Modeling Uncertainty

In many practical problems there are uncertain parameters.

Ex:
- multiperiod investment problems
  (uncertain returns, stock developments ...)
- inventory management problems
  (uncertain demands, ...)

Today: how to incorporate uncertainty into LP model

Introductary Example Facility location

F : set of facilities $\forall i \in F$, have opening cost $f_i \geq 0$
capacity $u_i \geq 0$

C : set of clients $\forall j \in C$, have demand $d_j \geq 0$

$\forall i \in F, j \in C$: c_{ij} : per-unit demand service cost

Goal: 1) decide on subset $F' \subseteq F$ of facilities to open
2) assign each client $j \in C$ to a facility $i(j) \in F'$
s.t.

a) $$\sum_{\substack{j \in C \\ i(j) = i}} d_j \leq u_i \quad \forall i \in F'$$

b) $$\boxed{\text{total opening cost}} + \boxed{\text{total service cost}}$$

$$= \sum_{i \in F'} f_i + \sum_{j \in C} d_j \cdot c_{i(j),j}$$

is minimized.

Write LP $\quad \forall i \in F: y_i = \begin{cases} 1 & : \text{ open fac } i \\ 0 & : \text{ othw.} \end{cases}$

$$\forall i \in F, j \in C: \quad x_{ij} = \begin{cases} 1 & : \text{ client } j \text{ assigned to } i \\ 0 & : \text{ othw.} \end{cases}$$

$$\min \sum_{i \in F} y_i \cdot f_i + \sum_{i \in F} \sum_{j \in C} x_{ij} \cdot d_j \cdot c_{ij}$$

s.t.

$$\sum_{i \in F} x_{ij} = 1 \qquad \forall j \in C$$

$$x_{ij} \le y_i \qquad \forall i \in F$$

$$\sum_{j \in C} d_j \cdot x_{ij} \le u_i y_i \qquad \forall i \in F$$

$$0 \le x_{ij}, y_i \le 1 \qquad \forall i, j$$

Now suppose that customer demand varies:

$$d_j \in [\, d_j^0 - \Delta_j,\ d_j^0 + \Delta_j \,] \qquad \forall j \in C$$

mean demand

worst case deviation

maybe obtained from historic data.

Solving LP with d_j replaced by d_j^0 may not lead to feas soln.

What to do with this?

1) Assume worst-case: substitute d_j by $(d_j^0 + \Delta_j)$ $\forall j \in C$. Very pessimistic!

2) Company might have prob. dist. for demand d_j (fitting hist. data). So, d_j is a r.v.

How do we use this in LP?

- replace d_j by $E[d_j]$ in obj func.
  $\Rightarrow$ loose info on variance of d_j!

- can't use $E[d_j]$ in constraints

2-stage stochastic optimization with recourse

Today: Open some facilities using distributional
Stage 1 info on demand

Tomorrow: Know exact demands.
Stage 2
1) Open extra facilities
2) Assign clients to fac opened in stages 1+2.

recourse actions

Recourse actions incur recourse costs.

Note: Opening fac today (in stage 1) may be cheaper than last minute patching in stage 2!
It may therefore be smart to prepare oneself in stage 1.

Scenarios: Uncertainty is modeled via scenarios.

$\mathcal{A}$: set of scenarios

For each $a \in \mathcal{A}$, have

- p_a : prob that it occurs

- $\{d_j^a\}_{j \in C}$ demand realizations
  $d_j^a \in [d_j^\circ - \Delta_j, d_j^\circ + \Delta_j]$
- facility-opening costs $\{f_i^a\}_{i \in F}$
- facility capacities $\{u_i^a\}_{i \in F}$

Goal: Find a stage one strategy $\{y_i\}_{i \in F}$,
a recourse action $\{y_i^a\}_i$ for each $a \in \mathcal{A}$,
and an assignment $\{x_{ij}^a\}_{ij}$ $\forall a \in \mathcal{A}$
that has min exp. cost.

