

As the notion of non-strict complementarity (and also the notion of degeneracy that we study in the next section) concerns an optimal primal-dual solution, the approximate optimal solution (x_k, y_k, z_k) we use must have high accuracy in order for its associated κ value to be of relevance. The approximate solutions delivered by SDPT3-3.1, however, are usually not accurate enough for the purpose of measuring non-strict complementarity and degeneracy of a primal-dual optimal solution. We therefore use a slightly different version of SDPT3-3.1 to compute more accurate approximate optimal solutions in this section as well as in Section 5. For ease of reference, we refer to that version as SDPT3-aug. The interior-point algorithms implemented in both versions are the same, except in the ways the search directions at each iteration are computed. For SDPT3-3.1, the search direction is computed from the Schur complement equation, which is a symmetric positive definite $m \times m$ linear system, whereas the search direction in SDPT3-aug is computed from a reduced augmented equation described in [21]. The reduced augmented equation is a symmetric indefinite linear system that has a larger dimension than the Schur complement equation. Because of the higher computational cost required to solve the reduced augment equation compared to the Schur complement equation, this alternative method of computing the search direction is not implemented in SDPT3-3.1. The reduced augmented equation, however, has empirically proven to have better stability properties than the Schur complement equation, thereby allowing SDPT3-aug to compute more accurate optimal approximate solutions than SDPT3-3.1 before numerical difficulties are encountered in the course of the IPM iterations.

Table 7 in the Appendix contains the IPM iterations obtained by SDPT3-aug for the 85 problems in the SDPLIB suite. The relative error (described in (16)) of the approximate optimal solution obtained for each problem is shown in the third column of the table. Note that in this case, we let the algorithm run until it cannot improve the accuracy of the approximate optimal solution or when numerical difficulties are encountered.

Using the tolerance value $T = \max\{100, \lambda_{\min}(w^s)\}$ for finding T^s and the value $T = \max\{100, \min_j(w_j^l)\}$ for finding T^l, we computed κ for all 85 problems in the SDPLIB suite; these values are reported in the fourth column of Table 7. Note that by the choice of value for T, both T^s and T^l have at least one element. Figure 4 shows a scatter plot of the number of IPM iterations taken by SDPT3 and κ in the SDPLIB suite. We computed the sample correlation of κ versus IPM iterations for the 85 problems, obtaining $\text{CORR}(\kappa,\ \text{IPM Iterations}) = 0.423$. Thus the near absence of strict complementarity (as measured by κ) is correlated with the IPM iteration counts, but less so than either $\log(C(d))$ or $\log(g^m)$.

We also constructed and tested a variety of other continuous and discrete measures of near-non-strict complementarity, but none of our other measures showed a stronger correlation with IPM iteration counts than κ. The smaller correlation between κ and overall IPM iteration counts is understandable given that the strict complementarity of the optimal solution is a local property, which has been shown to be related to the rate of local convergence. If we define β to be the geometric mean of the rate of convergence

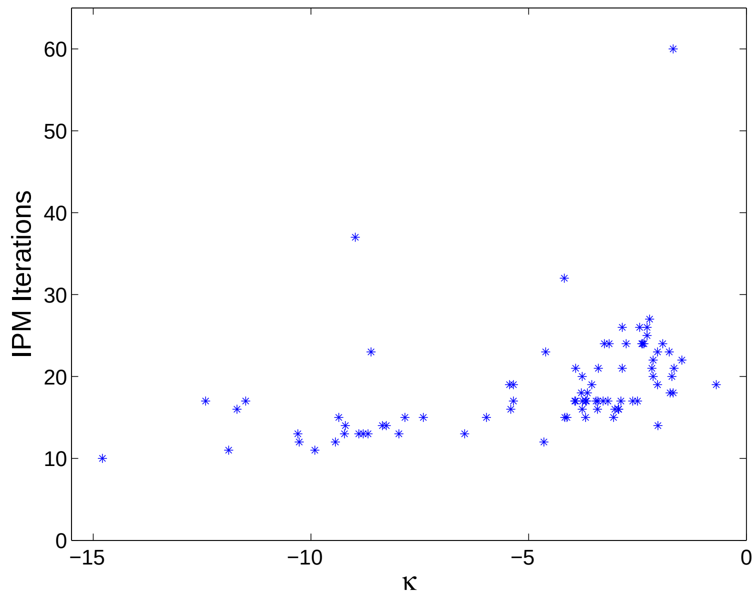

Figure 4: IPM iterations versus non-strict complementarity measure κ.

of the last 5 iterations of an IPM algorithm, i.e., $\beta = \sqrt[5]{\left(\frac{\mu^k}{\mu^{k-5}}\right)}$, then β is a proxy for the rate of local convergence. For the same 85 SDPLIB suite problems, it turns out that the correlation between κ and β is $\mathrm{CORR}(\kappa, \beta) = 0.458$. This result shows that κ is modestly correlated with the local rate of convergence on the SDPLIB problems. A more detailed study exploring the behavior of local rates of convergence is beyond the scope of the current paper.

We note that non-strict complementarity is not theoretically related to either $C(d)$ or to any of the four geometry measures, as it is straightforward to construct small examples with and without strict complementarity and with and without interiors of primal and/or dual feasible regions, for example.

