# C&O 350
# Assignment 4

Due: Friday March 17 (§2, §3), Thursday March 16 (§1 )

1. Consider the following two systems, where $A$ is a given $m \times n$ matrix, and $b$ is a given column $m$-vector:

   (i) $Ax = b,\ x \geq 0,\ x \in \Re^n$.

   (ii) $yA \geq 0,\ yb < 0,\ y \in \Re^m$.

   (a) Prove that it is not possible for both systems to have solutions.

   (b) Prove that if system (i) does not have a solution then system (ii) must have a solution. (Hint: consider the linear program: maximize $0x$ subject to $Ax = b$ and $x \geq 0$.)

2. Use complementary slackness to determine whether or not $(y_1, y_2) = (4, 5)$ is an optimal solution of the dual problem for the linear program

$$\left\{\begin{array}{lrcrcrcrcrcl} \text{maximize} & 10x_1 & + & 24x_2 & + & 20x_3 & + & 20x_4 & + & 25x_5 & & \\ \text{subject to} & x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & + & 5x_5 & \leq & 19, \\ & 2x_1 & + & 4x_2 & + & 3x_3 & + & 2x_4 & + & x_5 & \leq & 57, \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0. \end{array}\right.$$

3. Use the Revised Simplex Method to solve the problem

$$\left\{\begin{array}{lrcrcrcrcrcrcrcrcl} \text{maximize} & 7x_1 & - & 3x_2 & & & - & 4x_4 & + & 3x_5 & & & + & x_7 & + & 2x_8 & & \\ \text{subject to} & & & x_2 & + & 2x_3 & - & x_4 & - & 3x_5 & & & - & x_7 & & & \leq & 24, \\ & x_1 & + & x_2 & - & x_3 & & & + & 2x_5 & + & x_6 & & & - & x_8 & \leq & 18, \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & , & x_7 & , & x_8 & \geq & 0. \end{array}\right.$$

4. Consider the linear program

$$\left\{\begin{array}{lrcrcrcl} \text{maximize} & 5x_1 & + & 4x_2 & + & 3x_3 & & \\ \text{subject to} & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5, \\ & 4x_1 & + & x_2 & + & 2x_3 & \leq & 11, \\ & 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8, \\ & x_1 & , & x_2 & , & x_3 & \geq & 0. \end{array}\right.$$

   (i) After introducing slack variables $x_4, x_5, x_6$, verify that the optimal basis for this problem is $\{x_1, x_3, x_5\}$ by calculating the corresponding dictionary directly: **do not perform any pivots.**

(ii) Suppose the right-hand-side is changed to $(5, 11, 7)^T$. Use your answer to part (i) to derive an optimal dictionary for the new problem. (Use the dual simplex method if needed.)

5. **BONUS QUESTION**
Suppose that the following standard form linear program is feasible and bounded.

$$(P) \qquad \left\{ \begin{array}{lrcl} \text{maximize} & cx & & \\ \text{subject to} & Ax & \leq & b, \\ & x & \geq & 0. \end{array} \right.$$

Prove that there exists an $x$, an optimal solution for $(P)$, and a $y$, an optimal solution for the dual of (P), for which *strict* complementary slackness holds, i.e.

$$\begin{array}{rclcrcll} \sum_{i=1}^{n} a_{ij}x_j & = & b_i & \text{or} & y_i & = & 0, & \text{but not both} \quad (i = 1, 2, \ldots, m), \\ \text{and} & & & & & & & \\ \sum_{j=1}^{m} a_{ij}y_i & = & c_j & \text{or} & x_j & = & 0, & \text{but not both} \quad (j = 1, 2, \ldots, n). \end{array}$$

(Hint: Try to do this for one $i$ and then for one $j$. Then use convex combinations.)