# Minimum Cost Flow

This is the most general model we will look at. Like the maximum flow problem, it considers flows in networks with capacities. Like the shortest path problem, it considers a cost for flow through an arc. Like the transportation problem, it allows multiple sources and destinations. Like the transshipment problem, it allows nodes between sources and destinations. In fact, all of these problems (all the models you have seen except minimum spanning tree) can be seen as *special cases* of the minimum cost flow problem.

Consider a directed network with **n** nodes. The decision variables are $x_{ij}$, the flow through arc $(i,j)$.

The given information includes:

- $c_{ij}$: cost per unit of flow from **i** to **j** (may be negative),
- $u_{ij}$: capacity (or upper bound) on flow from **i** to **j**,
- $b_i$: net flow generated at **i**.

This last value has a sign convention:

- $b_i > 0$ if **i** is a supply node,
- $b_i < 0$ if **i** is a demand node,
- $b_i = 0$ if **i** is a transshipment node.

The objective is to minimize the total cost of sending the supply through the network to satisfy the demand.

Note that for this model, it is not necessary that every arc exists. We will use the convention that summations are only taken over arcs that exist. The linear programming formulation for this problem is:

$$\begin{array}{ll} \text{Minimize} & \sum_i \sum_j c_{ij} x_{ij} \\ \text{Subject to} & \sum_j x_{ij} - \sum_j x_{ji} = b_i \text{ for all nodes } i, \\ & 0 \le x_{ij} \le u_{ij} \text{ for all arcs } (i,j). \end{array}$$

Again, we will assume that the network is balanced, so $\sum_i b_i = 0$, since dummies can be added as needed. We also still have a nice integrality property. If all the $b_i$ and $u_{ij}$ are integral, then the resulting solution to the linear program is also integral.

Minimum cost network flows are solved by a variation of the simplex algorithm and can be solved more than 100 times faster than equivalently sized linear programs. From a modeling point of view, it is most important to know the sort of things that can and cannot be modeled in a single network flow problem:

Can do

1. Lower bounds on arcs. If a variable $x_{ij}$ has a lower bound of $l_{ij}$, upper bound of $u_{ij}$, and cost of $c_{ij}$, change the problem as follows:
   - Replace the upper bound with $u_{ij} - l_{ij}$,
   - Replace the supply at **i** with $b_i - l_{ij}$,
   - Replace the supply at **j** with $b_i + l_{ij}$,

   Now you have a minimum cost flow problem. Add $c_{ij} l_{ij}$ to the objective after solving and $l_{ij}$ to the flow on arc $(i, j)$ to obtain a solution of the original problem.

2. Upper bounds on flow through a node. Replace the node **i** with nodes $i'$ and $i''$. Create an arc from $i'$ to $i''$ with the appropriate capacity, and cost 0. Replace every arc $(j, i)$ with one from **j** to $i'$ and every arc $(i, j)$ with one from $i''$ to **j**. Lower bounds can also be handled this way.

3. Convex, piecewise linear costs on arc flows (for minimization). This is handled by introducing multiple arcs between the nodes, one for each portion of the piecewise linear function. The convexity will assure that costs are handled correctly in an optimal solution.

Can't do

1. Fixed cost to use a node.
2. Fixed cost to use an arc.
3. ''Proportionality of flow.'' That is, if one unit enters node **i**, then you insist that .5 units go to node **j** and .5 to node **k**.
4. Gains and losses of flow along arcs, as in power distribution.

Note that although these cannot be done in a single network, it may be possible to use the solutions to multiple networks to give you an answer. For instance, if there is only one arc with a fixed cost, you can solve both with and without the arc to see if it is advantageous to pay the fixed cost.

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