Sensitivity Analysis

Suppose that the optimal solution of a given LP problem

(P) maximize cx, subject to $Ax = b$, $x \geq 0$,

is available, and that *one* of the following modifications is made to the problem data:

1. The coefficient of a nonbasic variable in the objective function is changed.
2. The coefficient of a basic variable in the objective function is changed.
3. A new variable is added.
4. A component b_i of the right hand side b is changed.
5. A new constraint is added.

The goal is to solve the new LP problem starting from the optimal solution of the original LP problem, *without* starting from scratch.

Let A be an $m \times n$ matrix, and let the jth column of A be denoted A_j. Let B denote the basis heading of an optimal basic solution of (P), and let A_B be the basis matrix. Let N denote $\{1, \ldots, m\} \backslash B$, and let A_N be the matrix formed by $\{A_j : j \in N\}$. Recall that

$$x_B^* = A_B^{-1} b, \qquad y = c_B A_B^{-1}, \qquad \text{and}$$

$$\overline{c}_N = c_N - c_B A_B^{-1} A_N.$$

We assume that all variables are nonnegative; free variables can be handled by an extension of the methods given here.

Two types of questions arise:
(Q1) Suppose that *one* item in the problem data is chosen to be a (changeable) parameter. What is the range of values for the changing parameter such that the old optimal basis continues to be optimal for the new LP?

(Q2) Solve the new LP, i.e., either find an optimal solution for the new LP, or report that the new LP is unbounded, or report that the new LP is infeasible.

(**1**) Changing the coefficient of a nonbasic variable in the objective function.

Suppose that the coefficient of the nonbasic variable x_k in the objective function is changed from c_k to $\tilde{c}_k$.

Answering (Q1):
If the old optimal basis heading B is to remain optimal for the new LP, then the new reduced cost of x_k, which is given by the formula

$$\overline{c}_k^{new} = \tilde{c}_k - yA_k,$$

must be nonpositive. Clearly, if $\tilde{c}_k \leq yA_k$, then $\overline{c}_k^{new}$ is nonpositive, and so the old optimal basis heading B remains optimal for the new LP.

Answering (Q2):
If $\tilde{c}_k > yA_k$, then the old optimal basis heading B is **not** optimal for the new LP. In order to find a new optimal basis, we do a simplex pivot (this is the pivot introduced in Chapter 2) with x_k as the entering variable, and continue doing simplex pivots until either we find an optimal basis for the new LP (i.e., until $\overline{c}_j \leq 0$ for all nonbasic j) or we find the new LP to be unbounded.

(**2**) Changing the coefficient of a basic variable in the objective function.

Suppose that the coefficient of the basic variable x_h in the objective function is changed from c_h to $\tilde{c}_h$.

Answering (Q1):
If the old optimal basis heading B is to remain optimal for the new LP, then the new reduced cost of each nonbasic variable must remain nonpositive. That is, we must have

$$\overline{c}_N^{new} = c_N - c_B^{new} A_B^{-1} A_N \leq 0.$$

The nonpositivity of each nonbasic variable x_j may impose a constraint on the value of $\tilde{c}_h$, since we must have

$$\overline{c}_j = c_j - c_B^{new} A_B^{-1} A_j \leq 0.$$

Taking the most stringent upper bound and lower bound on $\widetilde{c}_h$ from among all these constraints gives a range of values for $\widetilde{c}_h$, and this is the answer.

Answering (Q2):
If $\widetilde{c}_h$ falls outside the above range, then the old optimal basis heading B is **not** optimal for the new LP since some $j \in N$ has $\overline{c}_j > 0$. In order to find a new optimal basis, we do a simplex pivot, and continue doing simplex pivots until either we find an optimal basis for the new LP (i.e., until $\overline{c}_j \leq 0$ for all nonbasic j) or we find the new LP to be unbounded.

(**3**) Adding a new variable.

Suppose that a new variable x_ℓ, $x_\ell \geq 0$, has to be added to the LP, where c_ℓ (the coefficient of x_ℓ in the objective function) and A_ℓ are given to us.

First, check whether the old optimal basis heading B remains optimal for the new LP by computing

$$\overline{c}_\ell = c_\ell - y A_\ell.$$

If $\overline{c}_\ell$ is nonpositive, then the old optimal basis remains optimal for the new LP. If $\overline{c}_\ell$ is positive, then the old optimal basis is **not** optimal for the new LP. Therefore we do a simplex pivot with x_ℓ as the entering variable, and continue doing simplex pivots until either we find an optimal basis for the new LP or we find the new LP to be unbounded.

(**4**) Changing b to $\widetilde{b}$ by changing b_i to $\widetilde{b}_i$.

The new basic solution given by the old optimal basis heading B is

$$x_B^{*\,new} = A_B^{-1} \widetilde{b}.$$

EXERCISE. If $\widetilde{b}_i = b_i + \Delta$, then shown that the new basic solution is given by

$$x_B^{*\,new} = x_B^{*\,old} + \Delta(i \text{ th col in } A_B^{-1}).$$

Answering (Q1):
If the old optimal basis heading B is to stay optimal after changing b_i to $\widetilde{b}_i$, then we must have $x_B^{*\,new} \geq 0$. The nonnegativity of each component of $x_B^{*\,new}$ may impose a constraint on $\widetilde{b}_i$. Taking the most stringent upper bound and lower bound on $\widetilde{b}_i$ from among these m (or fewer) constraints gives a range of values for $\widetilde{b}_i$, and this is the answer.

Answering (Q2):
If $\widetilde{b}_i$ falls outside the above range, then $x_B^{*\,new}$ is not a *feasible* solution and the old optimal basis heading B is **not** optimal for the new LP. We need the **dual** simplex method to proceed. We do a **dual** simplex pivot, and continue doing **dual** simplex pivots until either we find an optimal basis for the new LP (i.e., until $x_h^* \geq 0$ for all basic h) or we find the new LP to be infeasible.

(**5**) Adding a new constraint.

Suppose that the constraint

$$\sum_{j=1}^{n} a_{gj} x_j \leq b_g$$

has to be added to the LP. First, we introduce a slack variable x_{n+1}, $x_{n+1} \geq 0$, and obtain the equality constraint

$$x_{n+1} = b_g - \sum_{j=1}^{n} a_{gj} x_j.$$

In order to add this equality constraint to the optimal dictionary (or basis) of our example LP, we substitute the basic variables x_h, $h \in B$, by their equations in the old optimal dictionary to get an equivalent constraint. Then, we add the new equality constraint as a new row in the old optimal dictionary.

The resulting dictionary is dual feasible. If the resulting dictionary is **not** feasible, then we do **dual** simplex pivots until either we find an optimal dictionary for the new LP or we find that the new LP is infeasible.
