C & O 350 ABOUT THE FINAL Fall 1995

The topics covered by the exam will include chapters 1, 2, 3, and 5, plus the material on the revised simplex method, methods for solving general LPs, duals of general LPs, theorems of the Farkas lemma type, sensitivity analysis, and integer programming as covered in lectures.

You should know the matrix formulas, and should be able to use them. They feature in the revised simplex method, and sensitivity analysis.

A good understanding of duality is important (read chapters 5 and 9). You should have a clear idea of the relationships[1] between an LP and its dual. (You can often draw inferences concerning an LP from the properties of its dual. For example, if an LP is unbounded, then its dual is infeasible.)

Sensitivity analysis incorporates most of the theoretical ideas and techniques developed in the course. For this reason, an understanding of this topic is important.

There will be questions asking for definitions of concepts[2]. You will be required to know the statements of all the theorems, and have an intuitive understanding of their proofs. You should be able to use them to derive simple consequences. (Definitions of concepts, and statements of theorems should be stated in clear and precise language; intuitive explanations are not acceptable. There will be some questions in which you would be required to prove simple propositions. Your answers to such questions should be clear and logical. There will be no questions which require lengthy proofs.)

Computational questions will be selected so as not to involve "horrible fractions". (In some questions, you will only be required to do one iteration, so that you do not compound numerical errors.) You should display your calcultions carefully, and provide a brief explanation of your work.

Practice Problems[3]

(1) For each of the LPs expressed in matrix notation, write down the duals in matrix notation.

1. max $\mathbf{cx}$, s.t $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.
2. max $\mathbf{cx}$, s.t $\mathbf{Ax} \geq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.
3. max $\mathbf{cx}$, s.t $\mathbf{Ax} \geq \mathbf{b}$.
4. max $\mathbf{cx}$, s.t $\mathbf{Ax} \leq \mathbf{b}$.
5. min $\mathbf{cx}$, s.t $\mathbf{Ax} \geq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.
6. min $\mathbf{cx}$, s.t $\mathbf{Ax} \geq \mathbf{b}$.
7. max $\mathbf{cx}$, s.t $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.
8. max $\mathbf{cx}$, s.t $\mathbf{Ax} = \mathbf{b}$.

(2) State the duality theorem for general LPs (Theorem 9.1, page 141).

(3) State the complementary slackness conditions for general LPs, and do exercise 9.5, page 147.

(4) Must a Phase 1 LP always have an optimal solution? Explain.

(5) Prove theorem 5.5 (page 66). (This is the same question as question 3 on the last assignment. It is a good question to test your understanding of many important ideas, such as the matrix formulas, sensitivity analysis,economic significance of the dual variables, etc.)

[1] The weak duality relation, the duality theorem, and the complementary slackness conditions.

[2] Some of the important concepts are: **basis, basic solution, feasible solution, basic feasible solution, dictionarry, primal feasibility of a dictionary (of a basis), dual feasibility of a dictionary (of a basis).**

[3] Solutions will not be provided. Note that this list does not cover all the course material.

((6) and (7) are easy questions. They simply test your understanding of the definitions.)

(6) If a dual feasible dictionary for an LP is available, show that the LP cannot be unbounded.

(7) If a primal feasible dictionary for an LP is available, show that the LP cannot be infeasible.

(8) Exercise 3.10, page 44.

(9) If any LP has more than one feasible solution, show that it has infinitely many feasible solutions.

(10) If any LP has more than one optimal solution, show that it has infinitely many optimal solutions.

(11) Suppose that a computer program which is capable of testing whether or not a given system of linear inequalities[4] has a feasible solution is available. Show that such a program can be used for optimizing (i.e. for solving LPs.) (Hint: Cosider the equation $by - cx = 0$, the difference between the dual objective function and the primal objective function equated to zero, together with all the primal constraints, and the dual constraints. This is a system of linear inequalities in the $m + n$ variables $y_1, \cdots, y_m$, and $x_1, \cdots, x_n$. A feasible solution to this system provides, simultaneously, an optimal solution to the primal and to the dual.)

(12) A company has a demand to produce 380 units of product A and 425 units of product B. It has three factories with hourly capacities: factory 1 produces 8 units of A and 5 units of B at a cost of 47; factory 2 produces 6 units of A and 10 units of B at a cost of 54; factory 3 produces 4 units of A and 12 units of B at a cost of 56. The company can also buy these two products on the open market.
The company must solve the following problems:

1. Determine the number of hours each factory should run in order to meet the demand and minimize cost.

2. Determine the maximum amount that the company should pay if it chooses to buy 380 units of A and 425 units of B and the maximum per unit prices p_a, p_b it should pay for A and B, respectively, so that the purchasing cost of the yield of each factory does not exceed the total cost of production.

3. Formulate the above two problems as LP's.

4. Show that the optimum solution to the first problem is $x = (25, 30, 0)$ and find the optimum solution to the second problem. Do not perform any simplex operations but quote the relevant theorems to justify your calculations.

5. If a new factory that is being designed is expected to yield 5 units of A and 5 units of B at a cost of 30 per hour, calculate the expected reduction in cost per hour.

[4]Note that a linear equation may be regarded as being equivalent to two linear inequalities. Thus, by a system of linear inequalities, we mean a system of linear inequalities and equations.

