Transportation Problem

One of the main products of P&T Company is canned peas. The peas are prepared at three canneries (near Bellingham, Washington; Eugene, Oregon; and Albert Lea, Minnesota) and are then shipped by truck to four distributing warehouses in Sacremento, California; Salt Lake City, Utah; Rapid City, South Dakota; and Albuquerque, New Mexico. Because shipping costs are a major expense, management has begun a study to reduce them. For the upcoming season, an estimate has been made of what the output will be from each cannery, and how much each warehouse will require to satisfy its customers. The shipping costs from each cannery to each warehouse has also been determined. This is summarized in the next table.

Shipping cost per truckload	Warehouse 1	2	3	4	Output
Cannery 1	464	513	654	867	75
Cannery 2	352	416	690	791	125
Cannery 3	995	682	388	685	100
Requirement	80	65	70	85	

You should find it an easy exercise to model this as a linear program. If we let x_{ij} be the number of truckloads shipped from cannery **i** to warehouse **j**, the problem is to

$$\begin{array}{ll}
\text{minimize} & 464x_{11} + 513x_{12} + 654x_{13} + 867x_{14} + 352x_{21} + \ldots + 685x_{34} \\
\text{subject to} & \\
& x_{11} + x_{12} + x_{13} + x_{14} = 75 \\
& x_{21} + x_{22} + x_{23} + x_{24} = 125 \\
& x_{31} + x_{32} + x_{33} + x_{34} = 100 \\
& x_{11} + x_{21} + x_{31} = 80 \\
& x_{12} + x_{22} + x_{32} = 65 \\
& x_{13} + x_{23} + x_{33} = 70 \\
& x_{14} + x_{24} + x_{34} = 85 \\
& x_{ij} \geq 0 \text{ for all } i \text{ and } j.
\end{array}$$

This is an example of the **transportation model**. As has been pointed out, this problem has a lot of nice structure. All the coefficients are 1 and every variable appears in exactly two constraints. It is this structure that lets the simplex algorithm be specialized into an extremely efficient algorithm.

What defines a transportation model? In general, the transportation model is concerned with distributing (literally or figuratively) a commodity from a group of supply centers, called **sources** to a group of receiving centers, called **destinations** to minimize total cost.

In general, source **i** has a supply of s_i units, and destination **j** has a demand for d_j units. The cost of distributing items from a source to a destination is proportional to the number of units. This data can be conveniently represented in a table like that for the sample problem.

We will generally assume that the total supply equals the total demand. If this is not true for a particular problem, *dummy* sources or destinations can be added to make it true. The text refers to such a problem as a *balanced transportation problem*. These dummy centers may have zero distribution costs, or costs may be assigned to represent unmet supply or demand.

For example, suppose that cannery 3 makes only 75 truckloads. The total supply is now 25 units too little. A dummy supply node can be added with supply 25 to balance the problem, and the cost from the dummy to each warehouse can be added to represent the cost of not meeting the warehouse's demand.

The transportation problem has a couple of nice properties:

Feasibility
: As long as the supply equals demand, there exists a feasible solution to the problem.

Integrality
: If the supplies and demands are integer, every basic solution (including optimal ones) have integer values. Therefore, it is not necessary to resort to integer programming to find integer solutions. Linear programming suffices. Note that this does not mean that each destination will be supplied by exactly one source.

In the pea shipping example, a basic solution might be to ship 20 truckloads from cannery 1 to warehouse 2 and the remaining 55 to warehouse 4, 80 from cannery 2 to warehouse 1 and 45 to warehouse 2 and, finally, 70 truckloads from cannery 3 to warehouse 3 and 30 to warehouse 4. Even though the linear programming formulation of the pea shipping example has seven constraints other than nonnegativity, a basic solution has only six basic variables! This is because the constraints are linearly dependent: the sum of the first three is identical to the sum of the last four. As a consequence, the feasible region defined by the constraints would remain the same if we only kept six of them. In general, a basic solution to the transportation model will have a number of basic variables equal to the number of sources plus the number of destinations minus one.

We close this section by showing how the dual problem can be used to verify whether a solution, such as the one given above, is optimal. In our example, the dual linear program is

$$
\begin{array}{ll}
\text{Maximize} & 75u_1 + 125u_2 + 100u_3 + 80v_1 + 65v_2 + 70v_3 + 85v_4 \\
\text{Subject to} & \\
& u_1 + v_1 \leq 464 \\
& u_1 + v_2 \leq 513 \\
& u_1 + v_3 \leq 654 \\
& u_1 + v_4 \leq 867 \\
& u_2 + v_1 \leq 352 \\
& \ldots \\
& u_3 + v_4 \leq 685 \\
& u_i, v_j \text{ unrestricted for all } i \text{ and } j.
\end{array}
$$

The fact that one of the primal constraints is redundant means that one of the dual variables can be set to zero. For example we can set $u_1 = 0$. Now we apply the complementary slackness conditions. If the solution $x_{12} = 20, x_{14} = 55, x_{21} = 80, x_{22} = 45, x_{33} = 70, x_{34} = 30$ is optimal, then the dual constraints corresponding to these basic variables must be tight:

$$
\begin{aligned}
u_1 + v_2 &= 513 \\
u_1 + v_4 &= 867 \\
u_2 + v_1 &= 352 \\
u_2 + v_2 &= 416 \\
u_3 + v_3 &= 388 \\
u_3 + v_4 &= 685.
\end{aligned}
$$

Since we can choose $u_1 = 0$, this linear system of equations is very easy to solve. We get $v_2 = 513$, $v_4 = 867$. This implies $u_2 = -97$ and $u_3 = -182$. Finally $v_1 = 449$ and $v_3 = 570$. Is this dual solution a feasible dual solution? We must check the constraints.

$$
\begin{array}{lll}
u_1 + v_1 = 0 + 449 & \leq 464 \\
u_1 + v_3 = 0 + 570 & \leq 654 \\
u_2 + v_3 = -97 + 570 & \leq 690 \\
u_2 + v_4 = -97 + 867 & \leq 791 \\
u_3 + v_1 = -182 + 449 & \leq 95 \\
u_3 + v_2 = -182 + 513 & \leq 682.
\end{array}
$$

Since we have primal and dual feasible solutions which satisfy the complementary slackness conditions, these solutions are optimal.

Exercise 5 *Consider the following transportation problem*

Shipping cost		Destination 1	2	3	4	Supply
Source	1	6	0	2	11	15
	2	12	7	9	20	30
	3	0	4	0	8	5
Demand		10	15	15	10	

and the following solution: $x_{12} = 5$, $x_{14} = 10$, $x_{21} = 5$, $x_{22} = 10$, $x_{23} = 15$, $x_{31} = 5$. *Use duality to determine whether this is an optimal solution.*

Michael A. Trick
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