Sensitivity Analysis

- Discuss mainly the standard inequality case:

$$\begin{aligned} \max \quad & z = \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{i,j} x_j \le b_i, \quad i = 1,\dots,m \\ & x_j \ge 0, \quad j = 1,\dots,n \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & z = \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{i,j} x_j + x_{n+i} = b_i, \quad i = 1,\dots,m \\ & x_j \ge 0, \quad j = 1,\dots,n,n+1,\dots,n+m \end{aligned}$$

  - Maximize Profit given limited resources
    - each constraint associated to a resource
- Alternate interpretations of shadow price and reduced cost
- Sensitivity Analysis using GAMS output

Shadow Price

<u>Def</u>: ***Shadow Price*** of constraint (resource) *i*

- The ***<u>shadow price</u>*** <u>of resource *i*</u> is the amount that the **optimal objective value** will change with an **additional unit** of that resource

Other interpretations:

- **Opportunity cost** of using one unit of that resource (e.g. adding new activity)
- Marginal worth of an additional unit of that resource
- Market price of that resource (Duality)

Also know as:

- Dual Variable

Shadow Price

<u>Thm</u>: Suppose optimal solution of a Std Ineq or Eq LP is nondegenerate (i.e. all basic variable strictly positive)

If the RHS of the i^{th} constraint is changed from b_i to $b_i + \varepsilon$ (ε sufficiently small), then the i^{th} dual variable, y_i, is the shadow price of that constraint/resource.

Shadow Price/Dual Variable & Reduced Cost

In CO350, the formula for reduced cost is:

$$\bar{c}_j = c_j - A_j^T \mathbf{y},$$

Can we find the shadow price from the final tableau?

- In <u>standard inequality form</u>:
  - If s_i is the slack variable of row i, then the shadow price for row i is negative of the reduced cost of s_i

New Activity

GTC is thinking about making hammers

- Each hammer requires 2lb of steel, 1 hour of molding and 0.4 hours of assembly
- Price of hammers are $100 per thousand hammers

Without solving a new LP, can you tell if GTC should bother making hammers?

$$\begin{aligned} \max\ & 130x_1 + 100x_2 \\ s.t.\ & 1.5x_1 + x_2 \le 27 \quad \text{(steel)} \\ & x_1 + x_2 \le 21 \quad \text{(molding)} \\ & 0.3x_1 + 0.5x_2 \le 9 \quad \text{(assembly)} \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

Final Tableau:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
z	**2460**	0	0	60	40	0
x_1	12	1	0	2	- 2	0
x_2	9	0	1	-2	3	0
s_3	0.9	0	0	0.4	-0.9	1

Pricing Out an Activity

We need to find the reduced cost for hammers, given the current optimal solution:

⇨ i.e. find marginal contribution of hammers to objective value (revenue)

- Revenue: 1K unit of hammers gives us $100 in revenue
- Cost: *Opportunity cost* of diverting resources away from current optimal "production mix" to make hammers
  - Think of shadow prices as *opportunity cost* of using that resources (i.e., effect on opt value when RHS increases by -1)

Find Total Opportunity Cost:

Resource	Required to make hammer	Opportunity Cost	Total
Steel			
Molding			
Assembly			

Pricing Out an Activity

Net Marginal contribution = Contribution to Revenue – Total Opportunity Costs

Reduced Costs = Original Objective Coefficient – Shadow Price * Resources Required

Pricing Out an Activity

So producing 1K units of hammers will decreasing objective value by $-60, so GTC does <u>not</u> want to make hammers

- We didn't need to solve a new LP

How can we re-price hammers so that it would become attractive to produce hammers?

- Shadow Prices (Opportunity Costs) can be used to determine prices of new products

Pricing Out an Activity

If the Price of hammers were $200, what would we do next?

- Reduced Cost =
- Would it be attractive to produce hammers?

