Consider an integer program of the form

$$(IP) \qquad \max\ c^T x \qquad \text{s.t.}\ \ x \in X$$

where $X = \{x \in \mathbb{R}^n : Ax \le b,\ x \ge 0,\ x \text{ int}\}$

$A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m,\ c \in \mathbb{R}^n.$

The IP

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & \tilde{A}x \le \tilde{b} \\ & x \ge 0 \\ & x \text{ integer} \end{aligned}$$

is a formulation for (IP) if

$$X = \tilde{X} := \{x \in \mathbb{R}^n : \tilde{A}x \le \tilde{b},\ x \ge 0,\ x \text{ int}\}$$

Have argued that, ideally we want

$$\tilde{X} = \operatorname{conv}(X)$$

but this may be computationally intractable.

More modest goal: add a few useful inequalities to formulation (IP).

How do we compare two formulations

$$(P_1) \quad \max \ c^T x \qquad\qquad (P_2) \quad \max \ c^T x$$

$$\text{s.t.} \quad A_1 x \le b_1 \qquad \text{and} \qquad \text{s.t.} \quad A_2 x \le b_2 \qquad ?$$

$$x \ge 0 \qquad\qquad x \ge 0$$

$$x \text{ int} \qquad\qquad x \text{ int}$$

We know that $\max \ c^T x$ s.t. $x \in \text{conv}(X)$

has an optimal integral solution.

A formulation is good if

1) it contains all points in X
2) it does not contain much else

Def: Formulation (P_1) is (strictly) stronger than (P_2) if

$$\forall x \in X_1 \Rightarrow x \in X_2$$

$$(\exists \, x \in X_2 \Rightarrow x \notin X_1)$$

$$(X_i = \{ x \in \mathbb{R}^n : A_i x \le b_i \, , \, x \ge 0 \})$$

Ex: (Facility location)

Problem: want to build warehouses at some locations to serve customers

- F: possible locations for warehouses / facilities

- $\mathcal{C}$: locations of customers
- $\forall i \in \mathcal{F}$: incur cost $\$ f_i$ for building facility at location i
- each customer j needs to access a warehouse. If customer j accesses warehouse $i \in \mathcal{F}$, she incurs cost c_{ij}.

  (think: $c_{ij} \equiv$ distance between i and j)

- For company there are 2 conflicting goals
  1) spend as little as possible on building facilities
  2) minimize avg. cost of consumer for accessing warehouse

Model this by: company wants to minimize

total building cost

$+$

total access cost

Here's an IP formulation that can be used to solve this problem:

$$y_i = \begin{cases} 1 & : \text{build fac at } i \in \mathcal{F} \\ 0 & : \text{othw.} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & : \text{customer } j \text{ is served by } i \in \mathcal{F} \\ 0 & : \text{othw.} \end{cases}$$

(IP_1) $$\min \sum_{i \in \mathcal{F}} f_i \cdot y_i + \sum_{j \in \mathcal{C}} \sum_{i \in \mathcal{F}} c_{ij} \cdot x_{ij}$$

s.t.

$$\forall j \in \mathcal{C}: \quad \sum_{i \in \mathcal{F}} x_{ij} = 1$$

$$\forall i \in \mathcal{F}: \quad \sum_{j \in \mathcal{C}} x_{ij} \leq m \cdot y_i$$

$$x, y \quad 0,1\text{-int.}$$

(m = # of customers)

Alternate formulation:

(IP_2) $$\min \sum_{i \in \mathcal{F}} f_i \cdot y_i + \sum_{j \in \mathcal{C}} \sum_{i \in \mathcal{F}} c_{ij} \cdot x_{ij}$$

s.t.

$$\forall j \in \mathcal{C}: \quad \sum_{i \in \mathcal{F}} x_{ij} = 1$$

$$\forall i \in \mathcal{F}, j \in \mathcal{C}: \quad x_{ij} \leq y_i$$

$$x, y \quad 0,1\text{-int}$$

Which formulation is stronger? $\longrightarrow$ (IP_2)

1) x, y feasible for LP relax LP_2 of (IP_2)

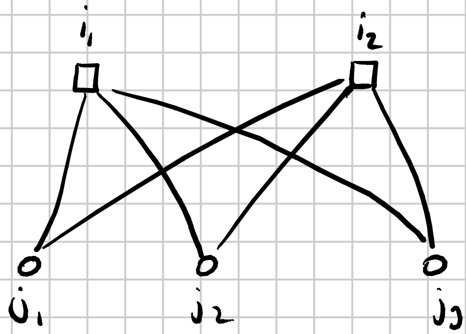
$$\Longrightarrow \quad x_{ij} \leq y_i \qquad \forall i \in \mathcal{F}, \forall j \in \mathcal{C}$$

Sum over all j for fixed i

$$\sum_{j \in \mathcal{C}} x_{ij} \leq |\mathcal{C}| \cdot y_i = m \cdot y_i$$

$\Longrightarrow$ x, y feas. for LP_1 (LP relax of (IP$_1$))

2) There are instances in which (LP$_2$) strictly stronger



$y_{i_1} = \frac{2}{3}$, $y_{i_2} = \frac{1}{3}$, $x_{i_1 j_1} = x_{i_1 j_2} = x_{i_2 j_3} = 1$

$x_{ij} = 0$ othw.

feasible for (P$_1$)

$$\sum_{j \in C} x_{i_1 j} = 2 \leq 3 \cdot y_{i_1} = 2$$

$$\sum_{j \in C} x_{i_2 j} = 1 \leq 3 \cdot y_{i_2} = 1$$

not feasible for (P$_2$)!

$x_{i_2 j_3} \leq y_{i_2}$ violated.