Faculty of Mathematics
University of Waterloo
MATH 235 – Linear Algebra II
FINAL EXAMINATION

Date: December 10, 2001

Time: 3 hours
9:00 - 12:00

Clearly indicate your section.

	Instructor	Section	Lecture Time
☐	H. Wolkowicz	01	12:30
☐	H. Wolkowicz	02	11:30
☐	C.T. Ng	03	8:30
☐	C.G. Hewitt	04	8:30
☐	C.G. Hewitt	05	9:30

Surname: (please print) ______________________ Initials: ______

Signature:______________________ Id. Number: ______________

Instructions:

1. Write your name, signature, and ID number on this page.
2. Check the box next to your section.
3. Answer each question in the space provided. Give reasons for your answers and show all your work.
4. If you need more space, then please use the back of the previous page.
5. No calculators are allowed.
6. Please check that you have all 12 pages of this examination.

For Marker Only	
1	/8
2	/10
3	/12
4	/12
5	/10
6	/12
7	/6
8	/10
9	/10
10	/10
Total	/100

[8] Question #1.

a) Find the adjunct and inverse of the matrix B:

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 5 & 6 & 0 \end{bmatrix}.$$

b) State Cramer's rule and use it to find y in the equation:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 5 & 6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}.$$

[10] Question #2.
Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$, where

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

[12] Question #3.

Let B be the matrix

$$B = \begin{bmatrix} 5 & 4 & -4 \\ -2 & -3 & 2 \\ 6 & 4 & -5 \end{bmatrix}.$$

a) Given that the characteristic polynomial of B is $p(t) = -t^3 - 3t^2 + t + 3$, find the eigenvalues of B.

b) Find det(B) and Trace(B).

c) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B.

Question #3 continued

d) If B is the matrix given in part a), then write down the general solution to the differential equation:

$$\frac{d}{dt}\mathbf{x} = B\mathbf{x}.$$

e) Given that $\mathbf{x}$ at time t=0 is $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, use you solution to part d) to solve the initial value problem:

$$\frac{d}{dt}\mathbf{x} = B\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}.$$

[12] Question #4.

a) Find the algebraic and geometric multiplicities of all the eigenvalues of C, where

$$C = \begin{bmatrix} 2 & 3 & 5 & 0 & 0 & 3 & 1 & 0 \\ 0 & 2 & 6 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

b) Is C diagonalizable? Explain your answer.

[10] Question #5.

By diagonalising the appropriate quadratic form, sketch the conic:

$$4x^2 + 4y^2 + 10xy = 1.$$

[12] Question #6.

Let $\langle \quad , \quad \rangle$ be the standard inner product over $\mathbb{C}^n$. Let A be an Hermitian $n \times n$ matrix.

a) Show that $\langle A\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, A\mathbf{w} \rangle \quad \forall \quad \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$.

b) Show that every eigenvalue of A is real.

c) Show that if $(\lambda, \mathbf{v})$ and $(\mu, \mathbf{w})$ are eigenpairs of A, where λ and μ are distinct, then $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

[6] Question #7.

The matrix

$$A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix},$$

has characteristic polynomial $\Delta_A(t) = -(t+2)^2(t-4)$.

a) Let

$$p(t) = -(t+2)^2(t-4)(t^{50} + t^{40} + t^{30} + t) + t^2 - 2t - 8.$$

Evaluate $p(A)$.

b) Is A diagonalizable? Explain your answer.

[10] Question #8.

a) Let T be a linear operator on a vector space V. If W is a vector subspace of V, define what it means for W to be T-invariant.

b) Let $(V, \langle \ \ , \ \ \rangle)$ be an inner product space and let T be a linear operator on V. Prove the following result (which is used in the proof of Schur's theorem) :

If $\mathbf{u}$ is an eigenvector of T^* and if W=span$(\{\mathbf{u}\})$, then $W^\perp$ is T-invariant.

[10] Question #9.

Let T be a linear operator on a finite-dimensional vector space, and let λ be an eigenvalue of T.

a) Define the algebraic multiplicity, a_λ, of λ.

b) Define the eigenspace, E_λ, of λ.

c) Define the geometric multiplicity, g_λ, of λ.

d) Prove Lemma 3.7 from the notes, which states that:

$$1 \leq g_\lambda \leq a_\lambda.$$

[10] Question #10.

Let A be an $n \times n$ matrix, and suppose you are told that:

$$A^3 = A.$$

a) Explain why there are only seven possible candidates for the minimum polynomial of A. Write these polynomials down and label them $p_1(t)$, $p_2(t), .., p_7(t)$.

b) Is the above information, $A^3 = A$, sufficient to claim that A is diagonalizable?

c) Show that if λ is an eigenvalue of A then $\lambda \in \{-1, 0, 1\}$.

d) For each one the polynomials, $p_i(t)$, found in part a), provide a 3×3 real matrix A_i with the property that $p_i(t)$ is the minimum polynomial of A_i.