New LP model

Variables

$$y_i = \begin{cases} 1 & : \text{ open } i \in F \text{ in stage 1} \\ 0 & : \text{ other} \end{cases}$$

$\forall a \in \mathcal{A}$:

$$\begin{cases} y_i^a = \begin{cases} 1 & : \text{ open } i \in F \text{ in stage 2, scen } a \\ 0 & : \text{ other} \end{cases} \\ x_{ij}^a = \begin{cases} 1 & : \text{ assign } j \text{ to } i \text{ in stage 2, scen } a \\ 0 & : \text{ other} \end{cases} \end{cases}$$

$$\min \underbrace{\sum_{i \in F} f_i \cdot y_i}_{\text{stage 1 cost}} + \underbrace{\sum_{a \in \mathcal{A}} p_a \cdot \underbrace{\left(\sum_{i \in F} f_i^a y_i^a + \sum_{i \in F} \sum_{j \in C} d_j \, x_{ij}^a \cdot c_{ij} \right)}_{\text{cost of scen } a}}_{\text{exp. second stage cost}}$$

s.t.

$$\sum_{i \in F} x_{ij}^a = 1 \qquad \forall j \in C, \forall a \in \mathcal{A}$$

$$x_{ij}^a \leq y_i + y_i^a \qquad \forall i \in F, j \in C, \forall a \in \mathcal{A}$$

$$\sum_{j \in C} d_j^a x_{ij}^a \leq u_i^a y_i^a \qquad \forall i \in F, a \in \mathcal{A}$$

$0 \le x_{ij}^{a}, y_i^{a}, y_i \le 1 \quad \forall i, j, a$

Assume $\forall a \in A$:

$$\sum_{a \in A} d_j^a \le \sum_{i \in F} u_i^a$$

Setting:
- A farmer raises grain, corn and sugar beets on 500 acres of land.
- During Winter he decides how much of his land to devote to each crop
- He also raises cattle and needs 200t of wheat and 240t of corn to feed them

  Corn and wheat can either be bought or grown on the farm
- Production exceeding feeding req can be sold at $170 p.t of wheat

  $150 p.t of corn
- Purchase prices are 40% higher (delivery costs etc): wheat: $238 p.t corn: $210 p.t
- Sugar beets sell at $36 p.t

  Can sell at most 6000t p.year

  excess can be sold at $10 p.t.

- yield p. acre in tons is uncertain
  mean yields

wheat	2.5 t
corn	3 t
sugar beet	20 t

- Planting costs p. acre

wheat	\$150
corn	\$230
sugar beets	\$260

Setting up an LP model.

<u>1st attempt</u> Variables

x_w : acres of land reserved for wheet

x_c : —— " —— corn

x_s : —— " —— sugarbeets

s_w : tons of wheat sold

s_c : —"— corn sold

s_s^h : —"— sugar beets sold at high price

s_s^l : —— " —— low price

b_w : tons of wheat bought

b_c : —"— corn —"—

Objective function:

min $150 x_w + 230 x_c + 260 x_s +$ [planting]

$238 b_w + 210 b_c -$ [buying]

$(170 s_w + 150 s_c + 36 s_s^h + 10 s_s^l)$ [sales]

Constraints:

(i) can plant at most 500 acres

$$x_c + x_w + x_s \leq 500$$

(ii) need at least 200t of wheat:

$$\underbrace{2.5 \cdot x_w}_{\text{exp. \#t of wheat}} + b_w - s_w \geq 200$$

need 240t of corn:

$$3 \cdot x_c + b_c - s_c \geq 240$$

(iii) need to relate s_s^h, s_s^l and x_s. How?

$$\underbrace{20 \cdot x_s}_{\text{exp. \#t of sugar beets}} \geq s_s^h + s_s^l$$

(iv) can't sell more than 6000t of sug. b. at high price

$$s_s^h \leq 6000$$

(v) non-neg. $x, b, s \geq 0$

<u>Solution</u>

Culture	Wheat	Corn	Beets	
X	120	80	300	
E yield (t)	300	240	6000 t	
Sale (t)	100	—	6000 t	(high pr)
	—	—	—	(low pr)

Interpretation: • Sugar beets are most profitable (when sold at high pr)
→ devote most land to this until 6000t quota is reached

• then satisfy corn and wheat req
• rest goes to wheat

Solution follows simple heuristic: allocate land in order of decr. profit p. acre.

sug.b.	$20 \cdot 36 - 260 =$	460
corn	$3 \cdot 150 - 230 =$	220
wheat	$2.5 \cdot 170 - 150 =$	275

Is this model good? Depending on variance of yields, maybe not.
Weather cond ⇒ yields deviating in ± 25% range not unusual

Scenarios

Assume that there are good, fair and bad years

good	yields 20% higher than means
fair	mean yields
bad	20% below mean

No price variations.

