# Traveling Salesperson Problem

Consider a traveling salesperson who must visit each of 20 cities before returning home. She knows the distance between each of the cities and wishes to minimize the total distance traveled while visiting all of the cities. In what order should she visit the cities?

Let there be **n** cities, numbered from 1 up to **n**. For each pair of cities $(i, j)$ let $c_{ij}$ be the cost of going from city **i** to city **j** or from city **j** to city **i**. Let's let $x_{ij}$ be 1 if the person travels between cities **i** and **j** (either from city **i** to city **j** or from **j** to **i**). This problem is known as the *symmetric* TSP. In the *asymetric* TSP, the cost to travel in one direction may differ from the cost to travel in the other, and the decision variables must distinguish between the two directions. Clearly the asymetric problem is the more general.

The objective is to minimize $\sum_{i=1}^{n} \sum_{j=1}^{i-1} c_{ij} x_{ij}$. The constraints are harder to find. Consider the following set:

$$\sum_{j \neq i} x_{ij} = 2 \text{ for all } i.$$

These constraints say that every city must be visited. These constraints are not enough, however, since it is possible to have multiple cycles (*subtours*), rather than one big cycle (*tour*) through all the points. To handle this condition, we can use the following set of *subtour elimination constraints*:

$$\sum_{i \in S} \sum_{j \notin S} x_{ij} \geq 2 \text{ for all } S \subset N.$$

This set states that, for any subset of cities **S**, the tour must enter and exit that set. These, together with $x_{ij} \in \{0, 1\}$ is sufficient to formulate the traveling salesperson problem (TSP) as an integer program.

Note however, that there are a tremendous number of constraints: for our 20-city problem, that number is roughly 524,288. For a 300-city problem, this would amount to 101851798816724304313422284420468908052573419683296 8125318070224677190649881668353091698688 constraints. Try putting that into LINGO!

Despite the apparent complexity of this formulation, it lies at the heart of the most promising current approach to solving medium (100--1000 city) TSPs. We will see this again in the cutting plane section.

Other forms of subtour elimination constraints are possible, but they suffer from the same explosive rate of growth in number. Subtour elimination constraints for the asymetric problem are equally difficult.

The traveling salesman problem has come up twice in my recent consulting. The first had to do with operating cellular telephone manufacturing robots. An American producer of cellular telephones had reached its production capacity with a lot of unsatisfied demand. The bottleneck in the production process were a set of automatic placement machines that placed components onto printed circuit boards. The manufacturer of the machines had a method for sequencing the placements in order to decrease production time. Was their method optimal? Could it be improved? Currently, the producer had 30 machines. Each new machine cost approximately $1 million. The problem of optimally operating the machines is a traveling salesman problem. It takes a certain time to move from one placement to the next. Minimizing the time requires finding the optimal route that visits all the placements.

The second example occured just last week (October 1995). I was approached by Bayer to look at one of their production facilities. In this chemical plant, there were a number of products that had to be produced on a number of machines. There are a number of delays when production changed from one product to the next. In particular, there are cleanup times to get the machine clean enough for the next product. Depending on the products involved, this change may be as little as a spray with a hose to as long as a three man crew with toothbrushes. Determining the optimal ordering is a traveling salesman problem.

**Exercise 2** (Optional) *Orders from 5 different destinations are delivered from a central warehouse. Routes are assigned to different carriers. Suppose that there are 9 feasible routes with each route specifying the destinations to which orders are delivered.*

*Route 1: 1,3,4*
*Route 2: 1,4,5*
*Route 3: 1,2,5*
*Route 4: 2,3,5*
*Route 5: 2,4,5*
*Route 6: 3,4,5*
*Route 7: 1*
*Route 8: 2,3*
*Route 9: 4,5*

*The length of the routes are 10, 12, 12, 13, 11, 9, 7, 8 and 8 respectively. Formulate an integer program that will result in the shortest total distance routing. Each destination must be on at least one of the chosen routes.*

**Exercise 3** (Optional) *Consider a job-shop scheduling problem involving eight operations on a single machine with a total of two end products. Let* $p_j$ *be the processing time for the* $j$*th operation. Since each operation requires a machine setup, it is assumed that any operation once started must be completed before a new operation can be undertaken on the machine. The sequencing of the operations must satisfy the following precedence relationships: Operations 1 and 2 must precede Operation 4 (i.e. both Operations 1 and 2 must be completed before Operation 4 can be started), Operations 4 and 5 precede 7, Operation 3 precedes 6 and, finally, Operations 5 and 6 precede 8. Once Operation 7 is completed, Product 1 is available. Similarly, once Operation 8 is completed, Product 2 is available. The due date for Product 1 is* $d_1$*. The objective is to minimize the delivery date of Product 2 while meeting the due date on Product 1.*

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