Lecture 12
Geometrical problems

geometrical problems involving polyhedra and ellipsoids

- distance
- volume
- classification
- the S-procedure

Distance problems

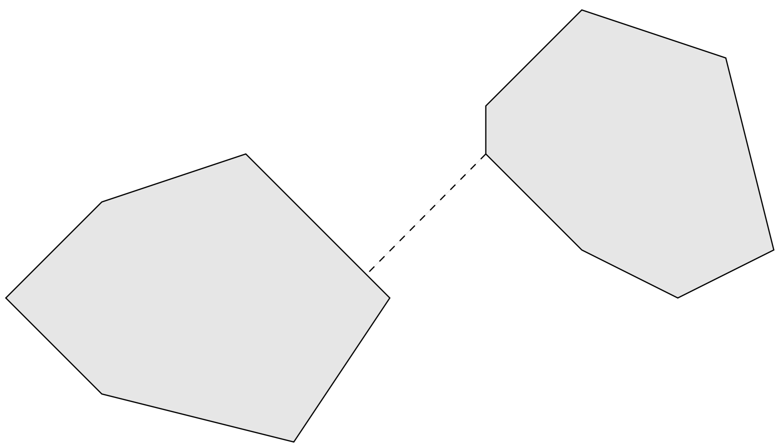
projection of a point x on a convex set A:

$$P_A(x) = \mathop{\textbf{argmin}}_y \{\|x - y\| \mid y \in A\}$$

($\|\cdot\|$ can be any norm)
computing $P_A(x)$ is a convex problem

distance between two convex sets A, $B \subseteq \mathbf{R}^n$:

$$\textbf{dist}\,(A, B) = \inf_{x \in A,\ y \in B} \|x - y\|$$



computing **dist** (A, B) is convex problem

Intersection & containment of polyhedra via LP

inequality description

$$\begin{aligned} \mathcal{P}_1 &= \{x \mid a_i^T x \le b_i, i = 1, \ldots, m\} = \{x \mid Ax \le b\} \\ \mathcal{P}_2 &= \{x \mid f_i^T x \le g_i, i = 1, \ldots, l\} = \{x \mid Fx \le g\} \end{aligned}$$

– $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$? Find x s.t. $Ax \le b$ and $Fx \le g$

– $\mathcal{P}_1 \subseteq P_2$? For $i = 1, \ldots, m$, check

$$\textbf{sup}\{a_i^T x \mid f_k^T x \le g_k, k = 1, \ldots, l\} \le b_i$$

vertex description

$$\mathcal{P}_1 = \textbf{Co}\,\{v_1, \ldots, v_K\}, \quad \mathcal{P}_2 = \textbf{Co}\,\{w_1, \ldots, w_L\}$$

– $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$? Find λ_1, …, λ_K, μ_1, …, $\mu_L \ge 0$ s.t.

$$\lambda_1 v_1 + \cdots + \lambda_K v_k = \mu_1 w_1 + \cdots + \mu_L w_l$$

$$\lambda_1 + \cdots + \lambda_K = \mu_1 + \cdots + \mu_L = 1$$

– $\mathcal{P}_1 \subseteq P_2$? For $k = 1, \ldots, K$ find $\mu_i \ge 0$, s.t.

$$\begin{aligned} v_k &= \mu_1 w_1 + \cdots + \mu_L w_L \\ 1 &= \mu_1 + \cdots + \mu_L \end{aligned}$$

Min volume ellipsoid covering a polytope

polytope $\mathcal{P} = \mathbf{Co}\,\{v_1, \ldots, v_L\} \subseteq \mathbf{R}^n$ (vertex description)

ellipsoid $\mathcal{E} = \{x \mid \|Ax - b\| \leq 1\}$, $A = A^T > 0$
center $A^{-1}b$, volume proportional to $\det A^{-1}$

$\mathcal{P} \subseteq \mathcal{E} \iff \|Av_i - b\| \leq 1,\ i = 1, \ldots, L$
convex (quadratic) constraint in A and b

minimum volume $\mathcal{E} \supseteq \mathcal{P}$:

$$\begin{array}{ll} \text{minimize} & \log\det A^{-1} \\ \text{subject to} & A = A^T > 0 \\ & \|Av_i - b\| \leq 1,\ i = 1, \ldots, L \end{array}$$

convex problem in variables A, b
(total dimension $n(n+1)/2 + n$)