5 Degeneracy

It is shown in [2] that if in addition to a strictly complementary solution, the optimal solution is primal and dual non-degenerate, then some IPM variants exhibit local Q-quadratic convergence. This suggests that IPM iteration counts might be related to the extent of primal and/or dual degeneracy at the optimal solution. We use the standard definitions of degeneracy for SDP adapted from [1].

Definition 2 *Let $x = (x^s, x^l)$ be a primal optimal solution of (SDP) with* $\text{rank}(x^s) = r$ *and $J := \{j : x^l_j > 0\}$. Let $Q_1 \in \Re^{|s| \times r}$ and $Q_2 \in \Re^{|s| \times (|s|-r)}$ be matrices whose columns form orthonormal bases of eigenvectors for the range space and null space of x^s, respectively. The point x is said to be primal non-degenerate if and only if the matrix*

$$B(x) \;:=\; \Big[\text{mat}(A^s)(Q_1 \otimes Q_1),\; \text{mat}(A^s)(Q_1 \otimes Q_2),\;\; A^l_J\Big]$$

has full row rank.

In this definition $\text{mat}(A^s)$ denotes the matrix representation of the linear map A^s, A^l_J denotes the sub-matrix obtained from A^l whose columns correspond to the index set J, and $Q_i \otimes Q_j$ denotes the Kronecker matrix product of Q_i, Q_j.

Definition 3 *Let y and $z = (z^s, z^l)$ be a dual optimal solution with* $\text{rank}(z^s) = \tilde{r}$ *and $\tilde{J} := \{j : z^l_j = 0\}$. Let $\tilde{Q}_1 \in \Re^{|s| \times (|s|-\tilde{r})}$ and $\tilde{Q}_2 \in \Re^{|s| \times \tilde{r}}$ be matrices whose columns form orthonormal bases of eigenvectors for the null space and range space of z^s, respectively. The pair (y, z) is said to be dual non-degenerate if and only if the matrix*

$$\tilde{B}(y, z) \;:=\; \Big[\text{mat}(A^s)(\tilde{Q}_1 \otimes \tilde{Q}_1),\;\; A^l_{\tilde{J}}\Big]$$

has full column rank.

As we already noted in the previous section, in interior-point methods for either LP or SDP we terminate the algorithm with a primal-dual solution (x_k, y_k, z_k) that is almost optimal but not actually optimal. Thus strictly speaking, the rank of x^s_k is $|s|$. But since we know that x_k is converging to an optimal primal solution x^s_* that has rank r, we can estimate r from the eigenvalues of x^s_k by counting the number of eigenvalues that are significantly larger than $\mu_k := \langle x_k, z_k \rangle / n$. The rank $\tilde{r}$ can similarly be estimated. In order to determine r, $\tilde{r}$, J, and $\tilde{J}$ unambiguously, we need (x_k, y_k, z_k) to be a highly accurate approximate optimal solution. One of the main difficulties we encounter in trying to determine the degeneracy of a primal-dual approximate optimal solution is in the numerical determination of r, $\tilde{r}$, J, and $\tilde{J}$. Unless there is a clear separation of the eigenvalues of x^s_k to indicate clearly those that correspond to the range space of x^s_*, it is hard to determine r without ambiguity; these remarks also pertain to $\tilde{r}$, J and $\tilde{J}$.

Besides having to determine r, $\tilde{r}$, J and $\tilde{J}$ numerically from an approximate optimal solution, the ranks of the matrices in Definitions 2 and 3 must also be determined numerically from $B(x_k)$ and $\tilde{B}(y_k, z_k)$. One of the most commonly used method to determine the rank of a matrix is to compute its singular value decomposition and to count those singular values that are significantly larger than machine precision. We adopt this method here by considering as zero singular values that are computed to be smaller than 10^{-13} times the largest computed singular value.

5.1 Computation of measure of degeneracy for the SDPLIB Suite

Out of the 85 problems in the SDPLIB suite, we are able to compute approximate optimal solutions that are accurate enough to determine r, $\tilde{r}$, J and $\tilde{J}$ unambiguously for 68 problems. A summary of the degeneracy status of these 68 problems is shown in Table 3. Note that 25 of the 68 problems are degenerate, the rest are nondegenerate. Table 7 in the Appendix contains more specific degeneracy information for the 85 SDPLIB problems. The column labeled "pd" contains the fraction whose numerator is the rank of $B(x_k)$ and whose denominator is m. The column labeled "dd" contains the fraction whose numerator and denominator are the column rank and the number of columns of $\tilde{B}(y_k, z_k)$, respectively. Blank entries in these two columns indicate that that we were not able to determine the degeneracies without ambiguity. We measure the degeneracy of the primal/dual solution triplet (x_k, y_k, z_k) by the following quantity:

$$\gamma := \max \left\{ 1 - \frac{\text{row rank}(B(x_k))}{m}, 1 - \frac{\text{col rank}(\tilde{B}(y_k, z_k))}{N} \right\} ,$$

where N denotes the number of columns of $\tilde{B}(y_k, z_k)$. Note that γ will be larger to the extent that the matrices $B(x_k)$ and $\tilde{B}(y_k, z_k)$ are far from full rank. The values of γ for the 68 problems whose degeneracies are unambiguous are shown in the fifth column of Table 7.

Table 3: Degeneracy Status for 68 Problems in the SDPLIB Suite.

