STRENGTHENED SEMIDEFINITE PROGRAMMING BOUNDS FOR CODES

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ABSTRACT. We give a hierarchy of semidefinite upper bounds for the maximum size $A(n,d)$ of a binary code of word length n and minimum distance at least d. At any fixed stage in the hierarchy, the bound can be computed (to an arbitrary precision) in time polynomial in n; this is based on a result of de Klerk, Pasechnik and Schrijver [2] about the regular $*$-representation for matrix $*$-algebras. The Delsarte bound for $A(n,d)$ is the first bound in the hierarchy, and the new bound of Schrijver [13] is located between the first and second bounds in the hierarchy. While computing the second bound involves a semidefinite program with $O(n^7)$ variables and thus seems out of reach for interesting values of n, Schrijver's bound can be computed via a semidefinite program of size $O(n^3)$, a result which uses the explicit block-diagonalization of the Terwilliger algebra. We propose two strengthenings of Schrijver's bound with the same computational complexity.

1. INTRODUCTION

We consider the problem of computing the parameter $A(n,d)$, defined as the maximum size of a binary code of word length n and minimum distance at least d. With $\mathcal{P}$ denoting the collection of all subsets of $\{1,\ldots,n\}$, we can identify code words in $\{0,1\}^n$ with their supports; so a code C is a subset of $\mathcal{P}$ and the Hamming distance of $I,J \in \mathcal{P}$ is equal to $|I\Delta J|$. The minimum distance of a code C is the minimum Hamming distance of distinct elements of C. If we define the graph $\mathcal{G}(n,d)$ with node set $\mathcal{P}$, two nodes $I,J \in \mathcal{P}$ being adjacent if $|I\Delta J| \in \{1,\ldots,d-1\}$, then a code with minimum distance d corresponds to a stable set in the graph $\mathcal{G}(n,d)$. Therefore, the parameter $A(n,d)$ is equal to the stability number of the graph $\mathcal{G}(n,d)$, i.e., the maximum cardinality of a stable set in $\mathcal{G}(n,d)$.

Schrijver [13] introduced recently an upper bound for $A(n,d)$ which refines the classical bound of Delsarte [3]. While Delsarte bound is based on diagonalizing the (commutative) Bose-Mesner algebra of the Hamming scheme and can be computed via linear programming, Schrijver's bound is based on block-diagonalizing the (non-commutative) Terwilliger algebra of the Hamming scheme and can be computed via semidefinite programming. In both cases the bounds can be formulated as the optimum of a (linear or semidefinite) program of size polynomial in n (size $O(n)$ for Delsarte bound and size $O(n^3)$ for Schrijver's bound).

Finding tight upper bounds for the stability number $\alpha(\mathcal{G})$ of a graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ has been the subject of extensive research. Lovász [9] introduced the theta number $\vartheta(\mathcal{G})$,

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which can be computed, e.g., via the semidefinite program:

$$
(1) \qquad \vartheta(\mathcal{G}) := \max \ \textstyle\sum_{i\in\mathcal{V}} X_{ii} \ \text{ s.t. } \quad \begin{array}{l} X = (X_{ij})_{i,j\in\mathcal{V}\cup\{0\}} \succeq 0, \ X_{00} = 1, \\ X_{0i} = X_{ii} \ (i \in \mathcal{V}), \ X_{ij} = 0 \ (ij \in \mathcal{E}). \end{array}
$$

The theta number can be computed (with arbitrary precision) in time polynomial in the number of nodes of the graph. Moreover, $\vartheta(\mathcal{G}) = \alpha(\mathcal{G})$ when $\mathcal{G}$ is a perfect graph (see [5]). Schrijver [12] introduced the strenghtening $\vartheta'(\mathcal{G})$ of $\vartheta(\mathcal{G})$ obtained by adding the nonnegativity constraint $X \geq 0$ to the program (1) and proved that $\vartheta'(\mathcal{G}(n,d))$ coincides with Delsarte bound.

Various methods have been proposed in the litterature for constructing tighter semidefinite upper bounds for the stability number of a graph, in particular, by Lovász and Schrijver [10] and more recently by Lasserre [6, 7]. In both cases a hierarchy of upper bounds for $\alpha(\mathcal{G})$ is obtained with the property that the bound reached at the $\alpha(\mathcal{G})$-th iteration coincides in fact with $\alpha(\mathcal{G})$. It turns out that Lasserre's hierarchy refines the hierarchy of Lovász and Schrijver (see [8]).

For $k \geq 1$, denote by $\ell^{(k)}(\mathcal{G})$ the bound in Lasserre's hierarchy at the k-th iteration; see Section 3.1 for the precise definition. It is known (and easy to see) that, for *fixed* k, one can compute (with arbitrary precision) the parameter $\ell^{(k)}(\mathcal{G})$ in time polynomial in the number of nodes of the graph $\mathcal{G}$. However, for the coding problem, the graph $\mathcal{G}(n,d)$ has 2^n nodes and such complexity is prohibitive for large n. A first contribution of this paper (see Section 3.2) is to show that, for fixed k, the bound $\ell^{(k)}(\mathcal{G}(n,d))$ can be computed (with arbitrary precision) in time polynomial in n. This result is based on a result of de Klerk, Pasechnik and Schrijver [2], recalled in Section 2.1, about reducing the size of invariant semidefinite programs using the regular $*$-representation for the algebra of invariant matrices under action of a group.

The first bound $\ell^{(1)}(\mathcal{G})$ in the hierarchy is equal to the theta number $\vartheta(\mathcal{G})$; its strengthening obtained by adding nonnegativity is equal to $\vartheta'(\mathcal{G})$ which, for the graph $\mathcal{G} = \mathcal{G}(n,d)$, coincides with the bound of Delsarte for the parameter $A(n,d)$. It turns out that the bound of Schrijver [13] for $A(n,d)$ lies between $\ell_+^{(1)}(\mathcal{G})$ and $\ell_+^{(2)}(\mathcal{G})$, the strengthenings of $\ell^{(1)}(\mathcal{G})$ and $\ell^{(2)}(\mathcal{G})$ obtained by adding certain bounds on the variables. While Schrijver's bound can be computed via a semidefinite program of size $O(n^3)$ and thus computed in practice for reasonable values of n, a practical computation of $\ell_+^{(2)}(\mathcal{G}(n,d))$ seems out of reach for interesting values of n since one would have to solve a semidefinite program with $O(n^7)$ variables.

In Section 3.3, we introduce two bounds $\ell_+(\mathcal{G}(n,d))$ and $\ell_{++}(\mathcal{G}(n,d))$ satisfying

$$
\ell_+^{(2)}(\mathcal{G}(n,d)) \leq \ell_{++}(\mathcal{G}(n,d)) \leq \ell_+(\mathcal{G}(n,d)) \leq \ell_+^{(1)}(\mathcal{G}(n,d));
$$

they are at least as good as Schrijver's bound, and their computation still relies on solving a semidefinite program of size $O(n^3)$. This complexity result follows from the fact that the new bounds, analogously to Schrijver's bound, require the positive semidefiniteness of certain matrices lying in the Terwilliger algebra (or a variation of it) whose dimension is $O(n^3)$ and for which the explicit block-diagonalization has been given by Schrijver [13].

Some notation. We group here some notation that will be used throughout the paper. We set $V := \{1, \ldots, n\}$ and $\mathcal{P} := \mathcal{P}(V)$ denotes the collection of all subsets of the set V. For a finite set $\mathcal{V}$ and an integer $k \geq 1$, we set $\mathcal{P}_k(\mathcal{V}) := \{I \subseteq \mathcal{V} \mid |I| \leq k\}$ and $\mathcal{P}_{=k}(\mathcal{V}) := \{I \subseteq \mathcal{V} \mid |I| = k\}$. We let $Sym(\mathcal{V})$ denote the set of all permutations of the set $\mathcal{V}$ and we set $Sym(n) := Sym(\mathcal{V})$ when $|\mathcal{V}| = n$. The letter $\mathcal{G}$ will be used to denote a graph, with node set $\mathcal{V}$ and edge set $\mathcal{E}$, while the letter G will be used to denote a group (e.g., of automorphisms of $\mathcal{G}$).

2. Algebraic preliminaries

2.1. **Preliminaries on invariant matrices.** Let G be a finite group acting on a finite set $\mathcal{X}$; that is, we have a homomorphism $h : G \to Sym(\mathcal{X})$, where $Sym(\mathcal{X})$ is the group of permutations of $\mathcal{X}$. For $\sigma \in G$, $h(\sigma)$ is a permutation of $\mathcal{X}$ and M_σ is the associated $\mathcal{X} \times \mathcal{X}$ permutation matrix with

$$(M_\sigma)_{x,y} = \begin{cases} 1 & \text{if } h(\sigma)(x) = y, \\ 0 & \text{otherwise.} \end{cases}$$

The set:

$$\mathcal{A} := \left\{ \sum_{\sigma \in G} \lambda_\sigma M_\sigma \mid \lambda_\sigma \in \mathbb{R} \ (\sigma \in G) \right\}$$

is a *matrix $*$-algebra*; that is, $\mathcal{A}$ is closed under addition, scalar and matrix multiplication, and conjugation.

Any $\sigma \in G$ acts on matrices indexed by the set $\mathcal{X}$. Namely, for a $\mathcal{X} \times \mathcal{X}$ matrix N and $\sigma \in G$, set

$$\sigma(N) := (N_{\sigma(x),\sigma(y)})_{x,y \in \mathcal{X}}.$$

The matrix N is said to be *invariant under the action of* G if $\sigma(N) = N$ for all $\sigma \in G$. Then the commutant algebra $\mathcal{A}^G$ of the algebra $\mathcal{A}$, defined by

$$\mathcal{A}^G := \{N \in \mathbb{C}^{\mathcal{X} \times \mathcal{X}} \mid NM = MN \ \forall M \in \mathcal{A}\},$$

consists precisely of the $\mathcal{X} \times \mathcal{X}$ matrices N that are invariant under the action of G; $\mathcal{A}^G$ is again a matrix $*$-algebra.

