(Reference: "A variation on Karmarkar's algorithm for solving linear programming problems", Earl R. Barnes, Mathematical Programming 36 (1986) 174-182.)

Consider the LP problem

$$\begin{array}{cc} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0, \end{array} \tag{0.1}$$

where $c, x \in \Re^n$, $b \in \Re^m$, and A is $m \times n$ matrix. The dual program is

$$\begin{array}{cc} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \end{array} \tag{0.2}$$

Let $z \in \Re^n$ be a feasible solution of (0.1) satisfying $z_j > 0,\ i = 1, \ldots, n$. If $0 < R < 1$, and $x_i \leq 0$, for some i, then

$$\sum_j^n \frac{(x_j - z_j)^2}{z_j^2} \geq \frac{(x_j - z_j)^2}{z_j^2} \geq 1 > R^2.$$

Therefore, the ellipsoid

$$\sum_j^n \frac{(x_j - z_j)^2}{z_j^2} \leq R^2 \tag{0.3}$$

lies in the interior of the positive orthant in $\Re^n$. Therefore we can obtain a feasible solution for (0.1) by solving the problem:

$$\begin{array}{cc} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ \sum_j^n \frac{(x_j - z_j)^2}{z_j^2} \leq R^2. & \end{array} \tag{0.4}$$

the nonnegativity constraint $x \geq 0$ has been replaced by the ellipsoidal constraint.

To solve (0.4) define the diagonal matrix $D = \mathrm{Diag}\,(z)$. D is a nonsingular diagonal matrix. Then for any x satisfying $Ax = b$, we have, by the Cauchy-Schwartz inequality,

$$\begin{aligned} c^T z - c^T x &= \{c - A^t y\}^T (y - x) = \left[D(c - A^T y)\right]^T D^{-1}(x - y) \\ &\leq ||D(c - A^t y)||||D^{-1}(x - y)|| \leq ||D(c - A^t y)||R. \end{aligned} \tag{0.5}$$

Equality holds throughout (0.5) if

$$D(c - A^t y) = \gamma D^{-1}(x - y) \tag{0.6}$$

for some constant γ, and if

$$||D^{-1}(x - y)|| = R.$$

This implies that

$$\gamma = \frac{||D(c - A^t y)||}{R}.$$

Substituting in (0.6) gives

$$x = z - R\frac{D^2(c - A^t y)}{D(c - A^t y)||}. \tag{0.7}$$

The conditions that $Ax = Az = b$ imply that $AD^2(c - A^t y) = 0$, or

$$y = (AD^2A^T)^{-1}AD^2c. \tag{0.8}$$

We can write (0.5) as

$$c^T x \geq c^T z - R||D(c - A^t y)||. \tag{0.9}$$

Combining the above, we get the following algorithm:

Algorithm 0.1 *Let $x^0 > 0$ satisfying $Ax^0 = b$ be given. In general, if x^k is known, define*

$$D_k = \mathrm{Diag}\,(x^k)$$

and compute

$$x^{k+1} = x^k - R\frac{D^2(c - A^t y^k)}{D(c - A^t y^k)||}, \tag{0.10}$$

where

$$y^k = (AD_k^2A^T)^{-1}AD_k^2c. \tag{0.11}$$