Name: ____________________

I.D. #: ____________________

Section: ____________________

C&O 350 Linear Programming
Mid-Term Examination
Winter 1996
Thursday, 15 Feb., 1996
7:00 - 9:00 p.m. (2 hours)

1. Print your name and i.d. and section number in the space provided above.
2. This examination consists of 6 questions and 10 pages plus the cover page. Make sure you have a complete paper.
3. No calculators are allowed.
4. Write answers in space provided. If you need more space, use back of previous page.

Section 1	☐ Wolkowicz	TR	1:00-2:30
Section 2	☐ Cunningham	TR	10:00-11:30

Question	Mark Value	Student Mark
1	24	
2	16	
3	17	
4	19	
5	14	
6	10	
Total	100	

1. Consider the following linear programming problem (LP):

$$
\begin{array}{rrcrcl}
\text{maximize} & 2x_1 & + & x_2 & & \\
\text{subject to} & x_1 & - & 2x_2 & \leq & 1 \\
 & & & x_2 & \leq & 2 \\
 & x_1 & & & \leq & 5 \\
 & -x_1 & + & x_2 & \leq & 1 \\
 & x_1 & , & x_2 & \geq & 0
\end{array}
$$

[12] (a) Solve the LP by applying the simplex method.
(Use the largest coefficient rule for choosing the entering variable and the smallest subscript rule for choosing the leaving variable.)

1. continued,

[4] (b) For the final basis in (a) give the the basic solution, the basis B, the basis matrix A_B and inverse A_B^{-1}.

[4] (c) Is the optimal basic solution degenerate? If so, find another basis determining it.

[4] (d) Draw the set of feasible solutions of (LP) and indicate the basic feasible solutions.

2. Consider the linear programming problem (P) below:

$$
\begin{array}{lrrrrr}
\text{maximize } z = & & +x_2 & x_3 & & \\
\text{subject to} & & & & & \\
& -x_1 & +x_2 & -2x_3 & = & 4 \\
& 2x_1 & -x_2 & +x_3 & = & -1 \\
& x_1, & x_2, & x_3 & \geq & 0.
\end{array}
$$

[4] (a) Formulate a phase I linear programming problem (P') for finding a feasible solution to (P), and find a feasible tableau (or dictionary) for (P').

[8] (b) Solve (P') by the simplex method.

2. (b) continued,

[4] (c) From your solution to (b), find a feasible tableau (or dictionary) for (P). **Do not solve** (P).

[3] 3. (a) Define the term "basic solution" of $Ax = b$. (You may assume that A is $m \times n$ with rank m.) Your definition should not use other terms from linear programming, but it may use terms from linear algebra.

[3] (b) Let x^* be a feasible solution of a linear programming problem (LP). Define what it means for x^* to be an **extreme point** of the set of feasible solutions.

[3] (c) Suppose that the constraints of problem (LP) of part (c) are $Ax = b$, $x \geq 0$, where A has m rows and rank m. State a theorem giving a condition for x^* to be an extreme point of the set of feasible solutions.

(d) For each of the following statements, tell whether the statement is true or false. You do not need to give a reason for your answer.

[2] (i) If the simplex method applied to a linear programming problem fails to terminate, then the problem must be degenerate.

[2] (ii) If a linear programming problem is unbounded, then it has infinitely many optimal solutions.

[2] (iii) If a linear programming problem has more than one optimal solution, then it has infinitely many optimal solutions.

[2] (iv) If in an iteration of the simplex method, x_k is chosen as the entering variable, and there is more than one choice for the leaving variable, then the new basic solution will be degenerate.

4. (a) Consider the tableau

$$
(T') \qquad
\begin{array}{rrrrrrrcl}
 & & 2x_2 & +x_3 & & -x_5 & +x_6 & = & 4 \\
 & & x_2 & -x_3 & +x_4 & +x_5 & & = & 2 \\
 & x_1 & +2x_2 & +2x_3 & & +3x_5 & & = & 6 \\
z & & -2x_2 & +2x_3 & & -3x_5 & & = & 5
\end{array}
$$

Notice that the bottom row could also be written

$$-z + 2x_2 - 2x_3 + 3x_5 = 5.$$

[2] (i) List all the pairs (x_k, x_r) such that x_k could be the entering variable and x_r could be the leaving variable on a simplex iteration beginning from (T').

[2] (ii) List all the pairs if the largest-coefficient rule for choosing the entering variable is being used.

[2] (iii) List all the pairs, if the smallest subscript rules for choosing the entering and leaving variables are being used.

[4] (iv) List all the pairs if the lexicographic rule for choosing the leaving variable is used. (Here you should use the fact that the initial feasible basis was $(4, 5, 6)$.)

4. continued,

(b) Consider the tableau

$$(T) \qquad \begin{array}{rcrcl} x_i & + & \sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i \quad (i \in B) \\ z & - & \sum_{j \in N} \bar{c}_j x_j & = & \bar{v}. \end{array}$$

Notice that another way to write the last row is

$$-z + \sum_{j \in N} \bar{c}_j x_j = -\bar{v}.$$

[3] (i) Give the basic solution associated with (T).

[3] (ii) Suppose that x_k is chosen as the entering variable, and x_r as the leaving variable. Give a formula for the value of x_k in the new basic solution.

[3] (iii) What does it mean for an iteration or pivot of the simplex method beginning from (T) to be "degenerate"?

5. Consider the problem

$$
(P) \qquad \begin{array}{lrcll}
\text{Maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & & & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, 2, \ldots, m) \\
& x_j & \geq & 0 & (j = 1, 2, \ldots, n).
\end{array}
$$

[2] (a) Write the dual, (D), of (P).

[3] (b) State the Weak Duality relation (or property) for (P), (D) .

[4] (c) Prove the Weak Duality relation for (P), (D).

5. continued,

[2] (d) Consider the linear programming problem

$$
(P) \quad \begin{array}{lrcrcrcr}
\text{maximize} & 6x_1 & + & 3x_2 & + & 4x_3 & & \\
\text{subject to} & x_1 & + & 2x_2 & + & x_3 & \leq & 8 \\
 & +5x_1 & + & 4x_2 & + & 3x_3 & \leq & 25 \\
x_j \geq 0, \ \forall\, j & & & & & & &
\end{array}
$$

Write the dual (D') of (P').

[3] (e) Notice that the point $x = (1/2, 0, 15/2)^t$, is feasible for (P'). Is it optimal? Justify your answer. (Hint: Consider the point $y = (1, 1)$.)

[5] 6. (a) Suppose that x^* is the unique optimal solution of a linear programming problem maximize cx subject to $Ax = b$, $x \geq 0$, where A has m rows and rank m. Must x^* be a basic solution? Justify your answer.

[5] (b) Consider the linear constraints $Ax \leq b$, $x \geq 0$. We want to determine whether the set of feasible solutions will change if we drop the first constraint. Show how this can be determined by solving one linear programming problem.