The *orbit* of $(x, y) \in \mathcal{X} \times \mathcal{X}$ under action of G is the set $\{(\sigma(x), \sigma(y)) \mid \sigma \in G\}$. Let $\mathcal{O}_1, \ldots, \mathcal{O}_N$ denote the orbits of the set $\mathcal{X} \times \mathcal{X}$ under the action of the group G and, for $i = 1, \ldots, N$, let $\tilde{D}_i$ be the $\mathcal{X} \times \mathcal{X}$ matrix:

$$(\tilde{D}_i)_{x,y} = \begin{cases} 1 & \text{if } (x, y) \in \mathcal{O}_i \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Then, $\tilde{D}_1, \ldots, \tilde{D}_N$ form a basis of the commutant $\mathcal{A}^G$ (as vector space) and $\tilde{D}_1 + \ldots + \tilde{D}_N = J$ (the all-ones matrix). We normalize the $\tilde{D}_i$ to

$$D_i := \frac{\tilde{D}_i}{\sqrt{\langle \tilde{D}_i, \tilde{D}_i \rangle}} \tag{3}$$

for $i = 1, \ldots, N$. (For two $N \times N$ matrices A, B, $\langle A, B\rangle := Tr(A^T B) = \sum_{i,j=1}^N A_{ij}B_{ij}$.) Then, $\langle D_i, D_j\rangle = 1$ if $i = j$ and 0 otherwise. The *multiplication parameters* $\gamma_{i,j}^k$ are defined by

$$D_i D_j = \sum_{k=1}^N \gamma_{i,j}^k D_k \tag{4}$$

for all $i, j = 1, \ldots, N$. Define the $N \times N$ matrices $L_1, \ldots, L_N$ by

$$(L_k)_{i,j} := \gamma_{k,j}^i \quad \text{for } k, i, j = 1, \ldots, N. \tag{5}$$

De Klerk, Pasechnik and Schrijver [2] show:

THEOREM 1. *The mapping $D_k \mapsto L_k$ is a $*$-isomorphism, known as the* regular $*$-representation *of $\mathcal{A}^G$. In particular, given real scalars $x_1, \ldots, x_N$,*

$$\sum_{i=1}^N x_i D_i \succeq 0 \Longleftrightarrow \sum_{i=1}^N x_i L_i \succeq 0. \tag{6}$$

This result has important algorithmic applications, as it permits to give more compact formulations for invariant semidefinite programs. Consider a semidefinite program:

$$\min \langle C, Y\rangle \ \text{ s.t. } \langle A_\ell, Y\rangle \le b_\ell \ (\ell = 1, \ldots, m), \ Y \succeq 0 \tag{7}$$

in the $\mathcal{X} \times \mathcal{X}$ matrix variable Y. Assume that the program (7) is *invariant under action of the group* G; that is, C is invariant under action of G and, for every matrix Y feasible for (7) and $\sigma \in G$, the matrix $\sigma(Y)$ is again feasible for Y. (This holds, e.g., if the class of constraints is invariant under action of G, i.e., if for each $\ell \in \{1, \ldots, m\}$ and $\sigma \in G$, there exists $\ell' \in \{1, \ldots, m\}$ such that $\sigma(A_\ell) = A_{\ell'}$ and $b_\ell = b_{\ell'}$.) Then, if Y is feasible for (7) then the matrix $Y_0 := \frac{1}{|G|}\sum_{\sigma\in G} \sigma(Y)$ too is feasible for (7), with the same objective value as Y. Therefore, in (7), one can assume without loss of generality that Y is invariant under action of G; that is, Y is of the form $Y = \sum_{i=1}^N x_i D_i$ with $x_1, \ldots, x_N \in \mathbb{R}$. Then the objective function reads $\langle C, Y\rangle = \sum_{i=1}^N c_i x_i$, after setting $C = \sum_{i=1}^N c_i D_i$, and the constraints in (7) become linear constraints in x. As a direct application of Theorem 1, we find:

COROLLARY 2. *Consider the program (7) in the $\mathcal{X} \times \mathcal{X}$ matrix variable Y. If (7) is invariant under the action of the group G, then it can be equivalently reformulated as*

$$\min \sum_{i=1}^N c_i x_i \quad s.t. \ a_\ell^T x \le b_\ell \ (\ell = 1, \ldots, m), \ \sum_{i=1}^N x_i L_i \succeq 0. \tag{8}$$

The program (8) involves $N \times N$ matrices and N variables. Here, N is the dimension of the algebra $\mathcal{A}^G$ (the set of $\mathcal{X} \times \mathcal{X}$ invariant matrices under the action of the group G), typically much smaller than $|\mathcal{X}|$.

To use computationally this result, one needs to know explicitly the matrices $L_1, \ldots, L_N$, which involves computing the cardinality of the orbits of $\mathcal{X} \times \mathcal{X}$ and the multiplication parameters $\gamma_{i,j}^k$ in (4). De Klerk, Pasechnik and Schrijver [2] apply this technique for computing tighter bounds for the crossing number of a complete bipartite graph. We apply it in Section 3.2 for reducing the size of the semidefinite programs permitting to compute the hierarchy of semidefinite bounds for the parameter $A(n,d)$.

EXAMPLE 3. Let $\mathcal{X} := \mathcal{P}$, the collection of all subsets of the set $V = \{1,\dots,n\}$, and $G := Sym(V)$, the group of permutations of V. Each $\pi \in G$ induces a permutation of $\mathcal{X}$, again denoted by π, by letting $\pi(I) := \{\pi(i) \mid i \in I\}$ for $I \in \mathcal{P}$. Two pairs $(I,J),(I',J')$ $(I,J,I',J' \in \mathcal{P})$ lie in the same orbit [i.e., $I' = \pi(I)$, $J' = \pi(J)$ for some $\pi \in G$] if and only if $|I| = |I'|$, $|J| = |J'|$ and $|I \cap J| = |I' \cap J'|$. Therefore, the commutant algebra $\mathcal{A}^G$ is generated by the matrices $M_{i,j}^t$ $(i,j,t \in \mathbb{Z}_+)$, where

$$(M_{i,j}^t)_{I,J} := \begin{cases} 1 & \text{if } |I| = i, |J| = j, |I \cap J| = t, \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

for $I,J \in \mathcal{P}$; $\mathcal{A}^G =: \mathcal{A}_n$ is known as the *Terwilliger algebra* of the Hamming scheme (Terwilliger [15]).

EXAMPLE 4. Consider again the set $\mathcal{X} := \mathcal{P}$, but now the group is $G := Aut(\mathcal{P})$, the automorphism group of $\mathcal{P}$. The group G consists of the permutations $\sigma \in Sym(\mathcal{P})$ preserving the symmetric difference, i.e., for which $|\sigma(I)\Delta\sigma(J)| = |I\Delta J|$ for all $I,J \in \mathcal{P}$. Thus,

$$G = \{\pi s_A \mid A \subseteq V, \pi \in Sym(V)\} \tag{10}$$

where, for a set $A \subseteq V$, s_A is the permutation of $\mathcal{P}$ mapping any $I \in \mathcal{P}$ to $s_A(I) := A\Delta I$; we have $|G| = 2^n n!$. Two pairs $(I,J),(I',J')$ $(I,J,I',J' \in \mathcal{P})$ lie in the same orbit [i.e., $I' = \sigma(I)$, $J' = \sigma(J)$ for some $\sigma \in G$] if and only if $|I\Delta J| = |I'\Delta J'|$. Therefore, the algebra $\mathcal{A}^G$ is generated by the matrices M_k $(k = 0,1,\dots,n)$ where

$$(M_k)_{I,J} := \begin{cases} 1 & \text{if } |I\Delta J| = k, \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

for $I,J \in \mathcal{P}$; $\mathcal{A}^G =: \mathcal{B}_n$ is known as the *Bose Mesner algebra* of the Hamming scheme. The Bose-Mesner algebra is a subalgebra of the Terwilliger algebra, as $M_k = \sum_{i,j=0}^n M_{i,j}^{(i+j-k)/2}$ for $k = 0,1,\dots,n$.

In fact, it is known from invariant theory and $C*$-algebra theory that the algebra $\mathcal{A}^G$ can be block-diagonalized. Therefore, there exists a semidefinite program equivalent to the invariant program (7), where the matrix Y is replaced by a block-diagonal matrix with possibly repeated blocks; see, e.g., Gaterman and Parrilo [4]. Such program is typically more compact than the program (8). However, finding explicitly the block-diagonalization is a nontrivial task in general. An advantage of the above mentioned reduction method, based on the regular $*$-representation, is that it involves the matrices L_i which are explicitly defined in terms of the matrices D_i generating the algebra. Nevertheless, Schrijver [13] was able to determine explicitly the block-diagonalization for the Terwilliger algebra; we recall this result in the next section as we will need it for the computation of our stronger bounds for the coding problem.

2.2. **Block-diagonalization of the Terwilliger algebra.** While the Bose-Mesner algebra $\mathcal{B}_n$ is a commutative algebra and thus can be diagonalized (see [3]), the Terwilliger algebra $\mathcal{A}_n$ is a non-commutative algebra. Its dimension is $\dim \mathcal{A}_n = \binom{n+3}{3}$, which is the number of triples (i,j,t) for which $M_{i,j}^t \neq 0$. As $\mathcal{A}_n$ is a matrix $*$-algebra containing the identity, it can be block-diagonalized, which means the following: There exists a unitary $\mathcal{P}\times\mathcal{P}$ complex matrix U (i.e., $U^*U=I$) and positive integers m and $p_0,q_0,\dots,p_m,q_m$ such that the set $U^*\mathcal{A}_nU := \{U^*MU \mid M\in\mathcal{A}_n\}$ is equal to the collection of block-diagonal matrices

$$\begin{pmatrix} C_0 & 0 & \dots & 0 \\ 0 & C_1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & C_m \end{pmatrix}$$

where each C_k ($k=0,1,\dots,m$) is a block-diagonal matrix with q_k identical blocks B_k of order p_k:

$$C_k = \begin{pmatrix} B_k & 0 & \dots & 0 \\ 0 & B_k & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & B_k \end{pmatrix};$$

thus $2^n=\sum_{k=0}^m p_kq_k$ and $\sum_{k=0}^m p_k^2 = \dim\mathcal{A}_n = \binom{n+3}{3}$. By deleting copies of identical blocks, it follows that $\mathcal{A}_n$ is isomorphic to the algebra

$$\bigoplus_{k=0}^{m}\mathbb{C}^{p_k\times p_k} = \left\{ \begin{pmatrix} B_0 & 0 & \dots & 0 \\ 0 & B_1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & B_m \end{pmatrix} \mid B_k\in\mathbb{C}^{p_k\times p_k} \text{ for } k=0,1,\dots,m \right\}. \tag{12}$$

An important fact for our purpose is that this isomorphism preserves positive semidefiniteness. The existence of a unitary matrix U with the above properties is standard $C*$-algebra theory (see, e.g., [14]). Schrijver [13] has constructed explicitly this matrix U and the image of a matrix $M\in\mathcal{A}_n$ in the algebra (12). We recall some facts from [13] needed for our treatment; we refer to [13] for details and proofs.

