C&O 666
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[10]

Suppose that the functions $f, g_j : \Re^n \to \Re$, $j = 1, \cdots m$, are all convex. Define $L(x,\lambda) = f(x) + \lambda^t g(x)$, where $0 \leq \lambda \in \Re^n$, $g = (g_j)$, and suppose $\nabla L(x^*, \lambda^*) = 0$, for given x^*, λ^*. Show the following:

1. L is a convex function of x.
2. x^* is a minimum point of L.
3. Suppose that $\lambda^t g(x^*) = 0$ and that $g(x) \leq 0$. Then $f(x) \geq f(x^*)$.
4. Deduce a set of first order sufficient optimality conditions for a minimum point of the problem:

$$\min_x f(x) \text{ s.t. } g(x) \leq 0.$$

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[10]

Consider the quadratic programming problem:

$$(QP) \quad \min_x (\max_x) h(x) = x^t Q x - b^t x \;\; \text{s.t.} \;\; Ax = b,$$

where $Q = Q^t$ is a fixed non-singular $n \times n$ matrix, and A is a $m \times n$ matrix. Suppose that $Ax^* = b$ and $x^* = Q^{-1}(b + A^t \lambda)$, for some $\lambda \in \Re^m$.

1. Show that one of the following holds:
   (a) x^* is a **global** minimum for (QP);
   (b) x^* is a **global** maximum for (QP);
   (c) x^* is a saddle point (neither a local minimum nor local maximum) for (QP).

2. Using the above results (or otherwise) derive a projected Newton method for finding the local minimum of the program:

$$\min_x f(x) \quad \text{s.t.} \quad Ax = b,$$

where $f : \Re^n \to \Re$ is in C^2.

3

[10]

Consider the problem: $\min_x f(x) = x^3 + \frac{y^3}{3} - x^2 - \frac{y^2}{2} + xy - 10x - 4y$.

1. Verify that $(2,2)^t$ is a local minimum point.
2. Using Newton's method, complete the first iteration from the initial point $(0,0)^t$. What can you say about the Newton direction?

4

[10]

1. Show that the following problem is a convex programming problem, i.e. show that the objective function and constraints are convex functions on the domain $x, y \geq 0$.

$$\begin{array}{lrcl} \min & f(x) = x^3 + \frac{y^3}{3} + \frac{x^2}{2} + y^2 + xy - 2x - y & & \\ \text{s.t.} & -4x - 3y + x^2 + y^2 + 5 & \leq & 0 \\ & -x - 2y + 3 & \leq & 0 \\ & x, y & \geq & 0 \end{array}$$

2. Show that $(1,1)^t$ is an optimum solution using weak duality.

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[10]

Consider the function

$$f(x,y) = (x^2 - 4)^2 + 4y^2.$$

1. Show that the points $(2,0)$ and $(-2,0)$ are minimum points of f and the origin is a saddle point of f.

2. Draw the level curves for $f(x, y) = e$ for $e = 12, 16, 25$.

3. Find the Hessian of f and determine where it is singular, positive definite, and indefinite.

4. State the Newton algorithm for f, i.e. write down the function g for the Newton iteration $x_{k+1} = g(x_k)$. Then determine the regions of the plane where the iteration converges and find which points it converges to.