Integer Programming and Gomory's Cutting Plane Method

Given an integer linear programming problem

$$(IP) \quad \text{maximize } \mathbf{cx} \quad \text{subject to } \mathbf{Ax{=}b}, \;\; \mathbf{x} \geq \mathbf{0}, \quad \mathbf{x} \text{ integral},$$

the *cutting plane method* takes the LP relaxation (P) of (IP), obtained by dropping the constraints that $\mathbf{x}$ be integral, and finds an optimal dictionary for (P) by using the simplex method. Let the optimal basic solution of (P) be $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_B, \tilde{\mathbf{x}}_N)$; the nonbasic variables are $x_j,\ j \in N$. If every component of $\tilde{\mathbf{x}}$ has an integer value, then $\tilde{\mathbf{x}}$ is also the optimal solution for (IP) – nothing more needs to be done.

Suppose that one of the basic variables x_h in the optimal basic solution for (P) has a noninteger value $\overline{b}_h$. We can rewrite the equation for x_h in the optimal dictionary for (P) by splitting $\overline{b}_h$ and each $\overline{a}_{hj}$ into its integer part and its fractional part:

$$x_h = \left\{ \lfloor \overline{b}_h \rfloor - \sum_{j \in N} \lfloor \overline{a}_{hj} \rfloor x_j \right\} + \left\{ (\overline{b}_h - \lfloor \overline{b}_h \rfloor) - \sum_{j \in N} (\overline{a}_{hj} - \lfloor \overline{a}_{hj} \rfloor) x_j \right\}. \tag{1}$$

(**Notation**: For a real number r, we use $\lfloor r \rfloor$ to mean the greatest integer less than or equal to r, e.g., $\lfloor 1.3 \rfloor = 1$ and $\lfloor -1.3 \rfloor = -2$.) For example the equation $x_2 = \frac{9}{5} - \frac{1}{10}x_3 + \frac{1}{10}x_4$ (here, according to our notation for dictionaries $\overline{a}_{23} = +\frac{1}{10}$ and $\overline{a}_{24} = -\frac{1}{10}$) would be rewritten as

$$x_2 = (1 - 0x_3 - (-1)x_4) + \left(\frac{4}{5} - \frac{1}{10}x_3 - \frac{9}{10}x_4 \right).$$

The general equation (1) has two parts on the right, an integral part and a fractional part. Focus on the fractional part: Since each $(\overline{a}_{hj} - \lfloor \overline{a}_{hj} \rfloor)$ $(j \in N)$ is nonnegative (by our definition of the $\lfloor r \rfloor$ notation), each x_j is nonnegative, and $(\overline{b}_h - \lfloor \overline{b}_h \rfloor)$ is a fraction strictly between 0 and 1, either the fractional part has value ≤ 0 (at most zero) or its value lies strictly between 0 and 1. This brings us to the important point:

For every integral solution $\mathbf{x}$ of (P), in equation (1), the left hand side must have an integer value, and the integral part on the right hand side, $\{\lfloor \overline{b}_h \rfloor - \sum_{j \in N} \lfloor \overline{a}_{hj} \rfloor x_j\}$, must have an integer value, hence, the fractional part *must* have an integer value. That is, the fractional part *must* have a value ≤ 0, since it *cannot* take a nonintegral value between zero and one.

This gives us a new constraint,

$$\left\{ (\overline{b}_h - \lfloor \overline{b}_h \rfloor) - \sum_{j \in N} (\overline{a}_{hj} - \lfloor \overline{a}_{hj} \rfloor) x_j \right\} \leq 0,$$

to add to (P). Such constraints are called *Gomory cuts.* The key property of these constraints is this: *Every feasible integral solution of (P) satisfies the Gomory cut, but the optimal solution $\tilde{\mathbf{x}}$ for (P) does not satisfy it because in the optimal dictionary for (P) $x_j = 0$ for each $j \in N$ and $(\overline{b}_h - \lfloor \overline{b}_h \rfloor) > 0$.*

Let the new linear programming problem be (P'), i.e., (P') is the LP relaxation of (IP) with the Gomory cut added as a new constraint. To find an optimal solution for (P'), we use the method for adding a new constraint in sensitivity analysis, and do dual simplex pivots until we find an optimal solution to (P') or until we find that (P') is infeasible. Another important point is this:

For every feasible integral solution $\mathbf{x}$ of (P), the new slack variable introduced in the Gomory cut must have an integer value.

We iterate the above steps until we find an optimal solution x^* for the current LP (P^*) such that every component of x^* has an integer value, or until we find that (P^*) is infeasible,

Example: Cutting Plane Method

$$
\begin{array}{llllll}
(IP) & & & & & \\
\text{maximize} & z = x_1 & + & 5x_2 & & \\
\text{subject to} & x_1 & + & 10x_2 & \leq & 20 \\
 & x_1 & & & \leq & 2 \\
 & x_1,\ x_2 & \geq & 0 & & \\
 & x_1,\ x_2 & & \text{integral} & &
\end{array}
$$

We introduce slack variables x_3 and x_4. If x_1 and x_2 are integers, then notice that x_3 and x_4 *must be integers*, because all coefficients a_{ij} in the constraints and all right hand side values b_i are integers.