It turns out that U is real valued, $m=\lfloor\frac{n}{2}\rfloor$ and, for $k=0,1,\dots,\lfloor\frac{n}{2}\rfloor$, the block B_k has order $p_k=n-2k+1$ and multiplicity $q_k=\binom{n}{k}-\binom{n}{k-1}$. In particular, the block B_0 has order $n+1$ and multilplicity 1. We now describe explicitly the matrix U. For this, for $k=1,\dots,\lfloor\frac{n}{2}\rfloor$, define

$$\mathcal{L}_k := \{b\in\mathbb{R}^{\mathcal{P}} \mid M_{k-1,k}^{k-1}b=0 \text{ and } b_I=0 \text{ if } |I|\neq k\}.$$

Let $\mathcal{B}_k$ be a basis of $\mathcal{L}_k$. Then $|\mathcal{B}_k|=\binom{n}{k}-\binom{n}{k-1}$ and $\sum_{I\in\mathcal{P}}b_I=0$ for $b\in\mathcal{L}_k$. Set $\mathcal{B}_0:=\{b_0\}$ where $b_0:=(1,0,\dots,0)^T\in\mathbb{R}^{\mathcal{P}}$ (the nonzero entry being indexed by $\emptyset\in\mathcal{P}$) and define

$$\mathcal{Q} := \{(k,b,i) \mid k\in\{0,\dots,\lfloor\frac{n}{2}\rfloor\}, b\in\mathcal{B}_k, i\in\{k,k+1,\dots,n-k\}\}.$$

Then $|\mathcal{Q}|=2^n=|\mathcal{P}|$. For $(k,i,b)\in\mathcal{Q}$, define the vector

$$u_{k,i,b} := \binom{n-2k}{i-k}^{-\frac{1}{2}} M_{i,k}^{k}b\in\mathbb{R}^{\mathcal{P}}.$$

Finally define U as the $\mathcal{P} \times \mathcal{Q}$ matrix whose columns are the vectors $u_{k,i,b}$ for $(k,i,b) \in \mathcal{Q}$. The following is shown in [13].

PROPOSITION 5. [13] *The matrix U is orthogonal, i.e., $U^T U = I$. Moreover, for a matrix $M = \sum_{i,j,t=0}^n x_{i,j}^t M_{i,j}^t \in \mathcal{A}_n$ (with $x_{i,j}^t \in \mathbb{R}$), the matrix $U^T M U$ is a block-diagonal matrix determined by the partition of $\mathcal{Q}$ into the classes $\mathcal{Q}_{k,b} := \{(k,i,b) \mid k \le i \le n-k\}$ (for $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, $b \in \mathcal{B}_k$). For a given integer $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, the blocks corresponding to the classes $\mathcal{Q}_{k,b}$ (for $b \in \mathcal{B}_k$) are all identical to the following matrix:*

$$B_k(x) := \left(\sum_t \binom{n-2k}{i-k}^{-\frac{1}{2}} \binom{n-2k}{j-k}^{-\frac{1}{2}} \beta_{i,j,k}^t x_{i,j}^t \right)_{i,j=k}^{n-k}, \tag{13}$$

after setting

$$\beta_{i,j,k}^t := \sum_{u=0}^n (-1)^{t-u} \binom{u}{t} \binom{n-2k}{n-k-u} \binom{n-k-u}{i-u} \binom{n-k-u}{j-u} \tag{14}$$

for $i,j,k,t \in \{0, \ldots, n\}$. As $\mathcal{A}_n$ is isomorphic to the algebra (12), we have:

$$\sum_{i,j,t=0}^n x_{i,j}^t M_{i,j}^t \succeq 0 \Longleftrightarrow B_k(x) \succeq 0 \ \text{ for } k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor. \tag{15}$$

The above property (15) is the key tool used in [13] and in the present paper, which allows reducing semidefinite programs involving matrices in the Terwilliger algebra to semidefinite programs of size $O(n^3)$.

We will deal in this paper with matrices of the form

$$\tilde{M} = \begin{pmatrix} d & c^T \\ c & M \end{pmatrix}, \ \text{ where } M = \sum_{i,j,t=0}^n x_{i,j}^t M_{i,j}^t, \ d \in \mathbb{R}, \ c = \sum_{i=0}^n c_i \chi^{\mathcal{P}_{=i}(V)}. \tag{16}$$

Recall that $\mathcal{P}_{=i}(V) = \{I \subseteq V \mid |I| = i\}$ and $\chi^{\mathcal{P}_{=i}(V)} \in \{0,1\}^{\mathcal{P}}$ whose I-th entry is 1 if and only if $I \in \mathcal{P}_{=i}(V)$.

LEMMA 6. *The matrix $\tilde{M}$ from (16) is positive semidefinite if and only if $B_k(x) \succeq 0$ for $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, and*

$$\tilde{B}_0(x) := \begin{pmatrix} d & \tilde{c}^T \\ \tilde{c} & B_0(x) \end{pmatrix} \succeq 0, \ \textit{where } \tilde{c} := (c_i \binom{n}{i}^{\frac{1}{2}})_{i=0}^n.$$

Proof. Setting

$$\tilde{U} := \begin{pmatrix} 1 & 0 \\ 0 & U^T \end{pmatrix},$$

we have:

$$\tilde{U}^T \tilde{M} \tilde{U} = \begin{pmatrix} d & c^T U \\ U^T c & U^T M U \end{pmatrix}.$$

It suffices now to verify that $(c^T U)_{k,i,b} = c^T u_{k,i,b} = 0$ for $(k,i,b) \in \mathcal{Q}$ with $k \ge 1$, and that $(c^T U)_{0,i,b_0} = c_i \binom{n}{i}^{\frac{1}{2}}$ for $i = 0, \ldots, n$. This is direct verification using the above definitions;

details are omitted. Hence, $\tilde{U}^T\tilde{M}\tilde{U}$ is block-diagonal, with blocks $\tilde{B}_0(x)$ (with multiplicity 1) and $B_k(x)$ (with multiplicity q_k) for $k=1,\ldots,\lfloor\frac{n}{2}\rfloor$. The lemma now follows. ∎

3. Semidefinite programming bounds for the stability number of a graph

3.1. **Lasserre's construction.** Let $\mathcal{G}=(\mathcal{V},\mathcal{E})$ be a graph. A *stable set* in $\mathcal{G}$ is a set $S\subseteq\mathcal{V}$ containing no edge and the *stability number* $\alpha(\mathcal{G})$ of $\mathcal{G}$ is the maximum cardinality of a stable set in $\mathcal{G}$. Recall $\mathcal{P}_k(\mathcal{V})=\{I\subseteq\mathcal{V}\mid |I|\le k\}$ for an integer k. Given a stable set S in $\mathcal{G}$, define $x=(x_I)_{I\in\mathcal{P}_k(\mathcal{V})}\in\{0,1\}^{\mathcal{P}_k(\mathcal{V})}$ and $y=(y_I)_{I\in\mathcal{P}_{2k}(\mathcal{V})}\in\{0,1\}^{\mathcal{P}_{2k}(\mathcal{V})}$ with $x_I=1$ (resp., $y_I=1$) if and only if $I\subseteq S$, for $I\in\mathcal{P}_k(\mathcal{V})$ (resp., for $I\in\mathcal{P}_{2k}(\mathcal{V})$). Then y and the matrix $Y:=xx^T$ satisfy:

$$Y\succeq 0 \tag{17}$$

$$Y_{I,J}=y_{I\cup J}\quad(\text{for } I,J\in\mathcal{P}_k(\mathcal{V})) \tag{18}$$

$$Y_{I,J}=y_{I\cup J}=0\quad\text{if } I\cup J \text{ contains an edge}\quad(\text{for } I,J\in\mathcal{P}_k(\mathcal{V})) \tag{19}$$

$$Y_{\emptyset,\emptyset}=y_\emptyset=1 \tag{20}$$

$$0\le y_I\le y_J\quad\text{if } J\subseteq I\ (\text{for } I,J\in\mathcal{P}_{2k}(\mathcal{V})). \tag{21}$$

We refer to (19) as the *edge condition* and to (18) as the *moment condition.* A matrix Y satisfying (18) is known as a moment matrix and is denoted as $Y=M_k(y)$ (see [6, 7, 8]). Under the assumption (17), the edge condition (19) is, in fact, equivalent to $y_{ij}=0$ (for $ij\in\mathcal{E}$). (Here and below, we set $y_{ij}:=y_{\{i\},\{j\}}$, $y_i:=y_{\{i\}}$, etc.) Under (17), relation (21) holds for $I\in\mathcal{P}_k(\mathcal{V})$; indeed, the principal submatrix of $M_k(y)$ indexed by $\{I,J\}$ has the form $\begin{pmatrix} y_I & y_I\\ y_I & y_J\end{pmatrix}$, whose positive semidefiniteness implies $0\le y_J\le y_I$. On the other hand, $M_1(y)\succeq 0$ implies $|y_{ij}|\le\max(y_i,y_j)$; indeed the principal submatrix of $M_1(y)$ indexed by $\{\{i\},\{j\}\}$ has the form $\begin{pmatrix} y_i & y_{ij}\\ y_{ij} & y_j\end{pmatrix}$, whose positive semidefiniteness implies $y_{ij}^2\le y_iy_j\le\max(y_i^2,y_j^2)$. Similarly, $M_2(y)\succeq 0$ implies that $|y_{ijk}|$ is at most the largest two values among y_{ij},y_{ik},y_{jk}; indeed the principal submatrix of $M_2(y)$ indexed by $\{\{i,j\},\{i,k\},\{j,k\}\}$ has the form $\begin{pmatrix} y_{ij} & y_{ijk} & y_{ijk}\\ y_{ijk} & y_{ik} & y_{ijk}\\ y_{ijk} & y_{ijk} & y_{jk}\end{pmatrix}$, whose positive semidefiniteness implies $y_{ijk}^2\le\min(y_{ij}y_{ik},y_{ij}y_{jk},y_{ik}y_{jk})\le y_{ik}^2,y_{jk}^2$ assuming, say, that $y_{ij}\le y_{ik}\le y_{jk}$.