Farmer runs above model for all 3 scenarios.

Scenario	Profit
bad	\$59,950
fair	\$118,600
good	\$167,667

Large variations also in solution structure.

Critical issue: how much land to devote to sugar beets

- highly profitable to sell at high price
- not so good to sell at low price

Planning problem: Decisions have to be made now while yields manifest themselves only later.

$\rightarrow$ sales and purchases depend on yields

Idea: Maximize expected profit

Assume: each scenario happens with prob $\frac{1}{3}$

Have set of variables for each scenario:

$s_{i\gamma}$: # t of crop $i \in \{w, c\}$ sold in scenario $\gamma \in \{g, f, b\}$

sim: $s^h_{s\gamma}, s^l_{s\gamma}$: $\gamma \in \{g, f, b\}$

$b_{i\gamma}$: # t of crop $i \in \{w, c\}$ purchased in sc. $\gamma \in \{g, f, b\}$

Now max exp. profit:

$$\max -(150 x_w + 230 x_c + 260 x_s) + \sum_{\gamma \in \{g,f,b\}} \left(238 b_{w\gamma} + 210 b_{c\gamma} - 170 s_{w\gamma} - 150 s_c - 36 s^h_{s\gamma} - 10 s^l_{s\gamma} \right)$$

(i) acre constr. unchanged:

$$x_c + x_w + x_s \leq 500$$

(ii) For each $r \in \{g, f, d\}$ include $\quad p_r = \begin{cases} .8 : & s=d \\ 1 : & s=f \\ 1.2 : & s=g \end{cases}$

$$p_r \cdot 2.5 \cdot x_{wr} + b_{wr} - s_{wr} \geq 200 \quad \text{(wheat)}$$

$$\cdot 3 \cdot x_{cr} + b_{cr} - s_{cr} \geq 240 \quad \text{(corn)}$$

$$p_r \cdot 20 \cdot x_{sr} \geq s^h_{sr} + s^l_{sr} \quad \text{(sugar beets)}$$

$$s^h_{sr} \leq 6000 \quad \text{(sugar beets cap)}$$

LP model like this is known as extensive form of stoch prog. as it describes 2nd stage variables $\forall$ scenarios.

Solution: $\quad x_c = 80 \;,\; x_w = 170 \;,\; x_s = 250$

Exp. profit: \$ 108,390

Obs: 1) planting area of sugar beets avoids selling at low price even in good scenario:

$$1.2 \cdot 20 \cdot 250 = 6000$$

2) Corn area satisfies req. in good and fair years, not in bad years

3) rest is allocated to wheat

Impossible to find a single first stage decision that is good in all scenarios!

Expected Value of Perfect Information (EVPI)

Suppose the farmer knew the weather in coming year. He could then solve LP 1 for the true yields.

Profit :	good	\$167 667 (i)	expected profit:
	fair	\$ 118 600 (ii)	$\frac{1}{3}$ ((i) + (ii) + (iii)) =
	bad	\$ 59,950 (iii)	\$ 115 406

EVPI = \$115,406 − \$ 108390 = \$7016

Loss of profit due to presence of uncertainty.

Value of Stochastic Solution (VSS)

Another strategy is to solve original LP with expected yields (what we did at beg) and use the optimal solution to far planting decisions:

$$x_w = 120 \qquad x_c = 80 \qquad x_s = 300$$

<u>Scenarios</u>

				buy	sell
<u>Bad</u>	yields	Wheat	240	0	40
		Corn	192	48	0
		Sugar Beets	4800		4800

net rev. from buying selling:

$$40 \cdot 170 + 4800 \cdot 36 - 48 \cdot 238 = 169520$$

Fair	yields	Wheat	300	0	100
		Corn	240	0	0
		Sugar Beets	6000	0	6000

net rev. from buying selling:

$100 \cdot 170 + 6000 \cdot 36 = 233{,}000$

Good	yields	Wheat	360	0	160
		Corn	288	0	48
		Sugar Beets	7200	0	6000 h 1200 L

net rev. from buying selling:

$160 \cdot 170 + 6000 \cdot 36 + 48 \cdot 150 + 1200 \cdot 10 =$

$262{,}400$

Expected profit:

$$\frac{1}{3}(169520 + 233{,}000 + 262400)$$
$$-(150 \cdot 120 + 230 \cdot 80 + 300 \cdot 260)$$
$$= 107240$$

VSS: $108390 - 107240 = 1150$

Robust Optimization

In the above example, no matter what the farmer's 1st stage decision was, he could buy / sell corn and wheat appropriately in 2nd stage to obtain a feasible solution.