Status		Dual Problem: Degenerate	Dual Problem: Nondegenerate	Total
Primal Problem	Degenerate	8	2	10
	Nondegenerate	15	43	58
	Total	23	45	68

Figure 5 shows a scatter plot of the number of IPM iterations taken by SDPT3-aug and γ for 68 problems in the SDPLIB suite. The figure seems to reveal little in the way of a pattern/relationship between the extent of degeneracy and IPM iterations. For completeness, we computed the sample correlation of γ versus IPM iterations for the 68 problems in the SDPLIB, obtaining CORR(γ, IPM Iterations) = 0.100. This finding is consistent with the theoretical result in [13] showing that local superlinear convergence of interior-point methods can be achieved even for degenerate problems.

Because only 25 problems exhibited degeneracy, we tried to construct measures of "closeness to degeneracy" for nondegenerate problems, such as the ratios of the largest

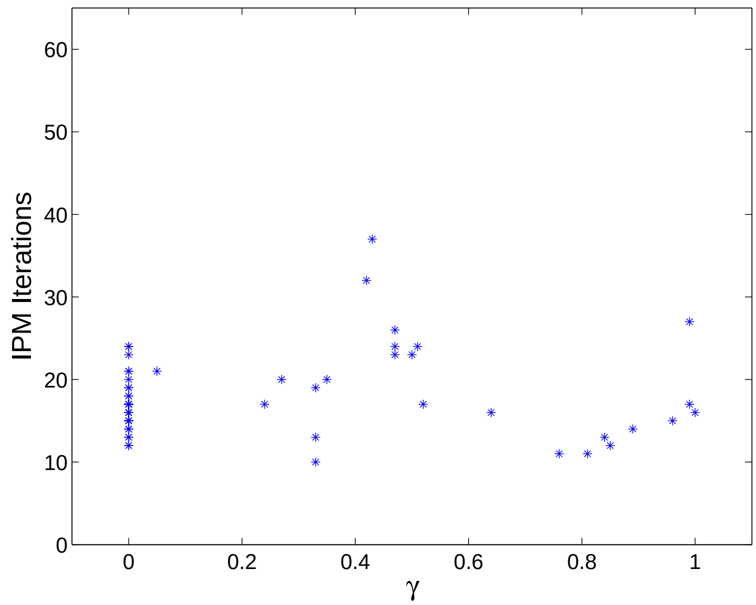

Figure 5: IPM iterations versus degeneracy measure γ.

to the smallest singular values of matrices $B(x_k)$ and $\tilde{B}(y_k, z_k)$. However, we found no evidence (using the SDPLIB suite) that such measures showed any noticeable relation to IPM iteration counts. Finally, we mention that degeneracy of the optimal solution, like strict-complementarity, is a local property, which has been shown to be related to the rate of local convergence. However, unlike strict complementarity, we found no evidence of correlation between the degeneracy measure γ and the local convergence measure β defined at the end of Section 4.

6 Summary Conclusions and Further Issues

6.1 Summary Conclusions

We observe that 53 of 85 SDPLIB problem instances have finite geometry measure g^m, and for these 53 instances we have $\text{CORR}(\log(g^m), \text{ IPM Iterations}) = 0.896$, indicating a significant linear relationship between IPM iterations and $\log(g^m)$ among these problem instances.

Regarding the condition measure $C(d)$, we observe that 32 of 80 SDPLIB problem instances are almost primal infeasible, i.e., $C(d) = +\infty$. Among the 48 SDPLIB instances with finite condition measure, we have $\text{CORR}(\log(C(d)), \text{IPM Iterations}) = 0.630$, which indicates a somewhat linear relationship between IPM iterations and $\log(C(d))$, that is less significant than for $\log(g^m)$.

The near absence of strict-complementarity, measured with the quantity κ developed in Section 4 and applied to the 85 SDPLIB problems under consideration, is weakly correlated with IPM iterations: $\text{CORR}(\kappa, \text{IPM Iterations}) = 0.423$.

Incidentally, traditional dimensional measures such as m, $\bar{n}$, $n := |s| + n_l$, or $\sqrt{n}$ are not well correlated with IPM iterations on the SDPLIB suite. For example, we observed $\text{CORR}(m, \text{IPM Iterations}) = 0.060$, $\text{CORR}(\bar{n}, \text{IPM Iterations}) = -0.107$, $\text{CORR}(n, \text{IPM Iterations}) = -0.008$, and $\text{CORR}(\sqrt{n}, \text{IPM Iterations}) = 0.043$.

We were able to determine the degeneracy status for 68 problems out of the 85 in the SDPLIB suite. Among these 68 problems, 25 are degenerate, and the degeneracy parameter γ developed in Section 5 is essentially uncorrelated with IPM iterations.

Table 4 shows more extensive correlation values among the finite values of the four behavioral measures we have studied. Comparisons between the correlation values in the table must be done with caution, since problem instances used to compute each correlation varied, and were limited to those problems with finite values for both measures in the pair.

Table 4: Summary Correlation Values for all Measures (with Number of Problem Instances in Boldface)

	Iterations	κ	$\log(g^m)$	$\log(C(d))$	γ
Iterations	1.000				
κ	0.423 (**85**)	1.000			
$\log(g^m)$	0.896 (**53**)	0.708 (**53**)	1.000		
$\log(C(d))$	0.630 (**48**)	0.631 (**48**)	0.831 (**48**)	1.000	
γ	0.100 (**68**)	-0.256 (**68**)	-0.000 (**42**)	0.030 (**38**)	1.000

Notice the high correlation between κ and $\log(g^m)$ (and less significantly to $\log(C(d))$). This is not indicated by any theory, since one can easily construct examples with high values of κ and low values of $\log(g^m)$ or $\log(C(d))$ and vice versa. Therefore the high correlation is specific to the 53 data instances in the SDPLIB suite, and suggests that the SDPLIB suite has some systematic behavioral patterns. Of course, this is not too surprising, since the SDPLIB suite contains large numbers of instances of a relatively few application domains of SDP.