Max volume ellipsoid inside polyhedron

polyhedron $\mathcal{P} = \{x \mid a_i^T x \leq b_i,\ i = 1, \ldots, L\}$
(inequality description)

ellipsoid $\mathcal{E} = \{By + d \mid \|y\| \leq 1\}$, $B = B^T > 0$
center d, volume proportional to $\det B$

$$\begin{aligned} \mathcal{E} \subseteq \mathcal{P} &\iff \forall y, \|y\| \leq 1 \ \Rightarrow\ a_i^T(By + d) \leq b_i \\ &\iff \sup_{\|y\| \leq 1} a_i^T By + a_i^T d \leq b_i \\ &\iff \|Ba_i\| + a_i^T d \leq b_i \end{aligned}$$

convex constraint in B and d

maximum volume $\mathcal{E} \subseteq \mathcal{P}$:

$$\begin{array}{ll} \text{maximize} & \log\det B \\ \text{subject to} & B = B^T > 0 \\ & \|Ba_i\| + a_i^T d \leq b_i,\ i = 1, \ldots, L \end{array}$$

convex problem in variables B, d
(total dimension $n(n + 1)/2 + n$)

Löwner-John ellipsoid

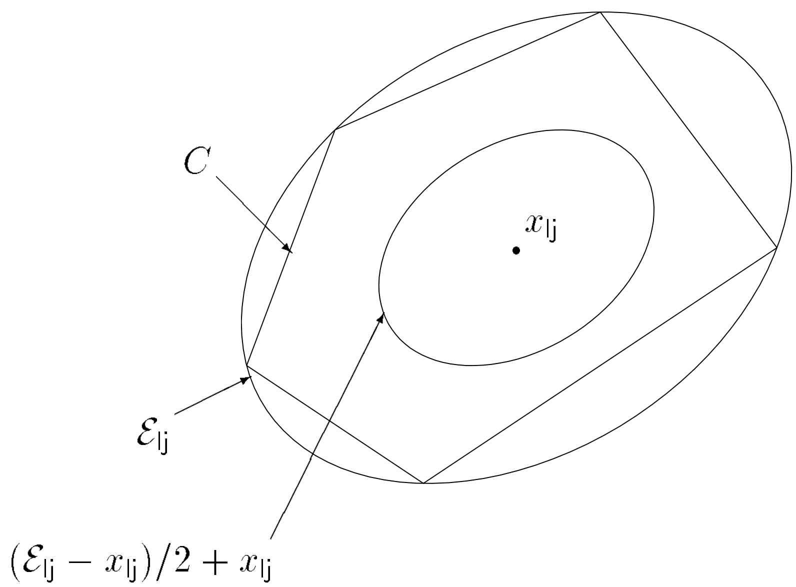
$C \subseteq \mathbf{R}^n$ convex, bounded

min-volume ellipsoid containing C is called *Löwner-John ellipsoid* $\mathcal{E}_{\mathrm{lj}}$

Fact. $\mathcal{E}_{\mathrm{lj}}$, shrunk by a factor n, lies inside C:

$$\mathcal{E}_{\mathrm{lj}} \supseteq C \supseteq \frac{1}{n}(\mathcal{E}_{\mathrm{lj}} - x_{\mathrm{lj}}) + x_{\mathrm{lj}}$$

where $x_{\mathrm{lj}} = \mathrm{center}(\mathcal{E}_{\mathrm{lj}})$
if C is symmetric about any point, can replace n by $\sqrt{n}$



– arbitrarily complex convex sets can be approximated by ellipsoids

– ellipsoidal approximation 'universal' and 'efficient'

– worst-case bound depends only on dimension n

– factor $1/n$ tight for C a simplex

$\Longrightarrow$ *any* norm on $\mathbf{R}^n$ can be approximated within $\sqrt{n}$ by a quadratic norm, *i.e.*, there is $A \in \mathbf{R}^{n \times n}$ s.t.

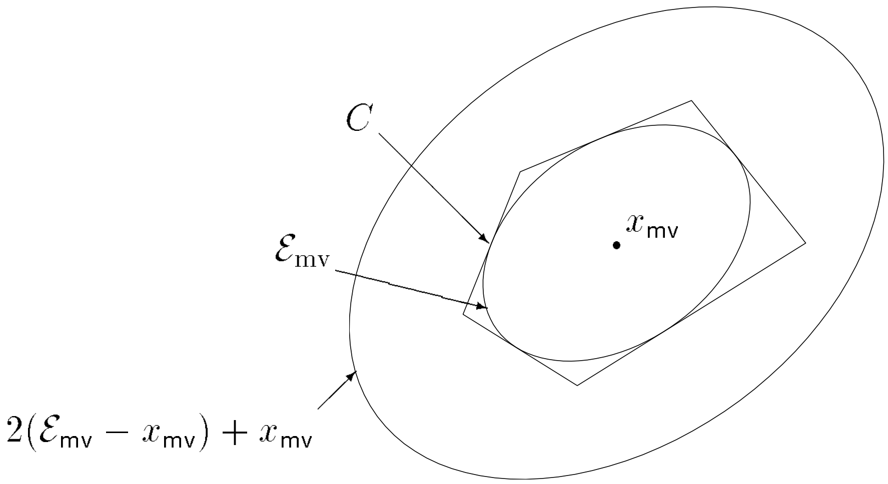
$$\|Az\| \leq \|z\|_X \leq \sqrt{n}\|Az\|$$