$\Longrightarrow$ uncertainty in data may lead to suboptimal decisions but never to infeasible ones!

Sometimes, no recourse is possible and data uncertainty may cause 1st-stage decisions to lead to infeasible solutions!

Suppose we want to solve

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \qquad A = \begin{bmatrix} A_1, \dots, A_n \end{bmatrix}, \; A_i = \begin{bmatrix} a_{i1} \\ \vdots \\ a_{im} \end{bmatrix}$$

Data uncertainty:

Suppose that the actual value of coeff. in row i and col j is uncertain. Particularly, assume that

$$\tilde{a}_{ij} \in [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$$

and define

$$\eta_{ij} = \frac{\tilde{a}_{ij} - a_{ij}}{\hat{a}_{ij}} \in [-1, -1]$$

Robust formulation of Soyster

<u>Goal:</u> want an LP whose solution is feasible even under data variations

Soyster '73

$$\hat{A} = \begin{bmatrix} \hat{a}_{11} & \cdots & \hat{a}_{1n} \\ \vdots & & \vdots \\ \hat{a}_{m1} & \cdots & \hat{a}_{mn} \end{bmatrix}$$

$$(P_1) \quad \begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax + \hat{A}y \le b_i \\ & -y_j \le x_j \le y_j \quad \forall j \in \{1..n\} \\ & y, x \ge 0 \end{aligned}$$

Let x^* be an optimal solution to (P_1). We may assume that $y_j = |x^*_j| \ \forall j$. Why?

Want to show: For any possible realization $\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & & \vdots \\ \tilde{a}_{m1} & \cdots & \tilde{a}_{mn} \end{bmatrix}$, x^* is feasible for

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \tilde{A}x \le b \\ & x \ge 0 \end{aligned}$$

$$\sum_j \tilde{a}_{ij} x_j^* = \sum_j a_{ij} x_j^* + \sum_j \eta_{ij} \hat{a}_{ij} x_j^*$$

$$\leq \sum_j a_{ij} x_j^* + \underbrace{\sum_j \hat{a}_{ij} |x_j^*|}_{\text{protection against constraint violation}} \leq b_i \qquad \forall i$$

Ex: In the farming example, suppose the farmer cannot buy wheat/corn. I.e., he must grow enough corn/wheat to satisfy his feeding requirements.
Give a robust formulation!

	yield range	
Wheat	$[.8 \cdot 2.5,\ 1.2 \cdot 2.5] = [2, 3]$	$\hat{a}_w = .5$
Corn	$[.8 \cdot 3,\ 1.2 \cdot 3] = [2.4, 3.6]$	$\hat{a}_c = .6$

Variables as before:

[obj]
$$\left.\begin{aligned} \max\ & -(150\, x_w + 230\, x_c + 260\, x_s) \\ & + 170\, s_w + 150\, s_c + 36\, s_s^h + 10\, s_s^l \end{aligned}\right\} \text{profit}$$

s.t.
$$x_w + x_c + x_s \leq 500$$
$$2.5 \cdot x_w - s_w - \hat{a}_w y_w \geq 200 \qquad \text{(i)}$$
$$3\, x_c - s_c - \hat{a}_c y_c \geq 240 \qquad \text{(ii)}$$
$$s_s^h + s_s^l \leq 20 \cdot x_s$$
$$s_s^h \leq 6000$$
$$-y \leq x \leq y$$

$$x, s, y \geq 0$$

As $x \geq 0$ we can simplify (i) and (ii) :

$$2 \cdot x_u - s_u \geq 200$$
$$2.4 \cdot x_c - s_c \geq 240$$

and delete y from model.

Soyster's model is very conservative as it assumes the worst case deviations in data.