In addition to being the most correlated with IPM iteration counts, the aggregate geometry measure g^m also poses the least computational challenge. In computing $C(d)$, in particular for determining $\rho_P(d)$, we must solve $2m$ SDP problems of size and structure comparable to the original SDP, as contrasted to solving just four such problems in order to determine g^m. In the case of the problem instance `control11` this translates to a few minutes to compute g^m versus over five days to compute $C(d)$. (Had we instead used Peña's method [15] for estimating $C(d)$, we still would face significant challenges in order to compute smallest eigenvalues of large dense positive definite matrices, see [5], which is why we did not adopt that approach.) As discussed in Section 4.1 the computation of κ (or γ) is made challenging by the need to compute an approximate solution with sufficiently high accuracy to identify which variables and eigenvalues are positive versus zero. Such high accuracy is not a prerequisite for reliable computation of g^m (or $C(d)$).

6.2 Further Issues

We chose to test the correlation of the behavioral measures on IPM iterations of a particular interior-point method, namely a "standard" primal-dual infeasible interior-point method that uses the HKM direction. Our intuition suggests that our conclusions would not change appreciably if we instead used the NT direction or the AHO direction, but might change if we used a homogeneous self-dual embedding model, see [4] for example.

Our agenda in this study was not to try to better predict IPM iterations, but rather to test the extent to which certain theoretically-motivated complexity measure and/or convergence measures might or might not be relevant to computational practice. Therefore we have not explored the extent to which the four behavioral measures (or others) might jointly better account for differences in IPM iteration counts. Nevertheless, presuming for the moment that the SDPLIB is a representative data set of the universe of relevant SDP instances, it would be interesting to see if certain combinations of different measures can do a better job of accounting for differences in IPM iterations. For example, might there be a systematic correlation between IPM iterations and, say, κ on those SDPLIB instances for which $C(d) = \infty$?

The high correlation between κ and $\log(g^m)$ shown in Table 4 clarifies the intuition that the SDPLIB suite has some systematic behavioral patterns. An overarching question is to construct or otherwise identify a reasonably-sized set of SDP problem instances that might be better suited to empirically examine issues related to the computational behavior of algorithms for SDP.

The measures of near absence of strict-complementarity κ and degeneracy γ considered in this paper are local properties of the optimal solution, shown in theory to be related to the local rate of convergence. We found preliminary computational results that suggest that κ is modestly correlated with the rate of local convergence, whereas γ is not. A detailed study of explanatory measures of local convergence rates of IPMs would hopefully shed further light on this issue.

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Appendix: Computation on the SDPLIB Suite

In this appendix we present tables with all computed measures for the SDPLIB suite. In these tables, floating point numbers are shown in scientific notation; for example we see from Table 5 below that D_d for problem `arch0` is 3.5×10^3, etc.

Table 5: Aggregate Geometry Measure g^m and IPM iteration counts obtained by SDPT3-3.1 for the SDPLIB Suite

Problem	Iterations	err	D_p	g_p	D_d	g_d	g^m
arch0	24		2.5 1	2.0 4	3.5 3	2.0 6	7.7 3
arch2	23		3.7 1	2.0 4	4.1 3	2.0 6	8.8 3
arch4	21		4.8 1	2.0 4	6.1 3	2.0 6	1.0 4
arch8	21		1.6 2	2.0 4	3.2 4	1.8 6	2.1 4
control1	17		1.9 1	9.3 4	8.7 5	5.0 3	9.4 3
control2	20		9.3 0	3.0 5	1.3 6	1.5 4	1.5 4
control3	20		1.5 1	7.7 5	5.7 6	3.2 4	3.8 4
control4	21		2.1 1	1.3 6	1.1 7	4.9 4	6.2 4
control5	22		1.8 1	2.0 6	1.7 7	6.2 4	7.8 4
control6	22	2.0 -6	3.8 1	3.1 6	5.2 7	9.2 4	1.5 5
control7	24		2.2 1	4.1 6	5.0 7	1.1 5	1.5 5
control8	24		2.1 1	5.5 6	5.8 7	1.4 5	1.7 5
control9	25		1.6 1	7.0 6	5.7 7	1.7 5	1.8 5
control10	24	1.1 -5	4.0 1	8.3 6	1.8 8	2.0 5	3.3 5
control11	26	1.1 -6	3.3 1	1.0 7	1.7 8	2.3 5	3.4 5
equalG11	16	2.0 -6	8.0 2	1.6 3	6.4 5	2.2 3	6.5 3
equalG51	17		1.0 3	2.0 3	1.0 6	3.1 3	8.9 3
gpp100	18		1.0 2	∞	∞	1.9 2	∞
gpp124-1	18	9.0 -6	1.2 2	∞	∞	1.9 2	∞
gpp124-2	17		1.2 2	∞	∞	2.4 2	∞
gpp124-3	15		1.2 2	∞	∞	2.8 2	∞
gpp124-4	17	5.0 -6	1.2 2	∞	∞	3.5 2	∞
gpp250-1	16	6.0 -5	2.5 2	∞	∞	4.0 2	∞
gpp250-2	17	7.0 -6	2.5 2	∞	∞	4.8 2	∞
gpp250-3	16	7.0 -6	2.5 2	∞	∞	5.9 2	∞
gpp250-4	19		2.5 2	∞	∞	7.2 2	∞
gpp500-1	24		5.0 2	∞	∞	7.9 2	∞
gpp500-2	15	5.0 -6	5.0 2	∞	∞	9.6 2	∞
gpp500-3	17		5.0 2	∞	∞	1.2 3	∞
gpp500-4	17		5.0 2	∞	∞	1.5 3	∞
hinf1	23		5.9 0	∞	∞	7.6 1	∞
hinf2	17		4.0 2	1.5 5	3.8 5	5.0 3	1.8 4
hinf3	19	5.0 -6	1.1 3	∞	∞	1.5 4	∞
hinf4	23		6.6 1	∞	∞	1.8 3	∞

