Math 235/235J Final Examination
Solutions and Marking Scheme

Wednesday, December 13, 2000
2:00 – 5:00 p.m.

University of Waterloo

[Marks]

[12] 1. (a) If A is a 2×2 matrix and $\mathrm{Tr}(A)$ its trace, verify that

$$(\mathrm{Tr}(A))^2 - \mathrm{Tr}(A^2) = 2\det(A).$$

Compute the following determinants in (b) to (e):

(b) $\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & 2 & 0 & 5 \end{bmatrix}.$

(c) $\det \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & -6 \\ -1 & 1 & -2 & 19 \\ 0 & 2 & 2 & 5 \end{bmatrix}.$

(d) $\det \begin{bmatrix} -1 & 3 & -3 & 3 \\ 2 & -6 & 6 & -6 \\ -1 & 0 & -2 & 9 \\ 0 & 2 & 12 & 5 \end{bmatrix}.$

(e) $\det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 5 \end{bmatrix}$. Compute this determinant by using row reduction to obtain an upper triangular matrix.

[8] 2. Assume that the matrices of the linear systems in (a) and (b) below are nonsingular. Use Cramer's Rule to solve these systems for y (only).

(a)
$$\begin{aligned} ax + by &= 1 \\ cx + dy &= 0 \end{aligned}$$

Solution: Value is 2:

$$y = \frac{\det \begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix}}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \frac{-c}{ad - bc}$$

-1 for wrong sign (in both parts a and b). ■

(b)
$$\begin{aligned} ax + by + cz &= 1 \\ dx + ey + fz &= 0 \\ gx + hy + iz &= 0 \end{aligned}$$

(Denote the determinant of the matrix of this system by D. You need not expand it.)

Solution: Value is 3:

$$y = \frac{\det \begin{pmatrix} a & 1 & c \\ d & 0 & f \\ g & 0 & i \end{pmatrix}}{D} = \frac{fg - di}{D}$$

-1 for wrong sign (in both parts a and b).

(c) Find the value of the parameter t which makes the vectors

$$[5, -1, 1], [-2, 1, -t], [9, 3, 2t]$$

linearly dependent.

Solution: Value is 3:
There are many ways to solve this, e.g. set the determinant to 0 or use Gaussian elimination. Marks were given if clear details were given for the equivalence of linear dependence. Arithmetic errors were penalized but judgement was used on how clear the rest of the solution was, e.g. no marks were given for just stating the equivalence in terms of linear dependence and not getting started on the arithmetic.
Set

$$\begin{aligned} 0 &= \det \begin{pmatrix} 5 & -2 & 9 \\ -1 & 1 & 3 \\ 1 & -t & 2t \end{pmatrix} \\ &= 5(2t+3t) + (-4t+9t) + (-6-9) \\ &= 30t - 15 \end{aligned}$$

or $t = 15/30 = 1/2$.

[10] 3. Let $A = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$.

(a) Calculate the characteristic polynomial of A, and determine the eigenvalues of A.

(b) Let λ be the eigenvalue of A with smallest absolute value. Determine all eigenvectors of A associated with λ.

(c) Is A diagonalizable? (Justify your claim.)

[12] 4. (a) Let A be a square matrix with eigenvalue λ. By using induction on k, show that λ^k is an eigenvalue of A^k for all integers $k \geq 1$.

(b) Assume that A satisfies the matrix equation $A^2 + A - 2I = 0$. Show that if λ is an eigenvalue of A then either $\lambda = 1$ or $\lambda = -2$.

(c) Assume further that A is $n \times n$ and that $\mathrm{Tr}(A) = 0$. Show that n is divisible by 3. (Hint: Use the result from part (b) and consider the (algebraic) multiplicities of the eigenvalues 1 and -2.)

[12] 5. Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

[12] 6. Answer one part only.

Part I. Let $Q : \mathbf{R}^2 \to \mathbf{R}$ be the quadratic form defined by $Q(x, y) = -3x^2 + 8xy + 3y^2$.

(a) Find the symmetric matrix A such that $Q(x, y) = [x, y]A \begin{pmatrix} x \\ y \end{pmatrix}$.

(b) Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

(c) Sketch in the x–y plane the conic defined by the equation $Q(x,y)=1$.

Part II. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & 4 & 3 \end{bmatrix}$. Determine the most general solution $x(t)$ to the system of ordinary differential equations $\dfrac{dx}{dt} = Ax$. Show clearly how the solution is obtained.

[12] 7. Let $\mathcal{M}$ be the space of 3 by 3 real matrices and $L : \mathcal{M} \to \mathcal{M}$ the linear operator defined by $L(X) = AX - XA$ where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $T : \mathcal{M} \to \mathbf{R}^3$ be the linear transformation which maps an arbitrary matrix $X = [x_{ij}] \in \mathcal{M}$ to $(x_{11} + x_{22} + x_{33}, x_{21} + x_{32}, x_{31})$.

(a) Compute $L(X)$ for an arbitrary $X \in \mathcal{M}$ and find a basis for the kernel of L.

(b) Show that range$(L) =$ kernel(T).

[12] 8. Let $(\cdot,\cdot)$ be the standard inner product over $\mathbf{C}^n$. Let A be a Hermitian matrix of size n.

(a) Show that $(A\mathbf{v}, \mathbf{w}) = (\mathbf{v}, A\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathbf{C}^n$.

(b) Show that every eigenvalue λ of A is real.

(c) Show that if $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of A corresponding to distinct eigenvalues λ and μ respectively, then $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

[10] 9. Determine if the statements below are **true** or **false**. Indicate your answers by entering **T** or **F** in the boxes to the right. You do not need to justify your answers.

Each correct answer is worth 2 marks. Each incorrect answer is worth -1 mark. If you do not give an answer, you get 0 marks for that statement. If the total score for this question is negative, it will be replaced by zero.

(a) If A and B are $n \times n$ real matrices, then $\det(AB) = \det(A^T B^T)$.

(b) Similar matrices have the same eigenvectors.

(c) All eigenvalues of a square matrix are nonzero.

(d) If A is a unitary matrix then $|\det(A)| = 1$.

(e) If $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent unit vectors in a complex inner product space, then $\mathbf{w} = \mathbf{v}$ or $\mathbf{w} = -\mathbf{v}$.