(13) Consider the LP:

$$
\begin{array}{rrrrl}
\max 27x_1 & +4x_2 & +7x_3 & +15x_4 & \\
2x_1 & x_2 & & +\ 7x_4 & \leq 11 \\
x_1 & +3x_2 & -\ x_3 & +\ 2x_4 & \leq 0 \\
5x_1 & & +2x_3 & -\ 2x_4 & \leq 16 \\
 & x_1, \ldots, x_4 \geq 0 & & &
\end{array}
$$

The coefficients of the equality LP obtained by adding slack variables x_5, x_6, and x_7 are as follows:

$$
\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 & 7 & 1 & 0 & 0 \\ 1 & 3 & -1 & 2 & 0 & 1 & 0 \\ 5 & 0 & 2 & -2 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} x_5^* \\ x_6^* \\ x_7^* \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 16 \end{pmatrix}
$$

$$
c = \begin{pmatrix} 27, & 4, & 7, & 15, & 0, & 0 & 0 \end{pmatrix}
$$

The optimal basis for this LP is $\{4, 1, 3\}$. The coefficients of the final dictionary, are displayed in the following matrices:

$$
\overline{\mathbf{A}} = \begin{pmatrix} 0 & -\frac{1}{9} & 0 & 1 & \frac{7}{45} & \frac{-4}{45} & \frac{-2}{45} \\ 1 & \frac{8}{9} & 0 & 0 & \frac{-2}{45} & \frac{14}{45} & \frac{7}{45} \\ 0 & \frac{-7}{3} & 1 & 0 & \frac{12}{45} & \frac{-39}{45} & \frac{3}{45} \end{pmatrix}, \quad \begin{pmatrix} x_4^* \\ x_1^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}
$$

$$
\overline{c} = \begin{pmatrix} 0, & -2, & 0, & 0, & -3, & -1 & -4 \end{pmatrix}
$$

1. What is the basis matrix $\mathbf{A}_B$ relative to the basis $B = \{4, 1, 3\}$?
2. What is the inverse of the matrix $\mathbf{A}_B$?
3. What is the $\mathbf{y}$ vector relative to the basis B?
4. Write down the dual of the LP.
5. Show that the above $\mathbf{y}$ is an optimal solution to the dual.
6. Over what range of values of the coefficient c_5 is $B = \{4, 1, 3\}$ an optimal basis?
7. Over what range of values of the coefficient c_4 is $B = \{4, 1, 3\}$ an optimal basis?
8. Over what range of values of the coefficient b_1 is $B = \{4, 1, 3\}$ an optimal basis?
9. If the constraint
$$x_1 + x_2 + x_3 + 2x_4 \leq 6$$
is added to the given LP, will the current optimal solution $\mathbf{x}^* = (2, 0, 4, 1)$ remain optimal? Explain. (If it is not optimal, find the optimal solution. Do not solve the problem from scratch.)

(14) Consider the system $Ax = b,\ x \geq 0$ where

$$
A = \begin{pmatrix} 3 & 3 & -3 & -1 \\ -6 & 3 & 3 & 4 \\ 4 & -8 & -1 & 4 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}
$$

Either find a solution to $Ax = b,\ x \geq 0$ or find a vector y s.t. $yA \geq 0,\ yb < 0$.
(b) Solve the integer linear program. (If you wish, you may rearrange terms to obtain and a dictionary and work with the dictionary.)

(15) For each of the following statements, decide whether or not the statement is true. If a statement is true, provide a proof of the statement, and if it is false, give a counter-example[5]:

1. Every optimal solution to an LP in standard form is a basic feasible solution.
2. If an LP in standard form has a unique optimal solution x^*, then x^* is a basic feasible solution.
3. The dual of an unbounded LP is unbounded.
4. An optimal solution to an ILP (integer LP) can always be obtained by rounding (up or down) the entries of an optimal solution to the LP relaxation of the ILP?

(16) Consider the LP

$$(1) \qquad \max \mathbf{cx} \; s.t. \; \mathbf{Ax} = \mathbf{b}, \; \mathbf{x} \geq \mathbf{0}$$

Let $\mathbf{x}^*$ be an optimal solution to (1) and let $\mathbf{y}^*$ be an optimal solution to its dual. We obtain a second LP (2) from (1) by multiplying both sides of the first equality constraint by 2, and leave the remaining unchanged. Find optimal solutions to (2) and its dual.

C & O 350	**Assignment 6**	**Do not Hand in Solutions**

(17) **CUTTING PLANES**
Consider the following pure integer programming problem.

$$(ILP) \quad \begin{array}{lrcrcrcrcl} \max & 18x_1 & + & 12x_2 & & & & & & \\ \text{subject to} & 2x_1 & - & x_2 & & +x_3 & & & = & 5 \\ & 2x_1 & + & 3x_2 & & & & +x_4 & = & 13 \\ & & & & & & & x_j & \geq & 0, \quad \text{and integer } \forall j \end{array}$$

(a) Draw the feasible set of the LP relaxation in two dimensions, i.e. note that x_3 and x_4 are slack variables.
(b) Solve the LP relaxation.
(c) Find the optimal integer solution of ILP by generating a sequence of cutting planes.

(18) **CUTTING PLANES**
Consider the following pure integer programming problem.

$$(ILP) \quad \begin{array}{lrcrcl} \max & -3x_1 & + & 4x_2 & & \\ \text{subject to} & 6x_1 & - & 4x_2 & \geq & 15 \\ & x_1 & + & x_2 & \geq & 5 \\ & 4x_1 & + & 2x_2 & \leq & 31 \\ & & & x_j & \geq & 0, \quad \text{and integer } \forall j \end{array}$$

(a) solve the problem graphically.
(b) Solve the LP relaxation graphically. Round the optimal solution to the LP relaxation in every possible way (i.e. round each noninteger value either up or down.) check these points for feasibility and compute the objective values of those that are feasible. compare these with the soluiton found in part (a).
(c) solve this problem by the cutting plane method. Solve the LP relaxation of each subproblem graphically.

Solutions to questions 17 an 18 will be placed in the library.

[5]Thinking about questions of this type is a good way of understanding the material. As you study the subject, pose such questions for yourself.