C&O 367
Assignment 0 (REVISION)

Due on Tuesday, Jan. 14, 2001. Instructor H. Wolkowicz

1 GEOMETRY

1.1 BACKGROUND MATERIAL

For a vector $x = (x_1, \cdots, x_n)^t \in \Re^n$, we define the *norm* and *inner product*

$$\|x\| := (x_1^2 + \cdots + x_n^2)^{1/2},$$

$$x \cdot y := x_1 y_1 + \cdots + x_n y_n.$$

Theorem 1 *(Cauchy-Schwarz)* $|x \cdot y| \leq \|x\|\|y\|$, *with equality if and only if* $x = \lambda y$, *for some* λ.

1.2 EXERCISES

1. (10) Explain the geometrical significance for the vectors x and y of:

   (a)
   $$x \cdot y = 0.$$

   (b)
   $$x \cdot y \leq 0.$$

   (c)
   $$\cos\theta = \frac{x \cdot y}{\|x\|\|y\|},$$

   where θ is an angle.

2. (10) Prove that the function $f : \Re^n \to \Re$ defined by $f(x) = a \cdot x + \alpha$ is continuous, where a is a given vector and $\alpha \in \Re$.

3. (10) Prove that the function $f : \Re^n \to \Re$ defined by $f(x) = \|x\|$ is continuous.

2 CALCULUS

2.1 BACKGROUND MATERIAL

For a function $g : \Re^n \to \Re$, the *gradient* is denoted

$$\nabla g := \begin{pmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \cdots \\ \frac{\partial g}{\partial x_n} \end{pmatrix}.$$

Suppose $f : \Re \to \Re$ is infinitely differentiable at $x = a$. The *Taylor Series* of f about a is:

$$f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \cdots.$$

2.2 EXERCISES

1. (5) Let $g(x) = \|7x + e\|$, where $x, e \in \Re^n$ and e is the vector of ones. Calculate the gradient of g, $\nabla g(x)$, when $7x + e \neq 0$.

2. (10) Suppose that $g : \Re^n \to \Re$, $a, b \in \Re^n$, and $f(t) := g(a - tb)$. Calculate the derivative $f'(t)$.

3. Write down the Taylor series of:

   (a) (5)
   $$f(x) = x^5 + x^3, \text{ about } x = 1.$$

   (b) (5)
   $$f(x) = \log(1 + x), \text{ about } x = 0.$$

   (c) (5)
   $$f(x) = x\cos(x), \text{ about } x = 0.$$

3 TOPOLOGY

3.1 BACKGROUND MATERIAL

The *open ball* $B(x; r) := \{y \in \Re^n : \|x - y\| < r\}$. Suppose that D is a subset of $\Re^n$.

Interior: $x \in \operatorname{int} D$ if there exists $r > 0$ with $B(x; r) \subset D$.

Closure: $x \in \operatorname{cl} D$ if there exists a sequence $x^k \in D$ with $x^k \to x$.

Boundary: $x \in \partial D$ if $x \in \operatorname{cl} D \backslash \operatorname{int} D$.

D is *open* if $D = \operatorname{int} D$. D is *closed* if $D = \operatorname{cl} D$.

3.2 EXERCISES

1. (15) For each of the following sets, find the interior, the closure, and the boundary. Then determine which of the sets are open, closed, neither, or both.

   (a)
   $$\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}.$$

   (b)
   $$\{(x_1, x_2) : x_1 > 0, x_2 > 0\}.$$

   (c)
   $$\{(x_1, x_2) : x_1 > 0, x_2 \geq 0\}.$$

   (d)
   $$\Re^n$$

   (e)
   $$\left\{(x_1, x_2) : x_1^2 + x_2^2 < 0\right\}.$$

   (f)
   $$\emptyset.$$

2. (a) (10) Prove that D is closed if and only if the complement D^c is open.

   (b) (10) Prove that $x \in \partial D$ if and only if for any $r > 0$ there exists a $y \in B(x; r) \cap D$ and a $z \in B(x; r) \cap D^c$.

4 MATRICES

4.1 EXERCISES

1. (10) Let
   $$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}$$

   (a) Calculate the determinant of A.

   (b) Calculate the trace of A.

   (c) Calculate the rank of A.

   (d) What is the rank of A^T.

2. (10) Let
   $$B = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$$
   Calculate the eigenvalues and eigenvectors of B.