Efficiency of max-volume inscribed ellipsoid:
$C \subseteq \mathbf{R}^n$ convex, bounded, nonempty interior
$\mathcal{E}_{\mathrm{mv}} =$ max-volume ellipsoid inside C

Fact. $\mathcal{E}_{\mathrm{mv}}$, expanded by factor n, covers C:

$$\mathcal{E}_{\mathrm{mv}} \subseteq C \subseteq n(\mathcal{E}_{\mathrm{mv}} - x_{\mathrm{mv}}) + x_{\mathrm{mv}}$$

where $x_{\mathrm{mv}} = \mathrm{center}(\mathcal{E}_{\mathrm{mv}})$



– another 'universal, efficient' ellipsoidal bound

– factor $1/n$ tight for C a simplex

Difficult variations

minor variations on these (convex, hence easy) problems are very difficult:

- farthest point from x_0 in polytope given by $Ax \leq b$
- minimum volume ellipsoid containing polytope $Ax \leq b$

in fact, just checking whether a given ellipsoid $\mathcal{E}$ covers a polytope described by $\{x \mid Ax \leq b\}$ is difficult (NP-hard)

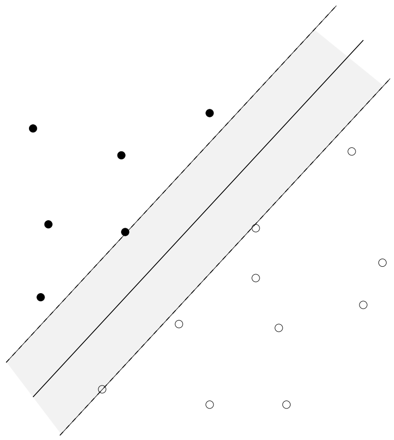
equivalent to *maximizing* convex quadratic s.t. linear inequalities

Linear classification

separate two sets of points x^i and y^j by a hyperplane:

$$\begin{array}{ll} a^T x^i > t & \text{for all } i \\ a^T y^j < t & \text{for all } j \end{array}$$

convex (LP) feasibility problem in variables a, t



robust separation: separate by a slab of maximum thickness

$$\begin{array}{ll} \text{maximize} & \mathbf{min}_i\, a^T x_i - \mathbf{max}_j\, a^T y_j \\ \text{subject to} & \|a\| \leq 1 \end{array}$$

or

$$\begin{array}{ll} \text{maximize} & t_1 - t_2 \\ \text{subject to} & a^T x_i \geq t_1 \text{ for all } i \\ & a^T y_j \leq t_2 \text{ for all } j \\ & \|a\| \leq 1 \end{array}$$

Nonlinear classification

Separate two sets of points by a quadratic surface:

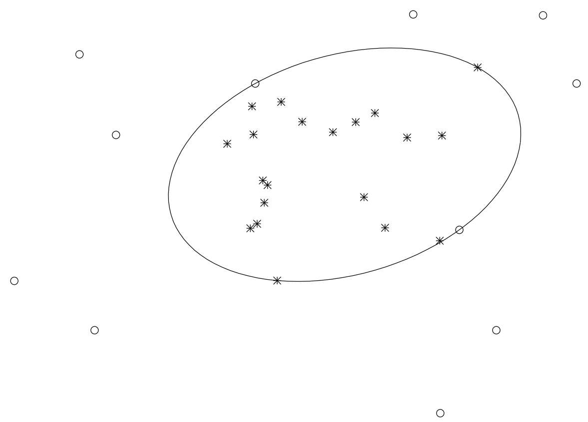
$$(x^i)^T A x^i + b^T x^i + c < 0 \text{ for all } i$$
$$(y^j)^T A y^j + b^T y^j + c > 0 \text{ for all } j$$

...a set of linear inequalities in variables $A = A^T$, b, c

extra constraints:

- *ellipsoid* containing x's, not y's: add constraint $A > 0$
- optimize size or shape

Example. find most spherical ellipsoid containing x's, not y's



– ratio of max. over min. semiaxis length is

$$(\lambda_{\mathsf{max}}(A)/\lambda_{\mathsf{min}}(A))^{1/2}$$

measures asphericity of ellipsoid

– can normalize $A \geq I$ $\left(\lambda_{\mathsf{min}}(A) \geq 1\right)$

– $A \leq \gamma I \Longleftrightarrow \lambda_{\mathsf{max}}(A) \leq \gamma$

Semidefinite program in $A = A^T$, b, c, and γ:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & (x^i)^T A x^i + b^T x^i + c \leq 0 \text{ for all } i \\ & (y^j)^T A y^j + b^T y^j + c \geq 0 \text{ for all } j \\ & I \leq A \leq \gamma I \end{array}$$

A useful result for quadratic functions

Two sets defined by quadratic functions:

$$S_i = \{x \mid x^T A_i x + 2b_i^T x + c_i \leq 0\}, \ i = 0, 1$$

(no assumption of convexity)

Problem: when do we have $S_1 \subseteq S_0$?