Consider the semidefinite program:

$$\ell^{(k)}(\mathcal{G}):=\max\sum_{i\in\mathcal{V}}y_i\quad\text{s.t. } M_k(y)\succeq 0,\ y_\emptyset=1,\ y_{ij}=0\ (ij\in\mathcal{E}). \tag{22}$$

Then, $\alpha(\mathcal{G})\le\ell^{(k)}(\mathcal{G})$, with equality if $k\ge\alpha(\mathcal{G})$ ([7, 8]). Define $\ell_+^{(k)}(\mathcal{G})$ as the parameter obtained by adding to (22) the constraints (21); thus,

$$\alpha(\mathcal{G})\le\ell_+^{(k)}(\mathcal{G})\le\ell^{(k)}(\mathcal{G}).$$

For $k=1$, $\ell^{(1)}(\mathcal{G})=\vartheta(\mathcal{G})$, the Lovász' theta number, and the stronger bound obtained by adding nonnegativity to (22) is $\vartheta'(\mathcal{G})$, the strengthening of $\vartheta(\mathcal{G})$ introduced by McEliece, Rodemich and Rumsey [11] and Schrijver [12]. The bound $\ell^{(2)}(\mathcal{G})$ is at least as good as the parameter obtained by optimizing over $N_+(\mathrm{TH}(\mathcal{G}))$, the convex relaxation of the stable set polytope of $\mathcal{G}$ obtained by applying the Lovász-Schrijver N_+-operator to the theta body $\mathrm{TH}(\mathcal{G})$ ([8]; or see (26)). For $k=2$, the program (22) has size $O(|\mathcal{V}|^4)$. We now formulate a bound $\ell(\mathcal{G})$, which is weaker than $\ell^{(2)}(\mathcal{G})$, but still at least as good as the bound obtained from $N_+(\mathrm{TH}(\mathcal{G}))$, although its computation is more economical since it can be expressed via a semidefinite program of size $O(|\mathcal{V}|^3)$.

Namely, for each $r\in\mathcal{V}$, consider the principal submatrix $Y_r(y)$ of $M_2(y)$ indexed by the set $\mathcal{P}_2(\mathcal{V};r):=\mathcal{P}_1(\mathcal{V})\cup\{\{r,i\}\mid i\in\mathcal{V}\}$; thus the matrices $Y_r(y)$ involve only variables y_I for $I\in\mathcal{P}_3(\mathcal{V})$. Define

$$\ell(\mathcal{G}):=\max\sum_{i\in\mathcal{V}}y_i\ \text{ s.t. } y_\emptyset=1,\ y_{ij}=0\ (ij\in\mathcal{E}),\ Y_r(y)\succeq 0\ (r\in\mathcal{V}) \tag{23}$$

and $\ell_+(\mathcal{G})$ as the parameter obtained by adding to (23) the constraints: $0\le y_{ijk}\le y_{ij}$ for distinct $i,j,k\in\mathcal{V}$ (coming from (21)). Obviously,

$$\ell^{(2)}(\mathcal{G})\le\ell(\mathcal{G})\le\ell^{(1)}(\mathcal{G});$$

analogously for the ℓ_+ parameters. We will see in Section 3.3 that, for the graph $\mathcal{G}=\mathcal{G}(n,d)$, the matrices involved in (23) lie in (a variation of) the Terwilliger algebra, which allows reformulating the parameters $\ell(\mathcal{G}(n,d))$, $\ell_+(\mathcal{G}(n,d))$ via semidefinite programs of size $O(n^3)$.

From the moment condition (18), the matrix $Y_r(y)$ has the block structure:

$$Y_r(y)=\begin{pmatrix}1 & a^T & b_r^T\\ a & A & B_r\\ b_r & B_r & B_r\end{pmatrix}, \tag{24}$$

where $A:=(y_{ij})_{i,j\in\mathcal{V}}$, $B_r:=(y_{\{i,j,r\}})_{i,j\in\mathcal{V}}$ are symmetric $\mathcal{V}\times\mathcal{V}$ matrices, and $a:=(y_i)_{i\in\mathcal{V}}$, $b_r:=(y_{ir})_{i\in\mathcal{V}}$. As b_r coincides with the r-th column of A and of B_r, by applying some column/row manipulation to $Y_r(y)$, one deduces that

$$Y_r(y)\succeq 0\Longleftrightarrow B_r\succeq 0 \text{ and } \tilde{C}_r:=\begin{pmatrix}1-y_r & a^T-b_r^T\\ a-b_r & A-B_r\end{pmatrix}\succeq 0, \tag{25}$$

which permits to reduce the size of the matrices involved in program (23). Setting

$$\mathrm{TH}(\mathcal{G})=\{x\in\mathbb{R}^{\mathcal{P}_1(\mathcal{V})}\mid\exists y\in\mathbb{R}^{\mathcal{P}_2(\mathcal{V})}\text{ s.t. } M_1(y)\succeq 0,\ y_{ij}=0\,(ij\in\mathcal{E}),\ x_I=y_I\ (I\in\mathcal{P}_1(\mathcal{V}))\},$$

$$\begin{aligned}N_+(\mathrm{TH}(\mathcal{G}))=\{x\in\mathbb{R}^{\mathcal{V}}\mid\exists y\in\mathbb{R}^{\mathcal{P}_2(\mathcal{V})}\text{ s.t. }\ & M_1(y)\succeq 0,\ y_\emptyset=1,\ x_i=y_i\ (i\in\mathcal{V}),\\ & (y_{I\cup\{r\}})_{I\in\mathcal{P}_1(\mathcal{V})},(y_I-y_{I\cup\{r\}})_{I\in\mathcal{P}_1(\mathcal{V})}\in\mathrm{TH}(\mathcal{G})\}\end{aligned}$$

one can verify that

$$\ell(\mathcal{G})\le\max_{x\in N_+(\mathrm{TH}(\mathcal{G}))}\sum_{i\in\mathcal{V}}x_i. \tag{26}$$

To see it, let y be feasible for (23); then $x := (y_i)_{i\in\mathcal{V}} \in N_+(\mathrm{TH}(\mathcal{G}))$. Indeed, the vector $(y_{I\cup\{r\}})_{I\in\mathcal{P}_1(V)}$ is equal to the first column of the principal submatrix of $Y_r(y)$ indexed by $\{r\}\cup\{\{r,i\}\mid i\in\mathcal{V}\}$, and $(y_I - y_{I\cup\{r\}})_{I\in\mathcal{P}_1(V)}$ is the first column of the matrix $\tilde{C}_r$ in (25).

3.2. **The semidefinite programming bounds $\ell^{(k)}(\mathcal{G})$ for the coding problem.** Let G be a group of automorphisms of the graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$; that is, $G\subseteq Sym(\mathcal{V})$ and each $\sigma\in G$ preserves edges, i.e., $ij\in\mathcal{E}\Longrightarrow \sigma(i)\sigma(j)\in\mathcal{E}$. Then G acts on the set $\mathcal{P}_k(\mathcal{V})$ indexing matrices in the program (22), by letting $\sigma(I)=\{\sigma(i)\mid i\in I\}$ for $\sigma\in G$, $I\in\mathcal{P}_k(\mathcal{V})$.

Lemma 7. *Let G be a group of automorphisms of $\mathcal{G}$. Then the program (22) is invariant under the action of G.*

Proof. Set $Y = M_k(y)$. The objective function is of the form $\sum_{i\in\mathcal{V}} y_i = \sum_{i\in\mathcal{V}} Y_{i,i} = \langle C, Y\rangle$, where C is invariant under action of G, since the set $\{(\{i\},\{i\})\mid i\in\mathcal{V}\}$ is a union of orbits of $\mathcal{P}_k(\mathcal{V})\times\mathcal{P}_k(\mathcal{V})$ (in fact, a single orbit if G is vertex-transitive). The constraint $y_\emptyset = Y_{\emptyset,\emptyset}=1$ is of the form $\langle A,Y\rangle = 1$ where A is invariant, since the set $\{(\emptyset,\emptyset)\}$ is an orbit. The class of edge constraints (19) is invariant under action of G: If $I\cup J$ contains an edge ij and $\sigma\in G$, then $\sigma(I)\cup\sigma(J)$ contains the edge $\sigma(i)\sigma(j)$ and thus the equation: $y_{\sigma(I)\sigma(J)} = Y_{\sigma(I),\sigma(J)} = 0$ is again an edge constraint. Similarly, the class of moment constraints (18) is also invariant under action of G. ∎

By Corollary 2, the parameter $\ell^{(k)}(\mathcal{G})$ can therefore be formulated as the optimum of a semidefinite program in N variables involving $N\times N$ matrices, where N is the number of orbits of the set $\mathcal{P}_k(\mathcal{V})\times\mathcal{P}_k(\mathcal{V})$ under the action of the group G. We now apply this technique to the graph $\mathcal{G}=\mathcal{G}(n,d)$ and to the group $G = Aut(\mathcal{P})$, the group of automorphisms of $\mathcal{P}$ (introduced in (10)). Recall that $\mathcal{G}(n,d)$ has node set $\mathcal{P}$, the collection of subsets of $\{1,\dots,n\}$, with an edge (I,J) if $|I\Delta J|\in\{1,\dots,d-1\}$ for $I,J\in\mathcal{P}$. Thus G also acts on the set $\mathcal{P}_k(\mathcal{P})=\{\mathcal{A}\subseteq\mathcal{P}\mid |\mathcal{A}|\le k\}$, indexing the matrix variable in program (22). We show:

Theorem 8. *For any fixed k, one can compute (to an arbitrary precision) the parameter $\ell^{(k)}(\mathcal{G}(n,d))$ from (22) in time polynomial in n. The same holds for the parameter $\ell_+^{(k)}(\mathcal{G})$ obtained by adding the constraints (21) to (22).*

Proof. Let k be fixed and let N_k denote the number of orbits of the set $\mathcal{P}_k(\mathcal{P})\times\mathcal{P}_k(\mathcal{P})$ under the action of the group G. As mentioned above, the parameter $\ell^{(k)}(\mathcal{G}(n,d))$ can be expressed via a semidefinite program of the form (8), involving $N_k\times N_k$ matrices and N_k variables. Hence, to show Theorem 8, it suffices to verify that N_k is bounded by a polynomial in n and that the new program equivalent to (22) can be constructed in time polynomial in n.

To begin with, it is useful to have a way to identify the orbits of the set $\mathcal{P}_k(\mathcal{P})\times\mathcal{P}_k(\mathcal{P})$.

