C&O 370

Cutting Stock Problem (CSP)/Relaxed Cutting Stock Problem (RCSP)

- **Reference**, Chvatal, Linear Programming, pp. 195-211

- **General** The reference contains an outstanding treatment of the (standard) Cutting Stock Problem (CSP). Included is:

  (i) a mathematical formulation (model) of the problem;

  (ii) a solution algorithm (revised simplex method with a knapsack (KP) subroutine for determining the entering column);

  (iii) a branch-and-bound algorithm for solving (KP);

  (iv) reductions of some non-standard problem versions to instances of the standard version;

  (v) a detailed solution to a sample problem.

  This handout contains a summary of the model formulation and solution algorithm in a GAMS-like format together with a suggested format for a Solution Summary display.

- **Model (Standard Cutting Stock Problem)**

  Sets & Variables

I	$:=$	finals-set $= /1, 2, \ldots, m/$
J	$:=$	cutting pattern-set $= /I/j(1), j(2), \ldots, j(n), \ldots / \cdots$ (implicit input)
$X(J)$	$:=$	rows-requirement variable $\cdots$ $(\forall_{j \in J}, X(j) :=$ number of rows required using cutting pattern $-j$.)
Z	$:=$	objective...(total number of rows required to fill all demands)

  Data

$w(I)$	$:=$	widths(units) of finals $= /w(1), w(2), \ldots, w(m)/$
$b(I)$	$:=$	demand for finals $= /b(1), b(2), \ldots, b(m)/$
$C(J)$	$:=$	unit "costs" of rows $= /1 * 1/ \cdots$ (implicit input all ones)
W	$:=$	width (same units as $w(I)$ of each row
$A(I, J)$	$:=$	cutting pattern matrix $= /a(I) \in /N^I : \mathrm{SUM}(I, w(I) * a(I)) \leq W/$ $\ldots (\forall_{(i,j)} \in (I, J), a_{ij} :=$ number of type $-i$ final widths specified by cutting pattern $-j$.) $\cdots$ (implicit input)

Program (RCSP)

(0) minimize $Z = \text{SUM}\ (J, C(J) * X(J)) \ldots$ (total number of rows required) subject to:

(1) $\text{SUM}\ (J, A(I,J) * X(J)) = b(I) \ldots$ (demands for finals)

(2) $X(J) \geq \underline{0}$.

Program CSP
Solve (RCSP) subject to:

(3) $X(J)$ integral.

- Solving (CSP) Approximately

  1. Solve (RCSP) by the Revised Simplex Algorithm, and let $[Z|X(J)] = [Z^*|X^*(B)|\underline{0}]$ be an optimum solution.
  2. If $[Z^*|X^*(B)]$ is integral, then it solves (CSP).
  3. Otherwise, find an approximate solution to (CSP) as follows:

     3.1 Solve (heuristically or otherwise) the much simpler (CSP) whose demands are $\bar{b}(I) := b(I) - \text{SUM}\ (B, A(I,B) * \lfloor X^*(B) \rfloor)$ and let $[\bar{Z}|\bar{X}(\bar{B})]$ be an optimum solution.

     3.2 Then, where $X^0(B|\bar{B}) := [\lfloor X^*(B) \rfloor | \bar{X}(\bar{B})]$ and $Z^0 = \text{SUM}\ (B, \lfloor X^*(B) \rfloor) + \bar{Z}$, $[Z^0|X^0(B|\bar{B})]$ is an approximate-optimum to (CSP). (Of course if you are lucky, and $Z^0 - Z^* < 1$, then $[Z^0|X^0(B|\bar{B})]$ is an optimum solution to (CSP). You should always check to see if this is the case.)

We describe how to solve 1. in the next section.

- Revised Simplex Algorithm for Solving (RCSP)

  0. Input

     0.0 (Explicit) Model Data: $I, w(I), b(I)$ and W

     0.1 Initial Basis: $B = I$

     0.2 Initial Basis Matrix:
     $A(I,B)$, given by: $\forall_{(i,j)} \in (I,B), a_{ij} = \begin{cases} \left\lfloor \frac{W}{w(i)} \right\rfloor, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$

     0.3 Initial Basic Solution: $X^*(B) - /b(B)/A(B,B)/$.

     0.4 Initial Iteration Setting: $\alpha = 1$.

  1. Solve $y(I) * A(I,B) = 1(B)$, obtaining $y(I) = y^*(I)$.

2. Determine Entering Column, $S \in J - B$

   2.1 Solve the (KP):

$$\begin{bmatrix} (\bar{0}) & \text{minimize } \xi = 1 - \text{SUM } (I, y^*(I) * a(I)) : \\ (\bar{1}) & \text{SUM } (I, w(I) * a(I)) \leq W \\ (\bar{2}) & a(I) \in /\mathrm{N}^I \end{bmatrix}$$

   obtaining optimum solution, $[\xi^* | a^*(I)]$.

   2.2 If $\xi^* \geq 0$, then go to step 7. Otherwise, set $s = j(\alpha)$, (a symbolic index), and set $A(I, s) = a^*(I)$.

