C&O 370

Midterm Examination

Winter 1996

1. A company consisting of three factories must produce 380 units of product p and 425 units of product q. Factory #1 can produce 8 units of p plus 5 units of q per hour at a cost of \$47. Factory #2 can produce 6 units of p plus 10 units of q per hour at a cost of \$54, and factory #3 can produce 4 units of p plus 12 units of q per hour at a cost of \$56.

   (a) [15 marks]
   Develop a GAMS-like model for determining the number of hours each factory should run in order to meet the demands and minimize total production cost.

   (b) [10 marks]
   Verify that an optimum solution to your model of part (a) is obtained by running factory #1 for 25 hours, factory #2 for 30 hours and not running factory #3 at all. As part of your verification argument, explicitly state the dual of your model and the complementary slackness equations resulting from this proposed solution.

2. Consider the linear program:

$$
\begin{array}{llrcrcl}
(P) & \text{maximize} \quad z = & x_1 & + & x_2 & & \\
 & \text{subject to:} & -x_1 & + & 2x_2 & \leq & 1 \\
 & & 12x_1 & + & 4x_2 & \leq & 9 \\
 & & x_1 & , & x_2 & \geq & 0.
\end{array}
$$

After introducing slack variables, a tableau for (P) is:

$$
\begin{array}{ccccccccc}
 & & x_2 & + & \frac{12}{28}x_3 & + & \frac{1}{28}x_4 & = & \frac{21}{28} \\
 & x_1 & & - & \frac{4}{28}x_3 & + & \frac{2}{28}x_4 & = & \frac{14}{28} \\
z & & & + & \frac{8}{28}x_3 & + & \frac{3}{28}x_4 & = & \frac{35}{28}.
\end{array}
$$

   (a) [5 marks]
   Verify, as in 1.(b), that the basic solution of this tableau is (P)-optimum.

(b) [5 marks]
Use the given tableau for (P) and sensitivity techniques to solve the linear program:

$$
\begin{array}{llcrcrcl}
(P)' & \text{maximize} & z = & x_1 & + & x_2 & & \\
 & \text{subject to:} & & -x_1 & + & 2x_2 & \leq & 1 \\
 & & & 12x_1 & + & 4x_2 & \leq & 9 \\
 & & & x_1 & + & 2x_2 & \leq & 1 \\
 & & & x_1 & , & x_2 & \geq & 0.
\end{array}
$$

Note that $(P)'$ is (P) with an added constraint.

(c) [5 marks]
Use the given tableau for (P) and sensitivity techniques to solve the linear program:

$$
\begin{array}{llcrcrcrcl}
(P)'' & \text{maximize} & z = & x_1 & + & x_2 & + & x_5 & & \\
 & \text{subject to:} & & -x_1 & + & 2x_2 & + & 3x_5 & \leq & 1 \\
 & & & 12x_1 & + & 4x_2 & + & 2x_5 & \leq & 9 \\
 & & & x_1 & , & x_2 & , & x_5 & \geq & 0.
\end{array}
$$

Note that $(P)''$ is (P) with an added column.

(d) [10 marks]
Consider the parametric linear program:

$$
\begin{array}{llcrcrcl}
(P)''' & \text{maximize} & z = & (1+\theta)x_1 & + & (1-\theta)x_2 & & \\
 & \text{subject to:} & & -x_1 & + & 2x_2 & \leq & 1+\theta \\
 & & & 12x_1 & + & 4x_2 & \leq & 9-\theta \\
 & & & x_1 & , & x_2 & \geq & 0.
\end{array}
$$

Use the given tableau for (P) and parametric techniques to determine all values of θ for which $B = (2, 1)$ is an optimum basis for $(P)'''$.
Note that $(P)'''$ is (P) with simultaneous parameterization of both the c and b vectors.

3. A manufacturer wants to plan his production schedule for a three month period. The following table contains all of the relevant cost, production capacity and end-of-month demand data. Note that a penalty cost is incurred for surplus (unused) production capacity. Production levels must be at least **50 units** in each of the three months, and inventory-holding levels must not exceed **40 units** in each of February and March. No inventory is to be held past the end of March.

Month	Costs ($/unit)			Production	End-of-Month
	Production	Holding	Penalty	Capacity	Demand
JAN	10	-	2	100	50
FEB	20	8	2	100	100
MAR	30	8	2	100	75

Table: Cost & "Demand" Data

The manufacturer wants you to determine a production schedule which meets these demands, subject to the conditions stated, and minimizes total cost.

You recognize this as an instance of a network programming problem of the form:

$$(P) \text{ minimize } z = cx : \; Ax = b, \quad \ell \leq x \leq u,$$

where A is the incidence matrix of some digraph (network) $G = (V, E)$.

(a) [5 marks]
Draw network $G = (V, E)$, labelled with the given costs, (nonzero) lower capacities, (finite) upper capacities and "demands".

(b) [5 marks]
Consider a production schedule requiring production levels of 90, 85 and 50, respectively for January, February and March. Extend this schedule to a vector, $x^* = x^*(E)$, which is feasible for (P).

(c) [5 marks]
Write down the dual, (D), of (P), **explicitly** for the given data. That is, do **not** use a GAMS-like formulation. In particular, for each arc, $(\alpha, \beta) \in E$, write down the dual constraint corresponding to (α, β).

(d) [10 marks]
Write down the complementary slackness equations corresponding to vector x^* from part (b), and solve them. Then, explain why vector x^* is optimum for (P).

Hint: When solving the complementary slackness equations, let the dual variables indexed by the start-of-month nodes be equal to 0.

4. In digraph $G = (V, E)$ below, the node-set, V, represents a set of relay stations; the arc-set, E, represents a set of (one-way) communication channels, and, the number, $u(\alpha, \beta)$, shown on each arc, $(\alpha, \beta) \in E$, represents the number of parallel communication channels through which a message can be sent directly from station α to station β.

   (a) [15 marks]
   An st-**vulnerability set**, C, is a subset of channels, the destruction of which would make communication from station s to station t impossible. Find an st-vulnerability set, C^*, of minimum cardinality (size).

   (b) [10 marks]
   Assuming the truth of any relevant theorems from lectures and/or homework readings, prove that your answer to part (a) is correct.