Problem	Iterations	err	D_p	g_p	D_d	g_d	g^m
hinf5	18	1.1 -4	2.5 3	∞	∞	1.0 5	∞
hinf6	24	9.0 -6	5.7 3	∞	∞	6.8 4	∞
hinf7	17	5.0 -6	3.7 4	∞	∞	3.5 5	∞
hinf8	21	6.0 -6	1.0 3	∞	∞	1.6 4	∞
hinf9	19	1.2 -5	1.1 5	3.1 2	1.8 4	1.0 6	2.8 4
hinf10	37		2.1 1	∞	∞	1.6 3	∞
hinf11	32		1.3 1	∞	∞	1.3 3	∞
hinf12	60	1.1 -5	1.0 0	∞	∞	1.4 3	∞
hinf13	23	6.8 -2	5.5 3	∞	∞	9.4 4	∞
hinf14	26	2.3 -4	3.1 2	∞	∞	3.3 3	∞
hinf15	24	1.1 -1	8.8 3	∞	∞	1.8 5	∞
maxG11	15		8.0 2	8.0 2	6.1 2	1.4 3	8.6 2
maxG32	16		2.0 3	2.0 3	1.6 3	3.6 3	2.2 3
maxG51	17		1.0 3	1.0 3	1.1 3	2.1 3	1.2 3
mcp100	12		1.0 2	1.0 2	9.2 1	1.9 2	1.1 2
mcp124-1	13		1.2 2	1.2 2	6.7 1	1.9 2	1.2 2
mcp124-2	13		1.2 2	1.2 2	1.1 2	2.3 2	1.4 2
mcp124-3	13		1.2 2	1.2 2	1.6 2	2.8 2	1.6 2
mcp124-4	13		1.2 2	1.2 2	2.3 2	3.5 2	1.8 2
mcp250-1	14		2.5 2	2.5 2	1.5 2	4.0 2	2.5 2
mcp250-2	13		2.5 2	2.5 2	2.3 2	4.8 2	2.9 2
mcp250-3	13		2.5 2	2.5 2	3.4 2	5.9 2	3.3 2
mcp250-4	13		2.5 2	2.5 2	4.7 2	7.2 2	3.8 2
mcp500-1	15		5.0 2	5.0 2	2.9 2	7.9 2	4.9 2
mcp500-2	16		5.0 2	5.0 2	4.6 2	9.6 2	5.8 2
mcp500-3	15		5.0 2	5.0 2	6.7 2	1.2 3	6.7 2
mcp500-4	14		5.0 2	5.0 2	1.0 3	1.5 3	7.8 2
qap5	11		6.0 0	∞	∞	1.3 3	∞
qap6	18		7.0 0	∞	∞	3.3 3	∞
qap7	21		8.0 0	∞	∞	4.1 3	∞
qap8	20		9.0 0	∞	∞	7.1 3	∞
qap9	17		1.0 1	∞	∞	1.1 4	∞
qap10	17		1.1 1	∞	∞	1.5 4	∞
qpG11	16		8.0 2	1.6 3	4.9 3	6.5 3	2.5 3
qpG51	17		1.0 3	2.0 3	2.4 4	2.6 4	5.9 3
ss30	19		2.2 2	1.0 3	1.8 4	2.4 5	5.6 3
theta1	12		1.0 0	5.0 1	1.1 3	1.1 3	8.8 1
theta2	14		1.0 0	1.0 2	3.2 3	3.3 3	1.8 2
theta3	15		1.0 0	1.5 2	6.2 3	6.3 3	2.8 2
theta4	15		1.0 0	2.0 2	9.9 3	1.0 4	3.8 2
theta5	15		1.0 0	2.5 2	1.4 4	1.4 4	4.7 2
theta6	14		1.0 0	3.0 2	1.9 4	1.9 4	5.7 2
thetaG11	19		8.0 2	2.4 3	2.0 2	9.5 2	7.8 2
truss1	10		1.9 1	4.6 2	6.1 1	1.3 1	5.1 1