Changes in LP Data

Suppose the price of wrenches or pliers change, or the resource availability changes:

- Changes in Objective Coefficient for **nonbasic** and **basic** variables:
  - Current Solution always remain *feasible* when we change the objective coefficients – but it may no longer be optimal
- Changes in RHS vector for **non-binding** and **binding** constraints:
  - The optimality (reduced costs) of the Primal is not affected, but the current basis may no longer be feasible.

Changes in Objective Coefficients of Nonbasic Variable

Final Tableau:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
z	**2460**	0	0	60	40	0
x_1	12	1	0	2	- 2	0
x_2	9	0	1	-2	3	0
x_5	0.9	0	0	0.4	-0.9	1

- What happens when we decrease c_3?

- What happens when we increase c_3?

Changes in Objective Coefficients of NBV

Let's link this to the Simplex Method:

$$z + 60x_3 + 40x_4$$
$$= [z - 130x_1 - 100x_2] - \sum_{i=1,2,3} y_i \left(\sum_{j=1,..,5} a_{i,j} x_j \right)$$

Changes in Objective Coefficients of Basic Variables

- What happens when we decrease c_1 ?

- What happens when we increase c_1 ?

Changes in Objective Coefficients of BV

Let's link this to the Simplex Method:

$$z + 60x_3 + 40x_4$$
$$= \left[z - 130x_1 - 100x_2\right] - \sum_{i=1,2,3} y_i \left(\sum_{j=1,..,5} a_{i,j} x_j \right)$$

$$\left[z - (130 + \Delta c_1)x_1 - 100x_2\right] - \sum_{i=1,2,3} y_i \left(\sum_{j=1,..,5} a_{i,j} x_j \right)$$

Changes in Objective Coefficients of BV

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
z	**2460**	$-\Delta c_1$	0	60	40	0
x_1	12	1	0	2	- 2	0
x_2	9	0	1	-2	3	0
x_5	0.9	0	0	0.4	-0.9	1

But since x_1 it's z-row coefficient must be 0

- Make its z-row coefficient 0 by adding Δc_1 times row 1 to z-row

New z-row becomes:

Changes in Right-Hand-Side

- Changing the RHS will clearly change the *values* of the current optimal solution, so the current basis may become infeasible
- However, it does not affect the optimality condition
  - If the basis is the same, then the *shadow price/dual variables* and *reduced costs* are the same

Analyze effects of changes RHS for:

- Nonbinding constraints (LHS < RHS)
- Binding constraints (LHS = RHS)

Changes in RHS of Nonbinding Constraint

- What happens when we change the RHS of the *assembly constraint:*

$$0.3x_1 + 0.5x_2 \le 9 + \Delta b_3$$

- In the optimal solution, x_5 =0.9 so the constraint is nonbinding, thus its shadow price is 0 (why?)

- What happens if we increase b_3 ?

- What happens if we decrease b_3 ?

Changes in RHS of Binding Constraint

- What happens when we change the RHS of the *steel constraint:*

$$1.5x_1 + x_2 \le 27 + \Delta b_1$$

- In the optimal solution, x_3 =0, so the constraint is binding.
- Changing the RHS by Δb_1 is the same as decreasing slack variable x_5 (nonbasic so current =0) by Δb_1
- How does this effect the current basic feasible solution?

Changes in RHS of Binding Constraint

Final Tableau:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
z	**2460**	0	0	60	40	0
x_1	12	1	0	2	- 2	0
x_2	9	0	1	-2	3	0
x_5	0.9	0	0	0.4	-0.9	1

Dual Simplex Method

New Final Tableau:

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
z	**2460 + 60Δb_1**	0	0	60	40	0
x_1	$12 + 2\ \Delta b_1$	1	0	2	- 2	0
x_2	$9 - 2\Delta b_1$	0	1	-2	3	0
x_6	$0.9 + 0.4\Delta b_1$	0	0	0.4	-0.9	1

- What happens if we increase b_1 by 5? (Δb_1 = +5)