Now discuss a model due to Bertsimas and Sim that is less conservative.

Let's assume that $J_i \subseteq \{1..n\}$ is the set of indices of coeff. in ith row that can vary.

$$\forall i \in \{1..m\}, \forall j \in J_i : \quad \tilde{a}_{ij} \in [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$$

Is it really likely that a_{ij} changes $\forall i \in \{1..m\}$ and $\forall j \in J_i$ at the same time?

Introduce param Γ_i that takes values in $\{0, \dots, |J_i|\}$

Intuition: Want to express fact that at most Γ_i coeff's in row i can change at the same time.

Define the protection of constraint i as

$$\beta_i(x,\Gamma_i) = \underbrace{\max_{S_i \subseteq J_i,\, |S_i| = \Gamma_i} \sum_{j\in S_i} \hat{a}_{ij}|x_j|}_{\circledast}$$

Then write robust program:

$$\max \quad c^T x$$

s.t.

$$\sum_j a_{ij}x_j + \beta_i(x,\Gamma_i) \le b_i \qquad \forall i$$

$$x \ge 0$$

This is not an LP for two reasons:

1) β_i has absolute values
2) β_i has max

Proposition 1:

$$\beta_i(x,\Gamma_i) = \max \sum_{j\in J_i} \hat{a}_{ij}\,|x_j|\,z_{ij} \qquad (P_2^i)$$

s.t.

$$\sum_{j\in J_i} z_{ij} \le \Gamma_i \qquad [z_i]$$

$$0 \le z_{ij} \le 1 \qquad [p_{ij}]$$

Pf: Let $J_i = \{1,2,\dots,p\}$ s.t.

$$\hat{a}_{i1}|x_1| \ge \hat{a}_{i2}|x_2| \ge \cdots \ge \hat{a}_{ip}|x_p|$$

An optimal soln to (P_2) is

$$z_{i_1} = \cdots = z_{i_{\lfloor \Gamma_i \rfloor}} = 1$$

The set $S_i = \{1, \dots, \Gamma_i\}$ achieves same val in ⊛.

Reverse direction similar: let S_i be maximizer of (*).

$$z_{ij} = 1 \quad \forall j \in S_i$$

achieves same value in LP.

∎

Consider the dual of (P_2)

$$(D_2^i) \quad d_i := \min \sum_{j \in J_i} p_{ij} + \Gamma_i \cdot z_i$$

s.t.

$$z_i + p_{ij} \geq \hat{a}_{ij} |x_j| \qquad \forall j \in J_i$$

$$p_{ij}, z_i \geq 0 \quad \forall j \in J_i$$

Strong duality: (P_2^i) has opt. sol. of val v

$\Leftrightarrow (D_2^i)$ has opt. sol of val v

(P_2^i) is indeed always feasible and bounded.

$$\Rightarrow \beta_i(x, \Gamma_i) = d_i$$

Can reformulate robust LP :

$$(P_3) \qquad \max \quad c^T x$$

s.t.

$$\sum_j a_{ij} x_j + d_i \leq b_i \qquad \forall i$$

(D_2^i)

$$\left.\begin{array}{ll} d_i = z_i \Gamma_i + \sum_{j \in J_i} p_{ij} & \\ z_i + p_{ij} \geq \hat{a}_{ij}\, y_j & \forall j \end{array}\right\} \quad \forall i$$

$$-y_j \leq x_j \leq y_j \qquad \forall j$$

$$x, p, z \geq 0$$

$\Longrightarrow$ Solution x^* is feasible for

$$\max \quad c^T x$$
$$\text{s.t.} \quad \tilde{A} x \leq b$$
$$x \geq 0$$

as long as, for all i :

$$\tilde{a}_{ij} \neq a_{ij} \quad \text{for at most } \Gamma_i \text{ indices } j \in J_i$$