Consider $(\mathcal{A},\mathcal{B})\in\mathcal{P}_k(\mathcal{P})\times\mathcal{P}_k(\mathcal{P})$ with $r:=|\mathcal{A}|$ and $s:=|\mathcal{B}|$. If $r=s=0$ then $\mathcal{A}=\mathcal{B}=\emptyset$, the empty subset of $\mathcal{P}$, and the orbit of $(\emptyset,\emptyset)$ just consists of the pair $(\emptyset,\emptyset)$. We can now assume that $r+s\ge 1$. Let $\vec{\mathcal{A}}=(A_1,\dots,A_r)$ be an ordering of the elements of $\mathcal{A}$; similarly, $\vec{\mathcal{B}} = (B_1, \dots, B_s)$ is an ordering of the elements of $\mathcal{B}$. Then one can define the $(r+s) \times n$ *incidence tableau* of $(\vec{\mathcal{A}}, \vec{\mathcal{B}})$, whose rows are the incidence vectors $\chi^{A_1}, \dots, \chi^{A_r}, \chi^{B_1}, \dots, \chi^{B_s}$ (in that order) of the sets $A_1, \dots, A_r, B_1, \dots, B_s$. Define the function $\varphi_{\vec{\mathcal{A}},\vec{\mathcal{B}}} : \{0,1\}^r \times \{0,1\}^s \longrightarrow \mathbb{Z}_+$ where, for $(u,v) \in \{0,1\}^r \times \{0,1\}^s$, $\varphi_{\vec{\mathcal{A}},\vec{\mathcal{B}}}(u,v)$ is the multiplicity of (u,v) as a column of the incidence tableau of $(\vec{\mathcal{A}}, \vec{\mathcal{B}})$. Thus $\varphi_{\vec{\mathcal{A}},\vec{\mathcal{B}}}$ belongs to the set $\Phi_{r,s}$ consisting of the functions $\phi : \{0,1\}^r \times \{0,1\}^s \longrightarrow \{0,1,\dots,n\}$ satisfying: $\sum_{u\in\{0,1\}^r, v\in\{0,1\}^s} \phi(u,v) = n$ and, for all $i \neq j \in \{1,\dots,r\}$ (resp., $i \neq j \in \{1,\dots,s\}$), there exists $(u,v) \in \{0,1\}^r \times \{0,1\}^s$ for which $\phi(u,v) \geq 1$ and $u_i \neq u_j$ (resp., $v_i \neq v_j$).

Let $\vec{\mathcal{A}'}$ (resp., $\vec{\mathcal{B}'}$) be another ordered sequence of r (resp., of s) distinct elements of $\mathcal{P}$ and let $\phi = \phi_{\vec{\mathcal{A}},\vec{\mathcal{B}}}$, $\phi' = \varphi_{\vec{\mathcal{A}'},\vec{\mathcal{B}'}}$. Then, $\vec{\mathcal{A}'} = (\sigma(A_1), \dots, \sigma(A_r))$ and $\vec{\mathcal{B}'} = (\sigma(B_1), \dots, \sigma(B_s))$ for some $\sigma \in G$ if and only if $\phi(u,v) + \phi(\mathbf{1}-u, \mathbf{1}-v) = \phi'(u,v) + \phi'(\mathbf{1}-u, \mathbf{1}-v)$ for all $(u,v) \in \{0,1\}^r \times \{0,1\}^s$. (Here, $\mathbf{1} := (1,\dots,1)$ denotes the all-ones vector of the suitable size.) Moreover, $\vec{\mathcal{A}'} = (A_{\alpha(1)}, \dots, A_{\alpha(r)})$ and $\vec{\mathcal{B}'} = (B_{\beta(1)}, \dots, B_{\beta(s)})$ for some permutations $\alpha \in Sym(r)$, $\beta \in Sym(s)$ if and only if $\phi'(u,v) = \phi(\alpha(u), \beta(v))$ for all $(u,v) \in \{0,1\}^r \times \{0,1\}^s$, setting $\alpha(u) := (u_{\alpha(1)}, \dots, u_{\alpha(r)})$, $\beta(v) := (v_{\beta(1)}, \dots, v_{\beta(s)})$. For two elements $\phi, \phi' \in \Phi_{r,s}$, write $\phi \sim \phi'$ if

$$\phi'(u,v) + \phi'(\mathbf{1}-u, \mathbf{1}-v) = \phi(\alpha(u), \beta(v)) + \phi(\mathbf{1}-\alpha(u), \mathbf{1}-\beta(v))) \ \forall (u,v) \in \{0,1\}^r \times \{0,1\}^s$$

for some $\alpha \in Sym(r)$, $\beta \in Sym(s)$. This defines an equivalence relation on $\Phi_{r,s}$.

We can now characterize orbits in the following way: Two pairs $(\mathcal{A}, \mathcal{B}), (\mathcal{A}', \mathcal{B}')$ belong to the same orbit of $\mathcal{P}_k(\mathcal{P}) \times \mathcal{P}_k(\mathcal{P})$ under action of G if and only if $|\mathcal{A}| = |\mathcal{A}'| =: r$, $|\mathcal{B}| = |\mathcal{B}'| =: s$ and $\varphi_{\vec{\mathcal{A}},\vec{\mathcal{B}}} \sim \varphi_{\vec{\mathcal{A}'},\vec{\mathcal{B}'}}$ for some respective orderings $\vec{\mathcal{A}}$, $\vec{\mathcal{B}}$, $\vec{\mathcal{A}'}$, $\vec{\mathcal{B}'}$ of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$, $\mathcal{B}'$. Thus each orbit of $\mathcal{P}_k(\mathcal{P}) \times \mathcal{P}_k(\mathcal{P})$ corresponds to an equivalence class of $\cup_{0\leq r,s\leq k} \Phi_{r,s}$. Hence the number N_k of orbits of $\mathcal{P}_k(\mathcal{P}) \times \mathcal{P}_k(\mathcal{P})$ is at most $1 + \sum_{\substack{0\leq r,s\leq k\\ r+s\geq 1}} (n+1)^{2^{r+s-1}-1}$, giving:

$$N_k \leq O(n^{2^{2k-1}-1}). \tag{27}$$

We now verify that the matrices L_i ($i = 1, \dots, N_k$) (as defined in (5)) can be constructed in time polynomial in n.

For this one first needs to be able to compute in time polynomial in n the cardinality of the orbits of $\mathcal{P}_k(\mathcal{P}) \times \mathcal{P}_k(\mathcal{P})$. Given $\phi_0 \in \Phi_{r,s}$ ($0 \leq r,s \leq k, r+s \geq 1$), one has to count the number L_{ϕ_0} of pairs $(\mathcal{A}, \mathcal{B}) \in \mathcal{P}_{=r}(\mathcal{P}) \times \mathcal{P}_{=s}(\mathcal{P})$ for which $\varphi_{\vec{\mathcal{A}},\vec{\mathcal{B}}} \sim \phi_0$ for some orderings $\vec{\mathcal{A}}$, $\vec{\mathcal{B}}$ of $\mathcal{A}$, $\mathcal{B}$. Given $\phi \sim \phi_0$, there are $\ell_\phi := n!/\prod_{\substack{u\in\{0,1\}^r\\ v\in\{0,1\}^s}} \phi(u,v)!$ pairs $(\vec{\mathcal{A}}, \vec{\mathcal{B}})$ for which $\varphi_{\vec{\mathcal{A}},\vec{\mathcal{B}}} = \phi_0$. Therefore, $L_{\phi_0} = \frac{1}{r!s!} \sum_{\phi\sim\phi_0} \ell_\phi$, which can be computed in time polynomial in n since one can enumerate the equivalence class of ϕ_0 in time polynomial in n.

Next we verify that one can compute in time polynomial in n the multiplication parameters $\gamma_{i,j}^k$ from (4), used for defining the matrices L_i in (5). For this, given $(\mathcal{A}, \mathcal{B}) \in \mathcal{P}_{=r}(\mathcal{P}) \times \mathcal{P}_{=s}(\mathcal{P})$ with respective orderings $\vec{\mathcal{A}}$, $\vec{\mathcal{B}}$, given an integer $0 \leq t \leq k$, and given $\phi_0 \in \Phi_{r,t}$, $\psi_0 \in \Phi_{s,t}$, one has to count the number L_{ϕ_0,ψ_0} of elements $\mathcal{C} \in \mathcal{P}_{=t}(\mathcal{P})$ for which $\varphi_{\vec{\mathcal{A}},\vec{\mathcal{C}}} \sim \phi_0$ and $\varphi_{\vec{\mathcal{B}},\vec{\mathcal{C}}} \sim \psi_0$ for some ordering $\vec{\mathcal{C}}$ of $\mathcal{C}$. Set $\xi := \varphi_{\vec{\mathcal{A}},\vec{\mathcal{B}}}$. Given $\phi \sim \phi_0$ and $\psi \sim \psi_0$, we first count the number $\ell_{\phi,\psi}$ of ordered sequences $\vec{\mathcal{C}}$ of t elements of

$\mathcal{P}$ for which $\varphi_{\vec{\mathcal{A}},\vec{\mathcal{C}}} = \phi$ and $\varphi_{\vec{\mathcal{B}},\vec{\mathcal{C}}} = \psi$. For this let $x(u,v,w)$ denote the multiplicity of $(u,v,w) \in \{0,1\}^r \times \{0,1\}^s \times 2^t$ as column of the incidence tableau of $(\vec{\mathcal{A}},\vec{\mathcal{B}},\vec{\mathcal{C}})$. The first $r+s$ rows of the tableau are given and one needs to determine its last t rows. Then, $x(u,v,w) \in \{0,1,\ldots,n\}$ satisfy the system

$$\begin{array}{ll} \sum_{v\in\{0,1\}^s} x(u,v,w) = \phi(u,w) & \forall u \in \{0,1\}^r, w \in \{0,1\}^t \\ \sum_{u\in\{0,1\}^r} x(u,v,w) = \psi(v,w) & \forall v \in \{0,1\}^s, w \in \{0,1\}^t \\ \sum_{w\in\{0,1\}^t} x(u,v,w) = \xi(u,v) & \forall u \in \{0,1\}^r, v \in \{0,1\}^s. \end{array} \tag{28}$$

As the system (28) has polynomially many variables and equations, its set S of solutions can be found by complete enumeration and $|S| \le (n+1)^{2^{r+s+t}}$. Therefore, $\ell_{\phi,\psi} = \sum_{x\in S}\sum_{u\in\{0,1\}^r,v\in\{0,1\}^s} \frac{\xi(u,v)!}{\prod_{w\in 2^t} x(u,v,w)!}$, the number of possible ways to assign the vectors $w \in 2^t$ as columns of the lower $t\times n$ part of the tableau. Now, $L_{\phi_0,\psi_0} = \frac{1}{t!}\sum_{\substack{\phi\sim\phi_0\\ \psi\sim\psi_0}} \ell_{\phi,\psi}$ can be computed in time polynomial in n since one can enumerate the equivalence classes of ϕ_0 and ψ_0.

Remains only to construct the linear constraints corresponding to the moment constraints (18) and the edge constraints (19). Label the orbits of $\mathcal{P}_k(\mathcal{P})\times\mathcal{P}_k(\mathcal{P})$ as $\mathcal{O}_1,\ldots,\mathcal{O}_{N_k}$ and determine a pair $(\mathcal{A}_i,\mathcal{B}_i)$ belonging to each orbit $\mathcal{O}_i$. Then the moment constraints read: $x_i = x_j$ if $\mathcal{A}_i \cup \mathcal{B}_i = \sigma(\mathcal{A}_j \cup \mathcal{B}_j)$ for some $\sigma \in G$ (which can be tested in time polynomial in n), and the edge constraints read: $x_i = 0$ if $\mathcal{A}_i \cup \mathcal{B}_i$ contains a pair (I,J) with $|I\Delta J| \in \{1,\ldots,d-1\}$.