Sufficient condition: there exists $\tau \geq 0$

$$\begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 \end{bmatrix} \preceq \tau \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix}$$

(easy) **proof.** condition means for all x,

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq \tau \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix},$$

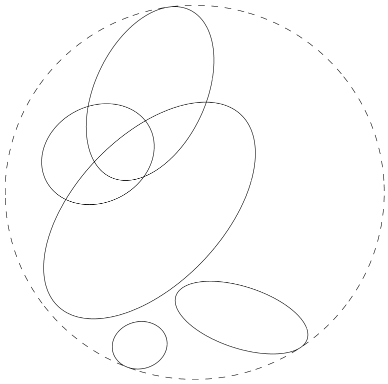
i.e., $x^T A_0 x + 2b_0^T x + c_0 \leq \tau(x^T A_1 x + 2b_1^T x + c_1)$
so $x^T A_1 x + 2b_1^T x + c_1 \leq 0 \implies x^T A_0 x + 2b_0^T x + c_0 \leq 0$

condition is also necessary if S_1 has nonempty interior (proof is not easy)

called S-procedure (in control theory)

Example

find smallest ball $\mathcal{B}$ that contains given ellipsoids $\mathcal{E}_i$



$\mathcal{B}$: $x^T x - 2x_c^T x + \gamma \leq 0$
$\mathcal{E}_i$: $x^T A_i x + 2b_i^T x + c_i \leq 0$

– $\mathcal{B}$ contains $\mathcal{E}_i$ iff there exists $\tau_i \geq 0$:

$$\begin{bmatrix} I & -x_c \\ -x_c^T & \gamma \end{bmatrix} \leq \tau_i \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix}$$

– $t \geq r^2 = x_c^T x_c - \gamma$ (radius of $\mathcal{B}$ squared) if

$$\begin{bmatrix} I & x_c \\ x_c^T & t+\gamma \end{bmatrix} \geq 0$$

Semidefinite program in t, x_c, γ, τ_i:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} I & -x_c \\ -x_c^T & \gamma \end{bmatrix} \leq \tau_i \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \\ & \tau_i \geq 0 \\ & \begin{bmatrix} I & x_c \\ x_c^T & t+\gamma \end{bmatrix} \geq 0 \end{array}$$

S-procedure for many quadratic functions

sublevel sets of $L+1$ quadratic functions:

$$S_i = \{x \mid x^T A_i x + 2b_i^T x + c_i \leq 0\}, \ i = 0, \dots, L$$

(no assumption of convexity)

Problem: $S_1 \cap \cdots \cap S_L \subseteq S_0$?

Sufficient condition: there exist $\tau_1, \dots, \tau_L \geq 0$ s.t.

$$\begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 \end{bmatrix} \preceq \tau_1 \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} + \cdots + \tau_L \begin{bmatrix} A_L & b_L \\ b_L^T & c_L \end{bmatrix}$$

Proof. condition implies for all x,

$$x^T A_0 x + 2b_0^T x + c_0 \leq \sum_{i=1}^{L} \tau_i (x^T A_i x + 2b_i^T x + c_i)$$

Hence, $x^T A_i x + 2b_i^T x + c_i \leq 0$, $i = 1, \dots, L$ implies $x^T A_0 x + 2b_0^T x + c_0 \leq 0$

- condition not necessary (except when $L = 1$)
- still, leads to useful approximations

Application in combinatorial optimization

general quadratically constrained quadratic optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \ i = 1, \ldots, L \end{array}$$

$f_i(x) = x^T A_i x + 2b_i^T x + c_i$

- very hard problem if A_i's are not positive semidefinite
- can compute lower bound via convex optimization

t is lower bound if

$$f_i(x) \leq 0, \quad i = 1, \ldots, L \implies f_0(x) \geq t$$

sufficient condition: exist $\tau_i \geq 0$, $i = 1, \ldots, L$, s.t.

$$\begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 - t \end{bmatrix} + \sum_{i=1}^{L} \tau_i \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \geq 0$$

linear matrix inequality in τ_i

finding best lower bound is a semidefinite program