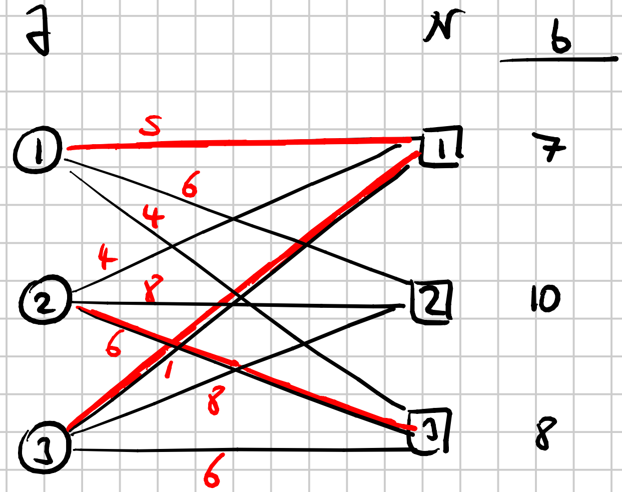
Example

3 jobs $J = \{1, 2, 3\}$ need to be assigned to 3 people $N = \{1, 2, 3\}$.

It takes person $i \in N$ p_{ij} hours to complete job $j \in J$.

Person $i \in N$ has b_i hours of time available to work on jobs.

feas assignm.



$x_{11} = x_{32} = x_{13} = 1$

$x_{ij} = 0$ othw.

Want to minimize total work time.

$$x_{ij} : \begin{cases} 1 & : \text{ job } j \text{ assigned to person } i \\ 0 & : \text{ othw.} \end{cases}$$

$$\min \sum_{i \in N} \sum_{j \in J} p_{ij} \cdot x_{ij}$$

s.t.

$$\sum_{j \in J} p_{ij} x_{ij} \leq b_i \qquad \forall i \in N \quad \leftarrow \text{ no worker overloaded}$$

$$\sum_{i \in N} x_{ij} = 1 \qquad \forall j \in J \quad \leftarrow \text{ each job assigned}$$

$$x \geq 0$$

In example:

$$\min \quad (5x_{11} + 6x_{21} + 4x_{31}) + (4x_{12} + 8x_{22} + 6x_{32}) + (x_{13} + 8x_{23} + 6x_{33})$$

s.t.

$$\begin{aligned}
5x_{11} + 4x_{12} + x_{13} &\le 7 \\
6x_{21} + 8x_{22} + 8x_{23} &\le 10 \\
4x_{31} + 6x_{32} + 6x_{33} &\le 8 \\
x_{11} + x_{21} + x_{31} &= 1 \\
x_{12} + x_{22} + x_{32} &= 1 \\
x_{13} + x_{23} + x_{33} &= 1
\end{aligned}$$

$$x \geq 0$$

Now assume that handling times are uncertain:

$$\tilde{p}_{ij} \in [p_{ij} - \hat{p}_{ij}, p_{ij} + \hat{p}_{ij}]$$

$$\hat{p}_{ij} = 1 \qquad \forall i,j$$

Observe: Example assignment becomes infeasible if only jobs 1, 3 take longer for person 1 and the other remains unchanged.

Soyster's model Only green constraints change.

$$\begin{aligned}
5x_{11} + 4x_{12} + x_{13} + 1\cdot y_{11} + 1\cdot y_{12} + 1\cdot y_{13} &\le 7 \\
6x_{21} + 8x_{22} + 8x_{23} + 1\cdot y_{21} + 1\cdot y_{22} + 1\cdot y_{23} &\le 10 \\
4x_{31} + 6x_{32} + 6x_{33} + 1\cdot y_{31} + 1\cdot y_{32} + 1\cdot y_{33} &\le 8
\end{aligned}$$

(*) $-y_{ij} \le x_{ij} \le y_{ij} \qquad \forall i,j$

In this example $x_{ij} \geq 0 \quad \forall i,j$.

If (x,y) feasible for soyster LP, $(x,\bar{y})$ feasible where

$$\bar{y}_{ij} = |x_{ij}| = x_{ij}$$

The LP can be rewritten:

$$\begin{aligned}
5x_{11} + 4x_{12} + x_{13} + (x_{11} + x_{12} + x_{13}) &\leq 7 \\
6x_{21} + 8x_{22} + 8x_{23} + (x_{21} + x_{22} + x_{23}) &\leq 10 \\
4x_{31} + 6x_{32} + 6x_{33} + (x_{31} + x_{32} + x_{33}) &\leq 8
\end{aligned}$$

$$\iff \begin{aligned}
6x_{11} + 5x_{12} + 2x_{13} &\leq 7 \\
7x_{21} + 9x_{22} + 9x_{23} &\leq 10 \\
5x_{31} + 7x_{32} + 7x_{33} &\leq 8
\end{aligned} \qquad (*)$$

Note: $x_{11} = x_{32} = x_{13} = 1$, $x_{ij} = 0$ other

not feasible for $(*)$. Red constraint is violated.

