Name: ______________________

I.D. #: ______________________

Section: ______________________

C&O 350 Linear Programming
Mid-Term Examination
Fall 1995
Thursday, 26 October, 1995
7:00 - 9:00 p.m. (2 hours)

1. Print your name and i.d. and section number in the space provided above.
2. This examination consists of 6 questions and 12 pages. Make sure you have a complete paper.
3. No calculating aids are allowed.
4. Write answers in space provided. If you need more space, use back of previous page.

Section 1	☐ Wolkowicz	TR	10:00-11:30
Section 2	☐ Murty	TR	11:30-1:00
Section 3	☐ Cunningham	TR	1:00-2:30

Question	Mark Value	Student Mark
1	20	
2	19	
3	18	
4	20	
5	18	
6	5	
Total	100	

[12] 1. (a) Solve the following linear programming problem by applying the simplex method. (Hint: Use the largest coefficient rule.)

$$\begin{array}{lrcrcrcr}
\text{maximize} & 6x_1 & + & 2x_2 & + & 2x_3 & & \\
\text{subject to} & 2x_1 & - & 2x_2 & + & x_3 & \leq & 4 \\
& 4x_1 & - & 4x_2 & & & \leq & 9 \\
& -2x_1 & + & 3x_2 & + & x_3 & \leq & 2 \\
& x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

1. continued,

[6] (b) Write the dual of the problem of (a) and give an optimal solution to the dual problem.

[2] (c) Suppose that in the problem of part (a), the right-hand side of the third constraint is changed from 2 to $2 + \epsilon$, where $|\epsilon|$ is very small. Give the optimal value (not an optimal solution) of the new problem.

[3] 2. (a) For the linear programming problem

$$(P) \quad \begin{array}{lrcll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, 2, \ldots, m) \\ & x_j & \geq & 0 & (j = 1, 2, \ldots, n) \end{array}$$

write the auxiliary problem (A) for finding an initial feasible solution.

[6] (b) Prove that (A) always has an optimal solution. You may use the fundamental theorem of linear programming.

2. continued,

[10] (c) For the following problem (P), construct the auxiliary problem and solve it. Hence give a feasible dictionary for (P). **(Do not solve (P).)**

$$
(P) \quad \begin{array}{lrcrcrcr}
\text{Maximize} & -3x_1 & - & x_2 & - & 4x_3 & & \\
\text{subject to} & -x_1 & - & x_2 & - & x_3 & \leq & -12 \\
 & -x_1 & - & 2x_2 & - & x_3 & \leq & -6 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

[2] 3. (a) Define the term "basic solution" of $Ax = b$. (You may assume that A is $m \times n$ with rank m.)

[2] (b) Define "degenerate basic solution" of $Ax = b$.

[6] (c) Consider the dictionary

$$(Dict) \qquad \begin{array}{rcl} x_i & = & \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \quad (i \in B) \\ z & = & \bar{v} + \sum_{j \in N} \bar{c}_j x_j. \end{array}$$

(i) Give the basic solution associated with (Dict).

(ii) Suppose that x_k is chosen as the entering variable, and x_r as the leaving variable. Give a formula for the objective value of the new basic solution.

(iii) What does it mean for an iteration or pivot of the simplex method beginning from (Dict) to be "degenerate"?

3. continued,

[8] (d) Consider the dictionary

$$(Dict)\qquad \begin{array}{rcrcrcrcr} x_6 &=& 4 &-& 2x_2 &-& x_3 &+& x_5 \\ x_4 &=& 2 &-& x_2 &+& x_3 &-& 2x_5 \\ x_1 &=& 6 &-& 2x_2 &-& 2x_3 &-& 3x_5 \\ z &=& 5 &+& 2x_2 &-& 2x_3 &+& 3x_5 \end{array}$$

(i) List all the pairs (x_k, x_r) such that x_k could be the entering variable and x_r could be the leaving variable on a simplex iteration beginning from (Dict).

(ii) List all the pairs if the largest-coefficient rule for choosing the entering variable is being used.

(iii) List all the pairs, if the smallest subscript rules for choosing the entering and leaving variables are being used.

(iv) List all the pairs if the lexicographic rule for choosing the leaving variable is used, and the constant terms of the first three equations of the dictionary are changed as follows:

$$\begin{array}{rclclcrcrcr} x_6 &=& 4 &+& 2\epsilon_1 - 2\epsilon_2 &-& 2x_2 &-& x_3 &+& x_5 \\ x_4 &=& 2 &+& \epsilon_1 + \epsilon_2 + 3\epsilon_3 &-& x_2 &+& x_3 &-& 2x_5 \\ x_1 &=& 6 &+& 2\epsilon_1 &-& 2x_2 &-& 2x_3 &-& 3x_5 \\ z &=& 5 & & &+& 2x_2 &-& 2x_3 &+& 3x_5 \end{array}$$

4. Consider the problem

$$(P) \qquad \begin{array}{lrcll} \text{Maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j & \le & b_i & (i = 1, 2, \ldots, m) \\ & x_j & \ge & 0 & (j = 1, 2, \ldots, n). \end{array}$$

[2] (a) Write the dual, (D), of (P).

[2] (b) State the Weak Duality relation (or property) for (P), (D).

[3] (c) State the Duality Theorem for (P), (D).

[3] (d) State the Complementary Slackness Conditions for (P), (D).

4. continued,

[10] (e) For the problem

$$\begin{array}{rrcrcrcrcl}
\text{Maximize} & 7x_1 & & & & & + & 9x_4 & & \\
\text{subject to} & 2x_1 & - & x_2 & + & x_3 & - & 2x_4 & \leq & 5 \\
& 3x_1 & + & x_2 & & & + & 3x_4 & \leq & 3 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}$$

use the Complementary Slackness Theorem to check whether each of the following are optimal solutions.

(i) $x^* = (1, 0, 3, 0)^T$

(ii) $x^* = (0, 0, 7, 1)^T$

5. For each of the following statements, tell whether the statement is true or false, and give a brief reason for your answer.

[3] (a) If a linear programming problem has an optimal solution, then its dual has a feasible solution.

[3] (b) If a linear programming problem is unbounded, then its dual is infeasible.

[3] (c) If a linear programming problem is infeasible, then its dual is unbounded.

[3] (d) If in an iteration of the simplex method, x_k is chosen as the entering variable, and there is more than one choice for the leaving variable, then the new basic solution will be degenerate.

5. continued,

[3] (e) If in an iteration of the simplex method, x_k is chosen as the entering variable and there is a unique choice for the leaving variable, then the new basic solution is not degenerate.

[3] (f) If a dictionary satisfies $\bar{b}_i \geq 0$ for all $i \in B$, and no simplex iteration starting from it gives a better objective value, then the basic solution associated with the dictionary is optimal.

[5] 6. Prove that if $x = 0$ is not an optimal solution of the problem

$$
\begin{array}{llrl}
 & \text{maximize} & cx & \\
(P) & \text{subject to} & Ax & \leq 0 \\
 & & x \geq & 0
\end{array}
$$

then (P) is unbounded.