Problem	Iterations	err	D_p	g_p	D_d	g_d	g^m
truss2	17		4.9 2	6.5 4	4.1 3	1.3 2	2.0 3
truss3	12		4.7 1	1.1 3	6.2 1	3.1 1	1.0 2
truss4	11		2.8 1	6.8 2	6.1 1	1.9 1	6.9 1
truss5	17		1.3 3	1.8 5	4.4 3	3.3 2	4.3 3
truss6	27		2.7 3	1.6 6	1.4 5	4.5 2	2.3 4
truss7	26		1.8 3	1.1 6	1.4 5	3.0 2	1.7 4
truss8	16		2.5 3	3.3 5	4.4 3	6.3 2	6.9 3

Table 6: Condition Measure $C(d)$ Computation for the SDPLIB Suite

			$\|d\|$		$\log C(d)$	
			Lower	Upper	Lower	Upper
Problem	$\rho_P(d)$	$\rho_D(d)$	Bound	Bound	Bound	Bound
arch0	9.9 -1	1.0 0	3.3 4	3.1 5	3.3 4	3.1 5
arch2	9.9 -1	1.0 0	3.4 4	3.1 5	3.4 4	3.1 5
arch4	9.8 -1	1.0 0	3.6 4	3.1 5	3.7 4	3.1 5
arch8	9.5 -1	1.0 0	3.9 4	3.1 5	4.1 4	3.3 5
control1	3.9 -1	5.3 -2	4.4 4	9.7 4	8.3 5	1.8 6
control2	2.5 -2	1.0 -1	9.9 4	3.2 5	3.9 6	1.3 7
control3	2.9 -1	6.8 -2	1.4 5	8.1 5	2.0 6	1.2 7
control4	1.7 -1	4.6 -2	1.8 5	1.4 6	3.9 6	3.1 7
control5	1.3 -1	5.6 -2	2.1 5	2.2 6	3.8 6	3.9 7
control6	1.9 -1	2.6 -2	2.9 5	3.4 6	1.1 7	1.3 8
control7	3.2 -1	4.6 -2	3.6 5	4.5 6	7.9 6	9.8 7
control8	2.8 -1	4.7 -2	3.7 5	6.1 6	7.8 6	1.3 8
control9	9.5 -2	6.3 -2	4.2 5	7.8 6	6.7 6	1.2 8
control10	1.5 -1	2.5 -2	4.5 5	9.1 6	1.8 7	3.6 8
control11		3.0 -2	5.5 5	1.1 7		
equalG11	1.3 -3	1.0 -0	8.0 2	1.6 3	6.4 5	1.3 6
equalG51		9.6 -1	1.0 3	2.0 3		
gpp100	0.0 0	1.0 0	1.0 2	2.0 2	∞	∞
gpp124-1	0.0 0	1.0 0	1.2 2	2.5 2	∞	∞
gpp124-2	0.0 0	1.0 0	1.2 2	2.5 2	∞	∞
gpp124-3	0.0 0	1.0 0	1.2 2	2.5 2	∞	∞
gpp124-4	0.0 0	2.5 0	1.2 2	2.5 2	∞	∞
gpp250-1	0.0 0	1.0 0	2.5 2	5.0 2	∞	∞
gpp250-2	0.0 0	1.0 0	2.5 2	5.0 2	∞	∞
gpp250-3	0.0 0	1.0 0	2.5 2	5.0 2	∞	∞
gpp250-4	0.0 0	2.1 0	2.5 2	5.0 2	∞	∞
gpp500-1	0.0 0	1.0 0	5.0 2	1.0 3	∞	∞
gpp500-2	0.0 0	1.0 0	5.0 2	1.0 3	∞	∞
gpp500-3	0.0 0	1.0 0	5.0 2	1.0 3	∞	∞