The initial dictionary for the LP relaxation (P) of (IP) is:

$$
\begin{array}{rcrcrcr}
x_3 & = & 20 & - & x_1 & - & 10x_2 \\
x_4 & = & 2 & - & x_1 & & \\
\hline
z & = & 0 & + & x_1 & + & 5x_2
\end{array}
$$

After two pivots of the simplex algorithm, we get the optimal dictionary for (P):

$$
\begin{array}{rcrcrcr}
x_2 & = & \frac{9}{5} & - & \frac{1}{10}x_3 & + & \frac{1}{10}x_4 \\
x_1 & = & 2 & & & - & x_4 \\
\hline
z & = & 11 & - & \frac{1}{2}x_3 & - & \frac{1}{2}x_4
\end{array}
$$

The optimum solution, $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = (2, \frac{9}{5}, 0, 0)$, is *not integral*. Therefore, the equation for x_2 in the optimal dictionary is used to generate the following Gomory cut:

$$\frac{4}{5} - \frac{1}{10}x_3 - \frac{9}{10}x_4 \leq 0.$$

(In terms of the decision variables x_1 and x_2, this constraint is $x_1 + x_2 \leq 3$, as can be seen by substituting for x_3 and x_4 from the initial dictionary.)
The slack variable x_5, $x_5 \geq 0$, is introduced, and the equation

$$-\frac{1}{10}x_3 - \frac{9}{10}x_4 + x_5 = -\frac{4}{5}$$

is added to the constraints of (P) to give the new LP (P').

EXERCISE: For every feasible solution x_1, x_2 of (P) that is integral, show that x_5 has an integer value.

Since the resulting dictionary

$$
\begin{array}{rcrcrcr}
x_2 & = & \frac{9}{5} & - & \frac{1}{10}x_3 & + & \frac{1}{10}x_4 \\
x_1 & = & 2 & & & - & x_4 \\
x_5 & = & -\frac{4}{5} & + & \frac{1}{10}x_3 & + & \frac{9}{10}x_4 \\
\hline
z & = & 11 & - & \frac{1}{2}x_3 & - & \frac{1}{2}x_4
\end{array}
$$

is dual feasible but not feasible, we apply the dual simplex method to do a pivot. The leaving variable is x_5 and the entering variable is x_4. The new dictionary, which is optimal for (P'), is as follows.

$$
\begin{array}{rcrcrcr}
x_2 & = & \frac{17}{9} & - & \frac{1}{9}x_3 & + & \frac{1}{9}x_5 \\
x_1 & = & \frac{10}{9} & + & \frac{1}{9}x_3 & - & \frac{10}{9}x_5 \\
x_4 & = & \frac{8}{9} & - & \frac{1}{9}x_3 & + & \frac{10}{9}x_5 \\
\hline
z & = & \frac{95}{9} & - & \frac{4}{9}x_3 & - & \frac{5}{9}x_5
\end{array}
$$

The optimum basic solution is $\tilde{\mathbf{x}}' = (\tilde{x}'_1, \tilde{x}'_2, \tilde{x}'_3, \tilde{x}'_4, \tilde{x}'_5) = (\frac{10}{9}, \frac{17}{9}, 0, \frac{8}{9}, 0)$, and it is *not integral*. Therefore, we generate a new Gomory cut, using the equation for x_2 in the optimal dictionary for (P'):

$$\frac{8}{9} - \frac{1}{9}x_3 - \frac{8}{9}x_5 \leq 0.$$

(In terms of the decision variables x_1 and x_2, this constraint is $x_1 + 2x_2 \leq 4$.)
The slack variable x_6, $x_6 \geq 0$, is introduced, and the equation

$$-\frac{1}{9}x_3 - \frac{8}{9}x_5 + x_6 = -\frac{8}{9}$$

is added to the constraints of (P') to give the new LP (P'').

The dictionary obtained by adding the equation for x_6 is dual feasible but not feasible, so we apply the dual simplex method to do a pivot. The leaving variable is x_6 and the entering variable is x_5. The new dictionary, which is optimal for (P''), is as follows.

$$
\begin{array}{rcrcrcr}
x_2 & = & 2 & - & \frac{1}{8}x_3 & + & \frac{1}{8}x_6 \\
x_1 & = & 0 & + & \frac{1}{4}x_3 & - & \frac{5}{4}x_6 \\
x_4 & = & 2 & - & \frac{1}{4}x_3 & + & \frac{5}{4}x_6 \\
x_5 & = & 1 & - & \frac{1}{8}x_3 & + & \frac{9}{8}x_6 \\
\hline
z & = & 10 & - & \frac{3}{8}x_3 & - & \frac{5}{8}x_6
\end{array}
$$

Since the optimal basic solution $\tilde{\mathbf{x}}'' = (0, 2, 0, 2, 1, 0)$ is *integral*, it follows that $(\tilde{x}''_1, \tilde{x}''_2) = (0, 2)$ is an optimal solution for (IP).