3. Solve $A(I, B) * d(B, s) = A(I, s)$, obtaining $d(B, s) = d^*(B, s)$.
4. Find the largest value, t^*, of t such that $X^*(B) - td^*(B, s) \geq \underline{0}$.
5. Determine Leaving Column, $r \in B$
   Choose $r \in B$ such that $d^*(r, s) > 0$ and $x^*(r) - t^* d^*(r, s) = 0$.
6. Update B and $X^*(B)$

   6.1 Set $X^*(s) = t^*$, and replace $X^*(B)$ by $X^*(B) - t^* d^*(B, s)$.

   6.2 Replace B by $B + s - r$.

   6.3 Replace α by $\alpha + 1$, and go to step 1.

7. Optimum Solution to (RCSP)

   7.1 Compute $V^* = \text{SUM } (B, X^*(B)) \ldots (= y^*(I) * b(I))$.

   7.2 $[Z | X(J)] = [V^* | X^*(B) | \underline{0}]$ is (RCSP)-optimum with corresponding cutting pattern matrix, $A(I, B)$.

- Explanatory Notes on Simplex Algorithms for (RCSP)

  (i) If (RCSP) had A as explicit input, then step 2 of the (revised) simplex algorithm would read:
  "if minimum $\{(1 - y^* A_j) : j \in J - B\} \geq 0$, then go to step 7."
  Although A is not explicit input for (RCSP), its column-set is precisely the solution-set to the (KP) constraints $(\bar{1})$ and $(\bar{2})$ in 2. Accordingly, we realize step 2 of the simplex algorithm for (RCSP) by solving the (KP) of 2.

  (ii) The strange looking iteration setting, initialized as $\alpha = 1$ in step 0.4, updated by $\alpha \leftarrow \alpha + 1$ in step 6.3 and used to index s in step 2.2, is merely a device for achieving a distinct index (name) for s, without having to scan previously indexed columns.

  (iii) As described, the simplex algorithm for (RCSP) is unrefined in the sense that it does not contain a tie-breaking rule for determining a specific pivot-pair (s, r), when more than one might qualify by satisfying the conditions of (step 2, step 5). We therefore have no assurance of its convergence for all instances of the input data.

This can be rectified by changing step 2 and step 5 so that Bland's Smallest-Subscript Rule can be implemented. The details are left as an exercise for you.

- Solution Summary Display (Input/Output)

	PATTERN-SET	ADD. PATTERN-SET	
FINAL WIDTHS	CUTTING PATTERNS (RCSP)	ADDITIONAL CUTTING PATTERNS (CSP)	FINAL DEMANDS
RAW WIDTH	RAWS-REQUIREMENT (RCSP)		TOTAL RAWS (RCSP)
	RAWS-REQUIREMENT (CSP)		TOTAL RAWS (CSP)

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

	B	$\bar{B}$	
$w(I)$	$A(I, B)$	$A(I, \bar{B})$	$b(I)$
W	$X^*(B)$		Z^*
	$X^0(B\|\bar{B}) = [\lfloor X^*(B) \rfloor \| X^0(\bar{B})]$		Z^0

- Revised Simplex Tableaus

  The following tableau format is sufficient for recording the relevant information determined at each iteration of the (revised) simplex algorithm for solving (RCSP).

B	$A(I, B)$	$X^*(B)$	$A(I, s) = a^*(I)$	$\frac{d^*(B,s)}{d^*(r,s)}$
Z	$y^*(I)$	v^*	ξ^*	t^*

Cutting Stock Example (Chvatal pp. 199-200)

- Solution Summary Display (Input/Output)

	B	$\bar{B}$	
$w(I)$	$A(I,B)$	$A(I,\bar{B})$	$b(I)$
W	$X^0(B)$		Z^*
	$X^0(B\|\bar{B}) = [X^0 * (B)\|X^0(\bar{B})]$		Z^0

=

	6 2 5 4		
25.5	2 0 2 0		78
22.5	1 4 0 0		40
20	0 0 2 0		30
15	1 0 0 6		30
91	24 4 15 1		44
	24 4 15 1		44

- Computation Summary

B	$A(I,B)$	$X^*(B)$	$A(I,s) = a^*(I)$	$d^*(B,s)$
Z	$y^*(I)$	v^*	ξ^*	t^*

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

1	3	0	0	0	26	2	2/3
2	0	4	0	0	10	0	0
3	0	0	4	0	7.5	2	1/2
4	0	0	0	6	5	0	0
Z	1/3	1/4	1/4	1/6		-1/6	15
1	3	0	2	0	16	2	2/3
2	0	4	0	0	10	1	1/4
5	0	0	2	0	15	0	0
4	0	0	0	6	5	1	1/6
Z	1/3	1/4	1/6	1/6		-1/12	24
6	2	0	2	0	24		
2	1	4	0	0	4		
5	0	0	2	0	15		
4	1	0	0	6	1		
Z	7/24	1/4	5/24	1/6	44	0	