Problem	$\rho_P(d)$	$\rho_D(d)$	$\|d\|$ Lower Bound	$\|d\|$ Upper Bound	$C(d)$ Lower Bound	$C(d)$ Upper Bound
gpp500-4	0.0 0	1.6 0	5.0 2	1.0 3	∞	∞
hinf1	0.0 0	8.3 -2	2.4 0	5.1 0	∞	∞
hinf2	1.0 -5	1.1 -3	3.5 0	5.6 0	3.5 5	5.6 5
hinf3	0.0 0	4.5 -4	2.2 1	3.4 1	∞	∞
hinf4	0.0 0	7.7 -3	6.4 1	1.0 2	∞	∞
hinf5	0.0 0	5.0 -5	1.2 2	1.8 2	∞	∞
hinf6	0.0 0	9.0 -5	3.3 1	5.8 1	∞	∞
hinf7	0.0 0	1.0 -5	1.7 2	2.7 2	∞	∞
hinf8	0.0 0	3.7 -4	6.4 1	1.3 2	∞	∞
hinf9	1.2 -2	4.7 -6	9.3 1	1.7 2	2.0 7	3.6 7
hinf10	0.0 0	1.2 -2	1.9 2	3.3 2	∞	∞
hinf11	0.0 0	2.8 -2	3.4 2	6.0 2	∞	∞
hinf12	0.0 0	9.1 -3	1.1 2	2.3 2	∞	∞
hinf13	0.0 0	8.0 -5	2.8 1	7.4 1	∞	∞
hinf14	0.0 0	1.6 -3	2.1 1	7.1 1	∞	∞
hinf15	0.0 0	4.0 -5	3.8 1	1.2 2	∞	∞
maxG11	1.3 -3	1.0 0	8.0 2	8.0 2	6.4 5	6.4 5
maxG32		1.0 0	2.0 3	2.0 3		
maxG51	1.0 -3	1.0 0	1.0 3	1.0 3	1.0 6	1.0 6
mcp100	1.0 -2	1.0 0	1.0 2	1.0 2	1.0 4	1.0 4
mcp124-1	8.1 -3	1.0 0	1.2 2	1.2 2	1.5 4	1.5 4
mcp124-2	8.1 -3	1.0 0	1.2 2	1.2 2	1.5 4	1.5 4
mcp124-3	8.1 -3	1.0 0	1.2 2	1.2 2	1.5 4	1.5 4
mcp124-4	8.1 -3	1.0 0	1.2 2	1.2 2	1.5 4	1.5 4
mcp250-1	4.0 -3	1.0 0	2.5 2	2.5 2	6.2 4	6.2 4
mcp250-2	4.0 -3	1.0 0	2.5 2	2.5 2	6.2 4	6.2 4
mcp250-3	4.0 -3	1.0 0	2.5 2	2.5 2	6.2 4	6.2 4
mcp250-4	4.0 -3	1.0 0	2.5 2	2.5 2	6.2 4	6.2 4
mcp500-1	2.0 -3	1.0 0	5.0 2	5.0 2	2.5 5	2.5 5
mcp500-2	2.0 -3	1.0 0	5.0 2	5.0 2	2.5 5	2.5 5
mcp500-3	2.0 -3	1.0 0	5.0 2	5.0 2	2.5 5	2.5 5
mcp500-4	2.0 -3	1.0 0	5.0 2	5.0 2	2.5 5	2.5 5
qap5	0.0 0	1.0 0	4.3 2	4.3 2	∞	∞
qap6	0.0 0	1.0 0	5.4 2	5.4 2	∞	∞
qap7	0.0 0	1.0 0	6.1 2	6.1 2	∞	∞
qap8	0.0 0	1.0 0	1.0 3	1.0 3	∞	∞
qap9	0.0 0	1.0 0	1.7 3	1.7 3	∞	∞
qap10	0.0 0	1.0 0	1.6 3	1.6 3	∞	∞
qpG11	1.3 -3	1.0 0	8.0 2	8.0 2	6.4 5	6.4 5
qpG51	1.0 -3	1.0 0	1.0 3	1.0 3	1.0 6	1.0 6
ss30	1.9 0	1.0 0	1.7 3	1.9 4	1.7 3	1.9 4
theta1	2.5 -1	1.0 0	5.0 1	5.2 1	2.0 2	2.1 2

Problem	$\rho_P(d)$	$\rho_D(d)$	$\|d\|$ Lower Bound	$\|d\|$ Upper Bound	$C(d)$ Lower Bound	$C(d)$ Upper Bound
theta2	2.5 -1	1.0 0	1.0 2	2.2 2	4.0 2	8.9 2
theta3	2.5 -1	1.0 0	1.5 2	4.1 2	6.0 2	1.6 3
theta4	2.5 -1	1.0 0	2.0 2	6.2 2	8.0 2	2.5 3
theta5	2.5 -1	1.0 0	2.5 2	8.7 2	1.0 3	3.5 3
theta6		1.0 0	3.0 2	1.1 3		
thetaG11		1.0 0	2.4 3	2.4 3		
truss1	1.3 -2	3.3 -1	3.0 0	4.0 0	2.2 2	3.0 2
truss2	5.1 -4	2.0 -1	8.5 0	1.3 1	1.7 4	2.6 4
truss3	5.4 -3	1.7 -1	4.0 0	1.0 1	7.4 2	1.9 3
truss4	9.0 -3	2.5 -1	3.2 0	6.9 0	3.6 2	7.7 2
truss5	1.9 -4	9.1 -2	1.1 1	3.3 1	5.9 4	1.8 5
truss6	9.0 -5	2.5 -1	3.2 0	6.4 0	3.6 4	7.1 4
truss7	1.4 -4	3.3 -1	3.0 0	3.2 0	2.1 4	2.3 4
truss8	1.0 -4	5.0 -2	1.9 1	5.1 1	1.9 5	5.1 5

Table 7: IPM Iterations, Non-Strict Complementarity Measure κ, Degeneracy Measure γ, and Solution Properties Obtained by SDPT3-aug on 85 Problems in the SDPLIB Suite

Problem	IPM Iterations	err	Solution Properties κ	γ	pd	dd
arch0	29	1.5 -10	-2.76	0.00	174/174	75/75
arch2	28	4.9 -11	-2.04	0.00	174/174	57/57
arch4	27	3.6 -11	-3.92	0.00	174/174	61/61
arch8	21	3.7 -8	-2.17	0.05	174/174	107/113
control1	22	5.6 -11	-2.88	0.24	21/21	16/21
control2	25	2.4 -11	-2.14	0.27	66/66	48/66
control3	26	8.0 -11	-1.71	0.35	136/136	88/136
control4	25	1.5 -10	-1.66			
control5	28	2.1 -10	-1.48			
control6	28	5.6 -10	-2.14			
control7	31	4.8 -10	-1.92			
control8	28	5.0 -10	-2.38			
control9	33	5.2 -10	-2.28			
control10	32	1.4 -9	-2.40			
control11	33	1.5 -9	-2.28			