The bounds (21) become: $x_i \ge 0$ $(i = 1,\ldots,N_k)$ and $x_i \le x_j$ if $\mathcal{A}_i \cup \mathcal{B}_i \supseteq \sigma(\mathcal{A}_j \cup \mathcal{B}_j)$ for some $\sigma \in G$ (which can be tested in time polynomial in n).

Therefore, the parameter $\ell^{(k)}(\mathcal{G}(n,d))$ (or $\ell_+^{(k)}(\mathcal{G}(n,d))$) can be computed as the optimum value of a semidefinite program of the form (8) involving $N_k \times N_k$ matrices, with N_k variables and $O(N_k^2)$ linear constraints. As $N_k = O(n^{2^{2k-1}-1})$, it can be computed in time polynomial in n (to any precision), which concludes the proof of Theorem 8. ∎

The result from Theorem 8 is mainly of theoretical value for $k \ge 2$. Indeed, for $k=2$, $N_k = O(n^7)$ and thus the semidefinite program defining $\ell^{(2)}(\mathcal{G}(n,d))$ is already too large to be solved in practice for interesting values of n by the currently available software for semidefinite programming.

3.3. **Refining Schrijver's bound.** We begin with observing that, when a graph $\mathcal{G}$ has a vertex-transitive group G of automorphisms then, in the program (23), it suffices to require the condition $Y_r(y) \succeq 0$ for *one* choice of $r \in \mathcal{V}$.

LEMMA 9. *Let G be a group of automorphisms of the graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$. The program (23) is invariant under action of G. If G is vertex-transitive then, in (23), it suffices to require the constraint $Y_r(y) \succeq 0$ for one choice of $r \in \mathcal{V}$ (instead of for all $r \in \mathcal{V}$).*

Proof. The first part of the proof is analogous to the proof of Lemma 7. Here, we use the fact that, for $r \in \mathcal{V}$, $\sigma \in G$, $Y_r(\sigma(y)) = \sigma(Y_{\sigma(r)}(y))$. Hence, if y is invariant under action of G, then $Y_r(y) \succeq 0$ for some $r \in \mathcal{V}$ implies that $Y_r(y) \succeq 0$ for all $r \in \mathcal{V}$. ∎

3.3.1. *A compact semidefinite formulation for the bound $\ell(\mathcal{G}(n,d))$.* In this section we consider the graph $\mathcal{G} = \mathcal{G}(n,d)$ and the group $G = Aut(\mathcal{P})$, whose action on the graph $\mathcal{G}(n,d)$ is indeed vertex-transitive. We set:

$$\mathcal{X} := \mathcal{P}_2(\mathcal{P};\emptyset) = \{\emptyset\} \cup \{\{I\} \mid I \in \mathcal{P}\} \cup \{\{\emptyset, I\} \mid I \in \mathcal{P}\}. \tag{29}$$

Applying Lemma 9, one can reformulate the parameter $\ell(\mathcal{G}(n,d))$ as

$$\begin{array}{rll} \ell(\mathcal{G}(n,d)) = & \max & \sum_{I\in\mathcal{P}} y_{\{I\}} \\ & \text{s.t.} & Y(y) \succeq 0,\ y_\emptyset = 1, \\ & & y_{\{I,J\}} = 0 \text{ if } |I\Delta J| \in \{1,\dots,d-1\} \\ & & y_{\mathcal{A}} = y_{\sigma(\mathcal{A})} \text{ for } \sigma \in G, \mathcal{A} \in \mathcal{X}, \end{array} \tag{30}$$

where the matrix variable $Y(y)$ is indexed by the set $\mathcal{X}$ and satisfies: $Y(y)_{\mathcal{A},\mathcal{B}} = y_{\mathcal{A}\cup\mathcal{B}}$ for $\mathcal{A}, \mathcal{B} \in \mathcal{X}$. By (24), $Y(y)$ has the form

$$Y(y) = \begin{pmatrix} 1 & a^T & b^T \\ a & A & B \\ b & B & B \end{pmatrix} \tag{31}$$

with $A = (y_{\{I,J\}})_{I,J\in\mathcal{P}}$, $B = (y_{\{\emptyset,I,J\}})_{I,J\in\mathcal{P}}$, $a = (y_{\{I\}})_{I\in\mathcal{P}}$, and $b = (y_{\{\emptyset,I\}})_{I\in\mathcal{P}}$. As y is invariant under action of G, it follows that $A_{I,J} = A_{I',J'}$ if $I' = \sigma(I)$, $J' = \sigma(J)$ for some $\sigma \in G$, i.e., if $|I\Delta J| = |I'\Delta J'|$. That is, the matrix A belongs to the Bose-Mesner algebra $\mathcal{B}_n$; say,

$$A = \sum_{k=0}^{n} x_k M_k \quad \text{for some real scalars } x_0, \dots, x_n \tag{32}$$

where the matrices M_k are as in (11). Moreover, $B_{I,J} = B_{I',J'}$ if $I' = \sigma(I)$, $J' = \sigma(J)$, $\emptyset = \sigma(\emptyset)$ for some $\sigma \in G$, i.e., if $|I'| = |I|$, $|J'| = |J|$ and $|I \cap J| = |I' \cap J'|$. That is, the matrix B belongs to the Terwilliger algebra $\mathcal{A}_n$; say,

$$B = \sum_{i,j,t\geq 0} x_{i,j}^t M_{i,j}^t \quad \text{for some real scalars } x_{i,j}^t \tag{33}$$

where the matrices $M_{i,j}^t$ are as in (9) and $x_{i,j}^t = x_{j,i}^t$ for all i,j,t. The variables x_k and $x_{i,j}^t$ are related by

$$x_k = x_{0,k}^0 \quad \text{for } k = 0, 1, \dots, n. \tag{34}$$

(since $x_k = A_{\emptyset,I} = B_{\emptyset,I} = x_{0,k}^k$ for $|I| = k$). Moreover,

$$x_{i,j}^t = x_{i',j'}^{t'} \quad \text{if } (i', j', i'+j'-2t') \text{ is a permutation of } (i, j, i+j-2t). \tag{35}$$

Equivalently, $x_{i,j}^t = x_{i+j-2t,i}^{i-t} = x_{i+j-2t,j}^{j-t}$. (Indeed, let $I, J \in \mathcal{P}$ with $i = |I|$, $j = |J|$, $t = |I \cap J|$. As $\sigma := s_J$ maps $\mathcal{A} := \{\emptyset, I, J\}$ to $\{\emptyset, J, I\Delta J\}$ and $y_{\sigma(\mathcal{A})} = y_{\mathcal{A}}$, then $x_{i,j}^t = y_{\{\emptyset,I,J\}} = y_{\{\emptyset,J,I\Delta J\}} = x_{j,i+j-2t}^{j-t}$.) The edge inequalities become:

$$x_{i,j}^t = 0 \quad \text{if } \{i, j, i+j-2t\} \cap \{1, \dots, d-1\} \neq \emptyset, \tag{36}$$

and the bounds (21) read:

$$0 \le x_{i,j}^t \le x_{i,0}^0 \text{ for } i,j,t = 0,\dots,n. \tag{37}$$

From (25), we know that $Y(y) \succeq 0$ if and only if

$$B = \sum_{i,j,t=0}^{n} x_{i,j}^t M_{i,j}^t \succeq 0 \;\text{ and } \tilde{C} := \begin{pmatrix} 1 - x_{0,0}^0 & c^T \\ c & C \end{pmatrix} \succeq 0,$$

where

$$C := A - B = \sum_{i,j,t=0}^{n} (x_{0,i+j-2t}^0 - x_{i,j}^t) M_{i,j}^t \;\text{ and }\; c := a - b = \sum_{i=0}^{n} (x_{0,0}^0 - x_{0,i}^0)\chi^{\mathcal{P}_{=i}(V)}.$$

(Recall $\mathcal{P}_{=i}(V) = \{I \subseteq V \mid |I| = i\}$.) Thus $\tilde{C}$ is of the form (16). For $k = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$, define the matrices:

$$A_k(x) := \left(\sum_t \binom{n-2k}{i-k}^{-\frac{1}{2}} \binom{n-2k}{j-k}^{-\frac{1}{2}} \beta_{i,j,k}^t x_{0,i+j-2t}^0 \right)_{i,j=k}^{n-k} \tag{38}$$

and $B_k(x)$ as in (13), where $\beta_{i,j,k}^t$ are as in (14). It follows from Lemma 6 that the positive semidefiniteness of $Y(y)$ is equivalent to

$$\begin{array}{ll} \text{(i)} & B_k(x) \succeq 0 \;\text{ for } k = 0,1,\dots,\lfloor \frac{n}{2} \rfloor \\ \text{(ii)} & A_k(x) - B_k(x) \succeq 0 \;\text{ for } k = 0,1,\dots,\lfloor \frac{n}{2} \rfloor \\ \text{(iii)} & \begin{pmatrix} 1 - x_{0,0}^0 & \tilde{c}^T \\ \tilde{c} & A_0(x) - B_0(x) \end{pmatrix} \succeq 0, \;\text{ setting } \tilde{c} := (\binom{n}{i}^{\frac{1}{2}} (x_{0,0}^0 - x_{0,i}^0))_{i=0}^n. \end{array} \tag{39}$$

(Of course, (39)(iii) implies (ii) for $k = 0$.) Summarizing, we have shown:

$$\ell(\mathcal{G}(n,d)) = \max\; 2^n x_{0,0}^0 \;\text{ s.t. }\; x_{i,j}^t \; (i,j,t = 0,\dots,n) \text{ satisfy } (35),(36),(39)(i)-(iii). \tag{40}$$

Similarly,

$$\ell_+(\mathcal{G}(n,d)) = \max\; 2^n x_{0,0}^0 \;\text{ s.t. }\; x_{i,j}^t \; (i,j,t = 0,\dots,n) \text{ satisfy } (35),(36),(37),(39)(i)-(iii). \tag{41}$$

Hence both parameters can be computed via a semidefinite program of size $O(n^3)$.

