C&O 350 Assignment 5, Due Thursday March 21, 1996

1. Prove that $Ax \leq b$ has a solution if and only if there does not exist y such that $yA = 0$, $y \geq 0$ and $yb < 0$. (Hint: Just imitate the proof of the Farkas Lemma. First, show that not both x and y can exist (this is easy). Then suppose that x does not exist, and use duality to prove that y must exist.)

2. (a) Solve by the (two-phase) revised simplex method:

$$\begin{array}{ll} \text{maximize} & 5x_1 - x_2 + x_3 - 10x_4 + 7x_5 \\ \text{subject to} & \\ & \begin{array}{lcl} 3x_1 - x_2 - x_3 & = & 4 \\ x_1 - x_2 + x_3 + x_4 & = & 1 \\ 2x_1 + x_2 + 2x_3 \quad + x_5 & = & 9 \\ x_j \geq 0,\ \forall j & & \end{array} \end{array}$$

   (b) Solve by the revised simplex method, beginning with $B = (1, 2, 6)$.

$$\begin{array}{ll} \text{minimize} & 14x_1 - 19x_2 \quad + 21x_4 + 52x_5 \\ \text{subject to} & \\ & \begin{array}{lcl} x_1 \qquad\quad - x_4 + x_5 + x_6 & = & 3 \\ x_1 + x_2 \qquad - x_4 + 3x_5 & = & 4 \\ x_1 + x_2 + x_3 - 3x_4 \quad + x_6 & = & 6 \\ x_j \geq 0,\ \forall j & & \end{array} \end{array}$$

3. Consider the (family) diet problem minimize(cx subject to $Ax \geq b$, $x \geq 0$), where the costs (in cents per kg) of 6 foods are given by $c = (35, 30, 60, 50, 27, 22)$; the minimum nutrient requirement for two nutrients, vitamins A and C, are $b = (9, 19)^t$; and the nutrient contents of the foods are given in the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 2 & 1 & 2 \\ 0 & 1 & 3 & 1 & 3 & 2 \end{bmatrix}$$

   The basis $B = \{5, 6\}$ is optimal for this problem. (Found after adding surplus variables x_7, x_8.)

   (a) Suppose a new primary food, food 7, is available in the market at 88 cents/kg. One kilogram of this food contains two units of vitamin A and four units of vitamin C. Should the family include this new food in its diet? If not, how much should the cost of this food decrease before the family can consider including it in its diet? What is the optimum diet if food 7 is actually available at 32 cents/kg?

(b) For what range of cost per kilogram of primary food 4 does the basis B remain optimal to the original family's diet problem? If the cost is slightly too small to be in this range, find the new primal solution after one revised simplex pivot.

(c) For what range of cost per kilogram of primary food 5 does the basis B remain optimal to the original family's diet problem? If the cost is slightly too large to be in this range, find the new primal solution after one revised simplex pivot.

(d) What is the marginal effect of increasing the minimum requirement of vitamin C on the cost of the optimal diet in the original family's diet problem? How much can the minimal requirement of vitamin C increase before the basis B becomes nonoptimal to the problem?

4. Consider the LP

$$\begin{array}{ll} \text{maximize} & 2x_1 + 4x_2 + x_3 + x_4 \\ \text{subject to} & \\ & \begin{array}{lcl} x_1 + 3x_2 \quad + x_4 & \leq & 8 \\ 2x_1 + x_2 & \leq & 6 \\ \quad x_2 + 4x_3 + x_4 & \leq & 6 \\ x_j \geq 0, \ \forall j & & \end{array} \end{array}$$

This problem represents the production plan of a company with 4 products, represented by the variables x_j, and utilizing 3 resources, with availability given by the right-hand side of the constraints. The slack variables x_5, x_6, x_7 correspond to the three constraints. The basis $B = \{1, 3, 5\}$ is optimal.

(a) Compute the inverse of the basis matrix, A_B^{-1}, and optimal primal and dual solutions.

(b) If the availability of only one of the resources can be marginally increased, which one should be picked? Why?

(c) For what range of values of b_1 (resource material 1) does the basis B remain optimal?