Problem	IPM Iterations	err	Solution Properties κ	γ	pd	dd
equalG11	16	1.1 -6	-2.94	0.00	801/801	28/28
equalG51	17	9.0 -7	-3.69	0.00	1001/1001	105/105
gpp100	18	6.9 -7	-3.65	0.00	101/101	15/15
gpp124-1	15	1.4 -6	-3.79	0.00	125/125	10/10
gpp124-2	20	7.1 -8	-3.18	0.00	125/125	10/10
gpp124-3	17	2.8 -8	-4.17	0.00	125/125	21/21
gpp124-4	26	4.2 -7	-3.41	0.00	125/125	21/21
gpp250-1	17	3.2 -6	-3.42	0.00	251/251	15/15
gpp250-2	19	3.6 -6	-3.76	0.00	251/251	28/28
gpp250-3	16	6.8 -6	-3.77	0.00	251/251	36/36
gpp250-4	20	7.6 -10	-5.44	0.00	251/251	36/36
gpp500-1	24	2.8 -8	-3.15	0.00	501/501	21/21
gpp500-2	15	5.3 -6	-3.69	0.00	501/501	36/36
gpp500-3	21	2.5 -10	-3.29	0.00	501/501	55/55
gpp500-4	17	2.4 -8	-3.45	0.00	501/501	55/55
hinf1	29	1.8 -8	-8.62	0.50	12/13	5/10
hinf2	23	1.5 -7	-2.50	0.52	13/13	10/21
hinf3	31	9.1 -8	-0.69			
hinf4	27	9.8 -10	-4.61	0.47	12/13	8/15
hinf5	26	2.9 -6	-1.68			
hinf6	34	1.5 -9	-2.35	0.47	12/13	8/15
hinf7	23	1.7 -7	-2.61			
hinf8	37	1.5 -8	-2.85			
hinf9	21	3.3 -6	-2.04			
hinf10	41	5.4 -8	-8.98	0.43	12/21	2/ 3
hinf11	36	1.7 -7	-4.18	0.42	18/31	4/ 6
hinf12	55	1.3 -5	-1.68			
hinf13	41	1.4 -5	-1.77			
hinf14	31	4.7 -7	-2.85	0.47	39/73	11/15
hinf15	44	2.2 -5	-3.26	0.51	45/91	11/15
maxG11	19	1.5 -11	-3.05	0.00	800/800	21/21
maxG32	21	4.6 -11	-2.93	0.00	2000/2000	45/45
maxG51	19	9.6 -13	-5.35	0.00	1000/1000	105/105

Problem	IPM Iterations	err	Solution Properties κ	γ	pd	dd
mcp100	16	1.2 -13	-9.44	0.00	100/100	15/15
mcp124-1	17	6.9 -14	-7.98	0.84	124/124	22/136
mcp124-2	17	5.6 -14	-10.30	0.00	124/124	21/21
mcp124-3	16	2.7 -13	-8.69	0.00	124/124	21/21
mcp124-4	16	2.0 -13	-9.23	0.00	124/124	15/15
mcp250-1	18	1.7 -13	-9.21	0.89	250/250	35/325
mcp250-2	17	3.9 -13	-8.90	0.33	250/250	30/45
mcp250-3	17	1.0 -13	-8.80	0.00	250/250	36/36
mcp250-4	16	8.9 -13	-6.47	0.00	250/250	36/36
mcp500-1	19	7.5 -9	-4.12	0.96	500/500	75/1830
mcp500-2	19	1.3 -13	-5.41	0.64	500/500	43/120
mcp500-3	17	7.0 -13	-5.97	0.00	500/500	45/45
mcp500-4	17	3.7 -13	-8.36	0.00	500/500	66/66
qap5	14	4.9 -13	-11.89	0.81	26/136	1/ 1
qap6	31	2.9 -9	-1.75	0.00	229/229	78/78
qap7	24	1.1 -8	-3.40	0.00	358/358	105/105
qap8	22	3.8 -8	-3.77	0.00	529/529	136/136
qap9	22	5.7 -8	-3.67	0.00	748/748	210/210
qap10	24	2.2 -8	-3.94	0.00	1021/1021	325/325
qpG11	18	7.6 -11	-3.02	0.00	800/800	21/21
qpG51	29	5.4 -12	-12.42	0.00	1000/1000	1/ 1
ss30	22	2.3 -10	-3.55	0.00	132/132	7/ 7
theta1	16	5.9 -14	-10.27	0.00	104/104	28/28
theta2	16	5.1 -13	-8.27	0.00	498/498	136/136
theta3	17	5.8 -14	-9.36	0.00	1106/1106	300/300
theta4	17	1.6 -13	-7.84	0.00	1949/1949	528/528
theta5	17	2.0 -13	-7.42	0.00	3028/3028	861/861
theta6	17	9.0 -12	-2.03	0.00	4375/4375	1225/1225
thetaG11	23	7.9 -13	-5.35	0.33	1600/2401	3/ 3
truss1	12	1.2 -13	-14.79	0.33	4/ 6	3/ 4
truss2	16	1.9 -12	-3.92			
truss3	13	9.9 -9	-4.65	0.85	27/27	20/137
truss4	11	8.0 -9	-9.91	0.76	12/12	9/37

Problem	IPM Iterations	err	Solution Properties κ	γ	pd	dd
truss5	18	4.3 -10	-11.50	0.99	208/208	208/25426
truss6	16	8.1 -8	-2.22	0.99	172/172	126/13204
truss7	27	1.1 -12	-2.45			
truss8	19	6.8 -10	-11.70	1.00	496/496	496/136504