3.3.2. *Comparison with Schrijver's bound.* Schrijver [13] introduced the following upper bound for the stability number $A(n,d)$ of the graph $\mathcal{G}(n,d)$:

$$\begin{array}{lll} \ell_{sch}(\mathcal{G}(n,d)) := & \max & \displaystyle\sum_{i=0}^{n} \binom{n}{i} x_{0,i}^0 \\ & \text{s.t.} & x_{i,j}^t \; (i,j,t = 0,\dots,n) \;\text{ satisfy } (35),(36),(37), \\ & & (39)(i)-(ii) \text{ and } x_{0,0}^0 = 1. \end{array} \tag{42}$$

As noted in [13], Schrijver's bound is at least as good as the Delsarte bound, which coincides with $\vartheta'(\mathcal{G}(n,d)) = \ell_+^{(1)}(\mathcal{G}(n,d))$. We now show:

LEMMA 10. *The bound $\ell_+(\mathcal{G}(n,d))$ is at least as good as Schrijver's bound $\ell_{sch}(\mathcal{G}(n,d))$ from (42); that is, $\ell_+(\mathcal{G}(n,d)) \le \ell_{sch}(\mathcal{G}(n,d))$.*

Proof. Let $(x_{i,j}^t)_{i,j,t=0}^n$ be feasible for the program (41). Define $y_{i,j}^t := x_{i,j}^t/x_{0,0}^0$ for all $i,j,t = 0,\dots,n$. Then the variables $y_{i,j}^t$ satisfy (35), (36), (37), (39) (i)-(ii), and $y_{0,0}^0 = 1$. Remains to verify that $2^n x_{0,0}^0 \leq \sum_{i=0}^n \binom{n}{i} y_{0,i}^0$, i.e., $2^n (x_{0,0}^0)^2 \leq \sum_{i=0}^n \binom{n}{i} x_{0,i}^0$. For this, recall that the conditions (39) (i)-(iii) are equivalent to the positive semidefiniteness of the matrix in (31). In particular, they imply

$$\begin{pmatrix} 1 & a^T \\ a & A \end{pmatrix} \succeq 0, \quad \text{i.e., } A - aa^T \succeq 0,$$

where A is as in (32), $a^T = (x_{0,0}^0, \dots, x_{0,0}^0)$, $x_k = x_{0,k}^0$ for $k = 0,\dots,n$. Thus, $aa^T = (x_{0,0}^0)^2 J$, where J is the all-ones matrix. As $A - (x_{0,0}^0)^2 J \succeq 0$, we deduce that $\langle J, A\rangle \geq (x_{0,0}^0)^2 \langle J, J\rangle = (x_{0,0}^0 2^n)^2$. But $\langle J, A\rangle = \sum_{k=0}^n x_k \langle J, M_k\rangle = \sum_{k=0}^n x_k 2^n \binom{n}{k}$, which gives $\sum_{k=0}^n x_{0,k}^0 \binom{n}{k} \geq 2^n (x_{0,0}^0)^2$. ∎

3.3.3. *Refining the bound* $\ell_+(\mathcal{G}(n,d))$. It is possible to define a new bound $\ell_{++}(\mathcal{G}(n,d))$, at least as good as the bound $\ell_+(\mathcal{G}(n,d))$, whose computation still involves a semidefinite program of size $O(n^3)$. Namely, let us now consider as matrix variable the principal submatrix $Y(y)$ of $M_2(y)$ indexed by the set

$$\mathcal{X}_+ := \{\emptyset\} \cup \{\{I\} \mid I \in \mathcal{P}\} \cup \{\{\emptyset, I\} \mid I \in \mathcal{P}\} \cup \{\{I,V\} \mid I \in \mathcal{P}\}. \tag{43}$$

Then, $Y(y)$ has the block structure:

$$Y(y) = \begin{pmatrix} 1 & a^T & b^T & c^T \\ a & A & B & C \\ b & B & B & D \\ c & C & D & C \end{pmatrix} \tag{44}$$

where $A = (y_{\{I,J\}})_{I,J\in\mathcal{P}}$, $B = (y_{\{\emptyset,I,J\}})_{I,J\in\mathcal{P}}$, $C = (y_{\{I,J,V\}})_{I,J\in\mathcal{P}}$, $D = (y_{\{\emptyset,I,J,V\}})_{I,J\in\mathcal{P}}$, $a = (y_{\{I\}})_{I\in\mathcal{P}}$, $b = (y_{\{\emptyset,I\}})_{I\in\mathcal{P}}$, and $c = (y_{\{I,V\}})_{I\in\mathcal{P}}$. The matrices A, B are given by (32), (33). The matrix C is a permutation of B; namely,

$$C = \sum_{i,j,t=0}^n x_{n-i,n-j}^{n+t-i-j} M_{i,j}^t.$$

The matrix D too belongs to the Terwilliger algebra:

$$D = \sum_{i,j,t=0}^n z_{i,j}^t M_{i,j}^t \text{ for some real scalars } z_{i,j}^t$$

satisfying $z_{i,j}^t = z_{j,i}^t$; indeed, $D_{I,J} = D_{I',J'}$ if there exists $\sigma \in G$ such that $\sigma(\emptyset) = \emptyset$, $\sigma(I) = I'$, $\sigma(J) = J'$ (then $\sigma(V) = V$), i.e., if $|I| = |I'|$, $|J| = |J'|$, $|I\cap J| = |I'\cap J'|$. We have the following relations for the variables $x_{i,j}^t$, $z_{i,j}^t$:

$$z_{i,j}^t = z_{n-i,n-j}^{n+t-i-j} \text{ for all } i,j,t = 0,\dots,n \tag{45}$$

since $D_{I,J} = y_{\{\emptyset,V,I,J\}} = y_{\{\emptyset,V,V\Delta I,V\Delta J\}} = D_{V\Delta I,V\Delta J}$, and

$$z_{i,i}^i = z_{0,i}^0 = z_{n,i}^i = x_{i,n}^i \text{ for } i = 0,\dots,n \tag{46}$$

since $y_{\{\emptyset,V,I\}} = D_{I,I} = D_{\emptyset,I} = D_{V,I} = B_{V,I}$. The edge condition for the z-variables reads:

$$z^t_{i,j} = 0 \text{ if } \{i,j,n-i,n-j,i+j-2t\} \cap \{1,\ldots,d-1\} \neq 0 \text{ for } i,j,t = 0,\ldots,n. \tag{47}$$

The bounds (21) imply:

$$0 \le z^t_{i,j} \le x^t_{i,j},\ z^t_{i,j} \le z^i_{i,i} \text{ for } i,j,t = 0,\ldots,n. \tag{48}$$

As each non-border block of the matrix $Y(y)$ in (44) belongs to the Terwilliger algebra, one can block-diagonalize $Y(y)$. Indeed, each non-border block in the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & U^T & 0 & 0 \\ 0 & 0 & U^T & 0 \\ 0 & 0 & 0 & U^T \end{pmatrix} Y(y) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & U & 0 & 0 \\ 0 & 0 & U & 0 \\ 0 & 0 & 0 & U \end{pmatrix} = \begin{pmatrix} 1 & a^TU & b^TU & c^TU \\ Ua & U^TAU & U^TBU & U^TCU \\ Ub & U^TBU & U^TBU & U^TDU \\ Uc & U^TCU & U^TDU & U^TCU \end{pmatrix}$$

is block-diagonal with respect to the same partition, with $\lfloor \frac{n}{2} \rfloor + 1$ distinct blocks labeled by $k = 0,1,\ldots,\lfloor \frac{n}{2} \rfloor$. It follows from Lemma 6 that $a^TU = (\tilde{a}^T, 0, \ldots, 0)$, $b^TU = (\tilde{b}^T, 0, \ldots, 0)$, $c^TU = (\tilde{c}^T, 0, \ldots, 0)$, where $\tilde{a} = x^0_{0,0} \sum_{i=0}^n \binom{n}{i}^{\frac{1}{2}} \chi^{\mathcal{P}_{=i}(V)}$, $\tilde{b} = \sum_{i=0}^n x^0_{0,i} \binom{n}{i}^{\frac{1}{2}} \chi^{\mathcal{P}_{=i}(V)}$ and $\tilde{c} = \sum_{i=0}^n x^0_{0,n-i} \binom{n}{i}^{\frac{1}{2}} \chi^{\mathcal{P}_{=i}(V)}$ are indexed by the positions corresponding to the 0-th block. Therefore, $Y(y) \succeq 0$ if and only if

$$\begin{pmatrix} 1 & \tilde{a}^T & \tilde{b}^T & \tilde{c}^T \\ \tilde{a} & A_0 & B_0 & C_0 \\ \tilde{b} & B_0 & B_0 & D_0 \\ \tilde{c} & C_0 & D_0 & C_0 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} A_k & B_k & C_k \\ B_k & B_k & D_k \\ C_k & D_k & C_k \end{pmatrix} \succeq 0 \text{ for } k = 1,\ldots,\lfloor \frac{n}{2} \rfloor \tag{49}$$

where $A_k = A_k(x)$ is as in (38), $B_k = B_k(x)$ is as in (13) and

$$C_k = \left(\sum_t \binom{n-2k}{i-k}^{-\frac{1}{2}} \binom{n-2k}{j-k}^{-\frac{1}{2}} \beta^t_{i,j,k} x^{n+t-i-j}_{n-i,n-j} \right)_{i,j=k}^{n-k},$$

$$D_k = \left(\sum_t \binom{n-2k}{i-k}^{-\frac{1}{2}} \binom{n-2k}{j-k}^{-\frac{1}{2}} \beta^t_{i,j,k} z^t_{i,j} \right)_{i,j=k}^{n-k}.$$

One can now define the bound

$$\ell_{++}(\mathcal{G}(n,d)) := \max 2^n x^0_{0,0} \text{ s.t. } \begin{array}{l} x^t_{i,j},\ z^t_{i,j}\ (i,j,t = 0,\ldots,n) \text{ satisfy} \\ (35),(36),(37),(45),(46),(47),(48),(49). \end{array} \tag{50}$$

Obviously,

$$A(n,d) \le \ell_{++}(\mathcal{G}(n,d)) \le \ell_{+}(\mathcal{G}(n,d)) \le \ell_{sch}(\mathcal{G}(n,d)),$$

and the bound $\ell_{++}(\mathcal{G}(n,d))$ is again expressed via a semidefinite program of size $O(n^3)$.

Summarizing, the parameters ℓ_{sch}, ℓ_+, ℓ_{++} can all be seen as variations of the Lasserre bound $\ell^{(2)}$. Namely, instead of considering the full matrix variable $M_2(y)$ indexed by the set $\mathcal{P}_2(\mathcal{P})$, one considers a principal submatrix of $M_2(y)$ indexed by a subset of $\mathcal{P}_2(\mathcal{P})$; namely, by the set $\mathcal{X} \setminus \{\emptyset\}$ for ℓ_{sch}, by the set $\mathcal{X}$ for ℓ_+, and by the set $\mathcal{X}_+ = \mathcal{X} \cup \{\{I,V\} \mid I \in \mathcal{P}\}$ for ℓ_{++}. (Recall the set $\mathcal{X}$ in (29).)

