II

C&O 350 – LINEAR PROGRAMMING

Midterm Examination - Winter 1995

Wednesday, 1 March, 1995, 7.00 – 9.00pm **(2 hours)**

This examination consists of 4 questions and 9 pages.

ANSWER ALL QUESTIONS.

Calculators permitted.

Tick off your section:

01	☐	Wolkowicz	TR 11:30
02	☐	Lewis	MWF 8:30
03	☐	Lewis	MWF 10:30

NAME: ______________________________

ID: ______________________________

SIGNATURE: ______________________________

For Marker Only	
1	/
2	/
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Total	/

1. (i) Solve

$$
(P)\quad \begin{cases}
\text{maximize} & 2x_1 + 3x_2 - 4x_3 \\
\text{subject to} & x_1 + x_2 - x_3 \leq 2, \\
& x_1 + 2x_2 + x_3 \leq 3, \\
& x_1 \;,\; x_2 \;,\; x_3 \geq 0,
\end{cases}
$$

by the simplex method.

1. (ii) State the (strong) duality theorem for linear programs in standard form.

(iii) Write down the dual problem (D) for (P), and deduce an optimal solution for (D) from your answer to part (i).

(iv) Use your answer to part (iii) to verify the duality theorem for (P).

2. (i) Use Phase I of the simplex method to find an initial feasible dictionary for

$$
(Q) \quad \left\{ \begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & -2x_1 & - & x_2 & \leq & -4, \\
 & x_1 & - & x_2 & \leq & -1, \\
 & -2x_1 & + & x_2 & \leq & 2, \\
 & x_1 & , & x_2 & \geq & 0.
\end{array} \right.
$$

Do not solve the problem.

2. (ii) Find a NEW objective function (in terms of x_1 and x_2) for the problem (Q) for which the feasible solution you found in part (i) is the unique optimal solution. Explain your answer.

3. (a) Let F be the set of x in $\mathbb{R}^k$ satisfying

$$(S) \qquad \begin{cases} \sum_{j=1}^k a_{ij}x_j &= b_i \quad (i = 1, 2, \ldots, m), \\ x_j &\geq 0 \quad (j = 1, 2, \ldots, k). \end{cases}$$

(i) Define 'x^* is a basic feasible solution for (S)'.

(ii) Define 'x^* is an extreme point of F'.

(b) Now let F be the set of x in $\mathbb{R}^5$ satisfying

$$\begin{cases} x_1 - 2x_2 - x_3 + x_4 = -2, \\ 2x_1 + 3x_2 + 5x_3 + x_5 = 10, \\ x_1\ ,\ x_2\ ,\ x_3\ ,\ x_4\ ,\ x_5 \geq 0, \end{cases}$$

For each of the following points x^*, either find a dictionary corresponding to x^*, or find a nonzero d in $\mathbb{R}^5$ with $x^* + d$ and $x^* - d$ in F:

(i) $x^* = (1, 1, 1, 0, 0)^T$.

3. (ii) $x^* = (0, 0, 2, 0, 0)^T$.

4. (a) Suppose that x and y are given such that

$$\begin{array}{rcccccl} \sum_{i=1}^{n} a_{ij}x_j & \leq & b_i & \text{and} & y_i & \geq & 0, \quad (i = 1, 2, \ldots, m), \\ \text{and} & & & & & & \\ \sum_{j=1}^{n} a_{ij}y_i & \geq & c_j & \text{and} & x_j & \geq & 0, \quad (j = 1, 2, \ldots, n). \end{array}$$

Prove from first principles (without using any results) that $\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$, with equality if and only if

$$\begin{array}{rcccccl} \sum_{i=1}^{n} a_{ij}x_j & = & b_i & \text{or} & y_i & = & 0, \quad (i = 1, 2, \ldots, m), \\ \text{and} & & & & & & \\ \sum_{j=1}^{n} a_{ij}y_i & = & c_j & \text{or} & x_j & = & 0, \quad (j = 1, 2, \ldots, n). \end{array}$$

4. (b) Use complementary slackness to determine if $x = (3,0,0,0)^T$ is optimal for the problem

$$\begin{cases} \text{maximize} & 2x_1 + 3x_2 + 5x_3 + 4x_4 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 + x_4 \le 5, \\ & x_1 + x_2 + 2x_3 + 3x_4 \le 3, \\ & x_1 \;,\; x_2 \;,\; x_3 \;,\; x_4 \ge 0, \end{cases}$$