Bertsimas & Sim

Now assume that only 1 job per person can change processing time.

In example: load on person 1 is currently 6, cap is 7

$\Rightarrow$ assignment remain feasible even under perturbation.

Bertsimas and Sim: $\quad \mathcal{J}_i = \{1,2,3\} \quad \forall i \in \mathcal{N}$

$\forall i \in \mathcal{N}$:

$$\beta_i(x,\Gamma_i) = \max_{\substack{S_i \subseteq \mathcal{J}_i \\ |S_i| = \Gamma_i}} \sum_{j \in S_i} \underset{\substack{\shortparallel \\ 1}}{\hat{a}_{ij}} \, \underbrace{|x_{ij}|}_{\geq 0}$$

$\Gamma_i \uparrow 1$

$$= \max_{j \in \mathcal{J}} x_{ij}$$

Once again only green part changes: $\qquad$ protection

$$5x_{11} + 4x_{12} + x_{13} + \beta_1(x,\Gamma_1) \leq 7$$
$$6x_{21} + 8x_{22} + 8x_{23} + \beta_2(x,\Gamma_2) \leq 10$$
$$4x_{31} + 6x_{32} + 6x_{33} + \beta_3(x,\Gamma_3) \leq 8$$

Rewriting $\beta_i(x,\Gamma_i)$.

$$\beta_i(x,\Gamma_i) = \max_{j \in \mathcal{J}} x_{ij}$$

$$= \max \sum_{j \in \mathcal{J}} x_{ij} \cdot z_{ij}$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}} z_{ij} \leq 1 \qquad [z_i]$$

at most one job can change time

$$0 \leq z_{ij} \leq 1 \quad \forall j \qquad [v_{ij}]$$

strong duality

$$= \min \sum_{j \in \mathcal{J}} z_{ij} + z_i$$

$$\text{s.t.} \quad z_i + z_{ij} \geq x_{ij} \quad \forall j \in \mathcal{J}$$

$$z \geq 0$$

Here:

$$\beta_i(x,\Gamma_i) = \min \; z_{i1} + z_{i2} + z_{i3} + z_i \quad =: d_i$$

$$\begin{aligned} \text{s.t.} \quad & z_i + z_{i1} \geq x_{i1} \\ & z_i + z_{i2} \geq x_{i2} \\ & z_i + z_{i3} \geq x_{i3} \\ & z \geq 0 \end{aligned}$$

Introduce var d_i for $\beta_i(x,\Gamma_i)$, replace $\beta_i(x,\Gamma_i)$ by d_i and add blue constr. above.

New LP:

$$\begin{aligned} \min \quad & (5x_{11} + 6x_{21} + 4x_{31}) + \\ & (4x_{12} + 8x_{22} + 6x_{32}) + \\ & (x_{13} + 8x_{23} + 6x_{33}) \end{aligned}$$

$$\begin{aligned} \text{s.t.} \quad & 5x_{11} + 4x_{12} + x_{13} + d_1 \leq 7 \\ & 6x_{21} + 8x_{22} + 8x_{23} + d_2 \leq 10 \\ & 4x_{31} + 6x_{32} + 6x_{33} + d_3 \leq 8 \\ & \left.\begin{aligned} & z_i + z_{i1} \geq x_{i1} \\ & z_i + z_{i2} \geq x_{i2} \\ & z_i + z_{i3} \geq x_{i3} \\ & d_i = z_{i1} + z_{i2} + z_{i3} + z_i \end{aligned}\right\} \forall i \in \{1,2,3\} \\ & 0 \leq z_{ij} \leq 1 \quad \forall ij \\ & x, z \geq 0 \end{aligned}$$

Consider again solution

$$x_{11} = x_{32} = x_{13} = 1, \quad x_{ij} = 0 \text{ otherw}$$

Set $z_1 = 1, z_3 = 1, z_2 = 0 \qquad z_{ij} = 0 \quad \forall ij$

$$\Longrightarrow \quad d_i = 1 \quad \forall\, i \in N$$

(x, z, d) is feasible!