3.3.4. *Reducing the number of variables.* The following observation from [13] can be used for reducing the number of variables in the programs (40), (41), (42), (50), and for further refining the corresponding bounds. A well known fact in coding theory is that, if d is odd then $A(n,d) = A(n+1, d+1)$, and if d is even then $A(n,d)$ is attained by a code with all code words having an even Hamming weight. Therefore, it suffices to compute $A(n,d)$ for d even. Moreover, for d even, $A(n,d) = \alpha(\mathcal{G}_{ev}(n,d))$, the stability number of the graph $\mathcal{G}_{ev}(n,d)$, defined as the subgraph of $\mathcal{G}(n,d)$ induced by the set

$$\mathcal{P}_{ev} := \{I \subseteq V \mid |I| \text{ is even}\}.$$

Therefore, for d even, one may add the constraints:

$$y_{\mathcal{A}} = 0 \quad \text{if } \mathcal{A} \not\subseteq \mathcal{P}_{ev} \tag{51}$$

for any $\mathcal{A} \in \mathcal{P}_{2k}(\mathcal{P})$ to the program (22) defining $\ell^{(k)}(\mathcal{G}(n,d))$, or for any $\mathcal{A} \in \mathcal{P}_3(\mathcal{P})$ to the program (23) defining $\ell(\mathcal{G}(n,d))$. Equivalently, one may add the constraints:

$$x_{i,j}^t = 0 \quad \text{if one of } i \text{ or } j \text{ is odd,} \tag{52}$$

to the programs (40), (41), (42), (50), as well as as the constraints:

$$z_{i,j}^t = 0 \quad \text{if one of } i, j, \text{ or } n \text{ is odd} \tag{53}$$

to (50), and the new programs still define upper bounds for $A(n,d)$. Namely, define:

$$\ell^0(\mathcal{G}(n,d)) := \max\ 2^n x_{0,0}^0 \quad \text{s.t.} \quad \begin{array}{l} x_{i,j}^t\ (i,j,t = 0,\ldots,n) \text{ satisfy} \\ (35),(36),(39)(i)-(iii),(52) \end{array} \tag{54}$$

and let ℓ_+^0, (resp., ℓ_{sch}^0, ℓ_{++}^0) be defined analogously by adding (52) (resp., (52), (52)-(53)) to (41) (resp., (42), (50)).

As $A(n,d) = \alpha(\mathcal{G}_{ev}(n,d))$, one may also bound $A(n,d)$ by the parameter $\ell(\mathcal{G}_{ev}(n,d))$ (and analogously by $\ell_+(\mathcal{G}_{ev}(n,d))$, $\ell_{++}(\mathcal{G}_{ev}(n,d))$). The subgroup $G_{ev} := \{\pi s_A \mid A \in \mathcal{P}_{ev}\}$ of the group G (introduced in (10)) acts vertex-transitively on $\mathcal{P}_{ev}$. Hence, applying Lemma 9, $\ell(\mathcal{G}_{ev}(n,d))$ can be formulated via the analogue of (30), where $Y(y)$ in (31) is now indexed only by *even* sets; that is, a, b, A and B in (31) are indexed by $\mathcal{P}_{ev}$. Again, A belongs to the Bose-Mesner algebra and B belongs to the Terwilliger algebra; that is, for some scalars x_k, $x_{i,j}^t$, A (resp., B) is equal to the principal submatrix of $\sum_{k \text{ even}} x_k M_k$ (resp., of $\sum_{i,j,t \text{ even}} x_{i,j}^t M_{i,j}^t$) indexed by $\mathcal{P}_{ev}$. Therefore, $\ell(\mathcal{G}_{ev}(n,d))$ can be computed via the program:

$$\ell(\mathcal{G}_{ev}(n,d)) = \max\ 2^{n-1} x_{0,0}^0 \quad \text{s.t.} \quad \begin{array}{l} x_{i,j}^t\ (i,j,t = 0,\ldots,n) \text{ satisfy} \\ (35),(36),(39)(i)-(iii),(52) \end{array} \tag{55}$$

where, in (39), we consider only the '*even half*' of the matrices $A_k(x)$, $B_k(x)$, i.e., their principal submatrices indexed by *even* indices i,j.

LEMMA 11. $A(n,d) \leq \ell(\mathcal{G}_{ev}(n,d)) \leq \ell^0(\mathcal{G}(n,d)) \leq \ell(\mathcal{G}(n,d))$ *and analogously for the parameters* ℓ_+, ℓ_{sch}, ℓ_{++}.

Proof. The right and left most inequalities are obvious. To compare the parameters $\ell(\mathcal{G}_{ev}(n,d))$ and $\ell^0(\mathcal{G}(n,d))$, it is easiest to use their formulation via (23); for the formulation of $\ell^0(\mathcal{G}(n,d))$, one should add to (23) the constraint (51) for any $\mathcal{A} \in \mathcal{P}_3(\mathcal{P})$. Consider a feasible solution y for the program (23) defining $\ell(\mathcal{G}_{ev}(n,d))$. Thus y is indexed by $\mathcal{P}_3(\mathcal{P}_{ev})$, $y_{\{I,J\}} = 0$ if $|I\Delta J| = 1, \ldots, d-1$ (for $I, J \in \mathcal{P}_{ev}$) and, for any $I \in \mathcal{P}_{ev}$, the matrix $Y_I(y)$ (indexed by $\mathcal{P}_2(\mathcal{P}_{ev}; I)$) is positive semidefinite. We define a feasible solution z for the program defining $\ell^0(\mathcal{G}(n,d))$ in the following way: For $\mathcal{A} \in \mathcal{P}_3(\mathcal{P})$, set $z_{\mathcal{A}} := y_{\mathcal{A}}$ if $\mathcal{A} \subseteq \mathcal{P}_{ev}$, and $z_{\mathcal{A}} := 0$ otherwise. It is easy to verify that, for each $I \in \mathcal{P}$, the matrix $Y_I(z)$ (indexed by $\mathcal{P}_2(\mathcal{P}; I)$) is positive semidefinite. Thus, $\ell^0(\mathcal{G}(n,d)) \geq \sum_{I\in\mathcal{P}} z_I = \sum_{I\in\mathcal{P}_{ev}} y_I$, implying $\ell^0(\mathcal{G}(n,d)) \geq \ell(\mathcal{G}_{ev}(n,d))$. The reasoning is analogous for the other parameters. ∎

The bound $\ell(\mathcal{G}_{ev}(n,d))$ is more economical to compute than $\ell^0(\mathcal{G}(n,d))$, since it involves smaller matrices; as a matter of fact, the bound computed by Schrijver [13] is the bound $\ell_{sch}(\mathcal{G}_{ev}(n,d))$. For n odd, in view of (53), all variables $z^t_{i,j}$ can be set to 0 for the computation of $\ell_{++}(\mathcal{G}(n,d))$; from this follows that $\ell_+(\mathcal{G}_{ev}(n,d)) = \ell_{++}(\mathcal{G}_{ev}(n,d))$ when n is odd.

3.3.5. *Some computational results.* We have tested the various bounds on several instances (n,d), in particular, on those where Schrijver's bound gave a improvement on the previously best known upper bound for $A(n,d)$. There are two instances: (20,8) and (25,6), for which we could find an upper bound for $A(n,d)$ (slightly) better than Schrijver's bound; namely, $\lfloor \ell_+(\mathcal{G}_{ev}(25,6)) \rfloor$ and $\lfloor \ell_{++}(\mathcal{G}_{ev}(20,8)) \rfloor$ improve the upper bound given by Schrijver by one. See Table 1 below (the values given there are the bounds rounded down to the nearest integer). For other instances (n,d), the bounds ℓ_+ and ℓ_{++} give an improvement over Schrijver's bound limited to some decimals, thus yielding no improved upper bound on $A(n,d)$. Our computations were made using the **NEOS Server for Optimization**, which can be accessed at `http://www-neos.mcs.anl.gov/`, and we used specifically the software DSDP for semidefinite programming.

We indicate in Table 2 the sizes of the semidefinite programs involved in our computations. (In the 'block sizes' column in Table 2, $-N$ indicates that the last block is a diagonal matrix of order N.)

(n,d)	Delsarte bound	Schrijver bound $\ell_{sch}(\mathcal{G}_{ev}(n,d))$	$\ell_+(\mathcal{G}_{ev}(n,d))$	$\ell_{++}(\mathcal{G}_{ev}(n,d))$	$\ell^0_+(\mathcal{G}(n,d))$	$\ell^0_{++}(\mathcal{G}(n,d))$
(20,8)	290	274	274	273	274	273
(25,6)	48148	47998	47997	47997	47998	47998

Table 1

bound	# var.	# blocks	block sizes
$\ell_+(\mathcal{G}_{ev}(25,6))$	131	27	13 14 12 12 11 11 10 10 9 9 8 8 7 7 6 6 5 5 4 4 3 3 2 2 1 1 -436
$\ell_+(\mathcal{G}_{ev}(20,8))$	43	23	11 12 9 9 9 9 7 7 7 7 5 5 5 5 3 3 3 3 1 1 1 1 -128
$\ell_{++}(\mathcal{G}_{ev}(20,8))$	68	12	34 27 27 21 21 15 15 9 9 3 3 -221

Table 2: # var. means 'number of variables', # blocks means 'number of blocks'

De Klerk and Pasechnik [1] have recently applied the bound of Schrijver [13] and our bound ℓ_+ for finding tighter upper bounds for the stability number of the orthogonality graph $\Omega(n)$; $\Omega(n)$ is the graph with node set $\mathcal{P}$, with an edge (I,J) if $|I\Delta J| = n/2$ (for $I,J \in \mathcal{P}$). Namely, to obtain an upper bound for the stability number of $\Omega(n)$, they propose to use the program (42) defining Schrijver's bound, or the program (41) defining the parameter ℓ_+, replacing the constraint (36) by the constraint:

$$x_{i,j}^t = 0 \quad \text{if } \{i, j, i+j-2t\} \cap \{n/2\} \neq \emptyset.$$

The only interesting case is when n is a multiple of 4, since $\Omega(n)$ is the empty graph for n odd and $\Omega(n)$ is a bipartite graph for $n = 2 \mod 4$. The computations made by de Klerk and Pasechnik [1], quoted in Table 3 below, indicate that the bound $\ell_+(\Omega(n))$ may give a much better upper bound for $\alpha(\Omega(n))$ than Schrijver's method. This contrasts with the situation encountered in the present paper, where the bound ℓ_+ gave only a moderate improvement upon Schrijver's bound for the instances of the coding problem we have tested.

n	$\ell_+(\Omega(n))$	Schrijver's bound
16	2304	2304
20	20,166.62	20,166.98
24	183,373	184,194
28	1,848,580	1,883,009
32	21,103,609	21,723,404

Table 3: [1] Comparing the bounds for the orthogonality graph $\Omega(n)$

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